THE SCHUR EXPANSION OF MACDONALD POLYNOMIALS

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Abstract. Building on Haglund's combinatorial formula for the transformed Macdonald polynomials, we provide a purely combinatorial proof of Macdonald positivity using dual equivalence graphs and give a combinatorial formula for the coefficients in the Schur expansion.

1. Introduction

Since Macdonald introduced them in 1988 [Mac88], Macdonald polynomials have been intensely studied and have been found to have applications in such areas as representation theory, algebraic geometry, group theory, statistics, quantum mechanics and more. The transformed Macdonald polynomials $\widetilde{H}_\mu(x; q, t)$ are uniquely characterized by certain triangularity and orthogonality conditions, from which their symmetry follows. Originally conjectured by Macdonald upon introducing these polynomials, one of the main problems has been to show that the coefficients obtained by expanding Macdonald polynomials in the Schur basis are polynomials in the parameters q and t with non-negative integer coefficients. Unfortunately, given the indirect definition of these polynomials, such questions require difficult technical machinery.

Following an idea outlined by Procesi, Garsia and Haiman [GH93] conjectured a representation theoretic interpretation for $\widetilde{H}_\mu(x; q, t)$. In 2001, Haiman [Hai01] resolved the conjecture and hence proved Macdonald positivity by analyzing the algebraic geometry of the Hilbert scheme of n points in the plane. This proof, however, is purely geometric and offers no combinatorial understanding for $\widetilde{H}_\mu(x; q, t)$. In 2004 came another breakthrough with Haglund's conjecture [Hag04] for a combinatorial formula for the monomial expansion of $\widetilde{H}_\mu(x; q, t)$. Proven in 2005 with Haiman and Loehr [HHL05], Haglund's formula does not establish Macdonald positivity, but it does open the door to combinatorial approaches.

In 2005, Haiman suggested looking at the dual equivalence relation on standard Young tableaux, and from it defining an edge-colored graph similar to a crystal graph. The result of this idea is a new combinatorial method for establishing the Schur positivity of a function which is expressed in terms of monomials. This paper describes the method of dual equivalence graphs and outlines how it may used in the case of Macdonald polynomials to obtain a combinatorial formula for the Schur expansion. Complete details of these results will appear in a forthcoming paper [Ass07] which treats the slightly more general case of Lascoux-Leclerc-Thibon polynomials.

2. Preliminaries

We introduce the basic objects in symmetric function theory, including Schur functions and Macdonald polynomials, for the most part following the notation in [Mac95].

2.1. **Partitions and tableaux.** A *partition* is a weakly decreasing (finite) sequence of positive integers: $\lambda = (\lambda_1, \ldots, \lambda_l), \lambda_1 \geq \cdots \lambda_l > 0$. A *composition* is any (finite) sequence of non-negative integers: $\pi = (\pi_1, \ldots, \pi_l), \pi_i \geq 0$.

We identify a partition with its Young diagram, that is the set of points (i, j) in the $\mathbb{Z}_+ \times \mathbb{Z}_+$ lattice quadrant such that $1 \leq i \leq \lambda_j$. We draw the diagram so that each point (i, j) is represented by the unit cell southwest of the point.

A *filling* of a diagram λ is a map $S : \lambda \rightarrow \mathbb{Z}_+$. A *semi-standard Young tableau* is a filling which is weakly increasing along each row and strictly increasing along each column. A filling is *standard* if it is a bijection from λ to $[n]$, where $[n] = \{1, 2, \ldots, n\}$. If a filling $S : \lambda \rightarrow \mathbb{Z}_+$ contains entries $1^{\pi_1}, 2^{\pi_2}, \ldots$ for

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some composition π, then we say S has *shape* λ and *weight* π. Let $\mathrm{SSYT}(\lambda)$ (respectively $\mathrm{SYT}(\lambda)$) denote the set of semi-standard tableaux (respectively standard tableaux) of shape λ.

The *reading word* of a filling S, denote w_S, is the word obtained by reading the rows of S from left to right, beginning with the top row.

2.2. **Symmetric functions.** A function $f(x) = f(x_1, x_2, \ldots)$ is *symmetric* if it is invariant under any permutation of the variables. Let $\Lambda_{\mathbb{F}}$ denote the ring of symmetric functions over the field $\mathbb{F}$. The dimension of the vector space of symmetric functions of degree n is equal to the number of partitions of n. The *Schur functions*, s_λ, form a basis for $\Lambda_{\mathbb{F}}$. They may be defined as the generating function for semi-standard tableaux as

$$s_\lambda(x) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T, \tag{2.1}$$

where x^T is the monomial $x_1^{\pi_1} x_2^{\pi_2} \cdots$ when T has weight π.

For $\sigma \in \{\pm 1\}^{n-1}$, define the *fundamental quasi-symmetric function* $Q_\sigma(x)$ by

$$Q_\sigma(x) = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_j = i_{j+1} \Rightarrow \sigma_j = +1}} x_{i_1} \cdots x_{i_n}. \tag{2.2}$$

The *descent signature of a standard filling* S is given by

$$\sigma(S)_i \;=\; \begin{cases} +1 & \text{if } i \text{ is left of } i{+}1 \text{ in } w_S \\ -1 & \text{if } i{+}1 \text{ is left of } i \text{ in } w_S \end{cases} \tag{2.3}$$

where w_S is the reading word S. The Schur function s_λ can be expressed in terms of quasi-symmetric functions as

$$s_\lambda(x) = \sum_{T \in \mathrm{SYT}(\lambda)} Q_{\sigma(T)}(x). \tag{2.4}$$

Comparing equation (2.1) with equation (2.4), using quasi-symmetric functions instead of monomials allows us to work with standard tableaux rather than semi-standard tableaux.

2.3. **Macdonald polynomials.** The transformed Macdonald polynomials [Mac88] form a basis for symmetric functions with two additional parameters. More precisely, $\{\widetilde{H}_\mu(x;q,t)\}$ is a basis for $\Lambda_{\mathbb{Q}(q,t)}$. Haglund's monomial (quasi-symmetric) expansion for Macdonald polynomials [Hag04, HHL05] gives an explicit combinatorial description of $\widetilde{H}_\mu(x;q,t)$ as the q,t-generating function of (standard) fillings of the diagram of μ.

For a cell x, let $l(x)$ (respectively $a(x)$) denote the number of cells directly north (respectively east) of x. A *descent* of a filling S is a cell c whose entry is greater than the entry of the cell immediately south of it. Denote by $\mathrm{Des}(S)$ the set of all descents of S. Define the *major index* by

$$\mathrm{maj}(S) \;\stackrel{\text{def}}{=}\; |\mathrm{Des}(S)| + \sum_{c \in \mathrm{Des}(S)} l(c). \tag{2.5}$$

An ordered pair of cells (c,d) is called *attacking* if c and d lie in the same row with c to the west of d, or if c is in the row immediately north of d and c lies strictly east of d. An *inversion pair* of S is an attacking pair (c,d) where the entry of c is greater than the entry of d. Denote by $\mathrm{Inv}(S)$ the set of inversion pairs of S. Define the *inversion number* by

$$\mathrm{inv}(S) \;\stackrel{\text{def}}{=}\; |\mathrm{Inv}(S)| - \sum_{c \in \mathrm{Des}(S)} a(c). \tag{2.6}$$

Note that if $c \in \mathrm{Des}(S)$, then for every cell e of the arm of c, the entry of e is either bigger than the entry of d or smaller than the entry of c (or both). In the former case, (e,d) will form an inversion pair, and in the latter case, (c,e) will form an inversion pair. Therefore $\mathrm{inv}(S)$ is a non-negative integer.

Definition 2.1. The transformed Macdonald polynomials are given by

$$\widetilde{H}_\mu(x;q,t) = \sum_{S:\mu \xrightarrow{\sim} [n]} q^{\mathrm{inv}(S)} t^{\mathrm{maj}(S)} Q_{\sigma(S)}(x). \tag{2.7}$$

It is a theorem in [HHL05] that equation (2.7) satisfies the conditions which uniquely characterize the transformed Macdonald polynomials as originally defined in [Mac88]. The proof is by an elegant and elementary combinatorial argument, so we take Haglund's formula as the definition.

3. Dual equivalence graphs

Comparing equations (2.4) and (2.7), in order to move from the quasi-symmetric expansion of $\widetilde{H}_\mu(x;q,t)$ to the Schur expansion, we must group together quasi-symmetric functions with the same maj and inv statistics which go into a single Schur function. The idea is to build a graph where each vertex corresponds to a quasi-symmetric function and the connected components group together quasi-symmetric functions which occur in a single Schur function. Then summing over the vertices of the graph gives the quasi-symmetric expansion, and summing over the connected components gives the Schur expansion, provided the statistics are fixed by the edges.

3.1. **Graph on standard tableaux.** Dual equivalence was first defined by Haiman [Hai92] as a relation on standard tableaux which is "dual" to *jeu de taquin* equivalence under the Schensted correspondence. An *elementary dual equivalence* on three consecutive letters $i-1, i, i+1$ of a standard word is given by switching the outer two letters whenever the middle letter is not i:

$$\cdots\ i\ \cdots\ i\pm 1\ \cdots\ i\mp 1\ \cdots \equiv^* \cdots\ i\mp 1\ \cdots\ i\pm 1\ \cdots\ i\ \cdots \tag{3.1}$$

Following Haiman's idea, construct a graph on standard tableaux in the following way. Whenever T and U have reading words which differ by an elementary dual equivalence for $i-1, i, i+1$, connect T and U with an edge colored by i, and to each T, assign the signature $\sigma(T)$ as defined in equation (2.3). The role of the signatures is to index the quasi-symmetric function corresponding to the given vertex.

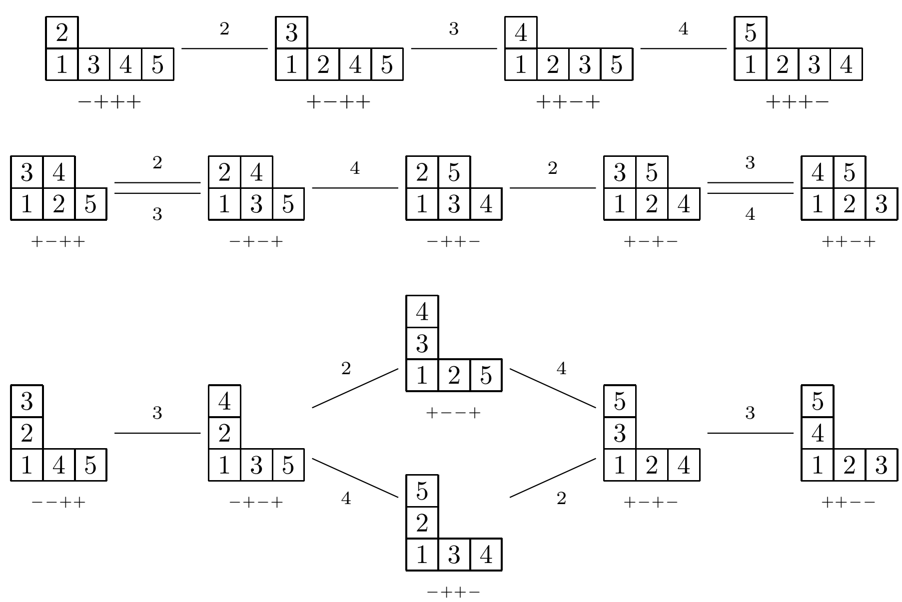


FIGURE 1. The standard dual equivalence graphs $\mathcal{G}_{(4,1)}, \mathcal{G}_{(3,2)}$ and $\mathcal{G}_{(3,1,1)}$.

Let $\mathcal{G}_\lambda$ denote the subgraph on tableaux of shape λ; see Figure 1. It follows from results in [Hai92] that the $\mathcal{G}_\lambda$ exactly give the connected components of the graph.

Define the generating function associated to $\mathcal{G}_\lambda$ by

$$\sum_{v \in V(\mathcal{G}_\lambda)} Q_{\sigma(v)}(x) = s_\lambda(x). \tag{3.2}$$

In particular, the generating function of any vertex-signed graph whose connected components are isomorphic to graphs $\mathcal{G}_\lambda$ is automatically Schur positive.

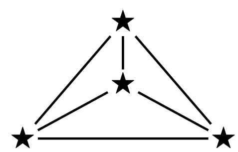


FIGURE 2. An illustration of axiom 4. Here ★'s represent connected components of $(V, \sigma, E_2 \cup \cdots \cup E_{i-1})$ and edges indicate when two components have vertices connected by an E_i edge.

3.2. **Local characterization.** A *signed, colored graph of degree* n consists of the following data: a vertex set V; a signature function $\sigma : V \to \{\pm 1\}^{n-1}$; for each $1 < i < n$, a collection E_i of pairs of distinct vertices of V. We denote such a graph by $\mathcal{G} = (V, \sigma, E_2 \cup \cdots \cup E_{n-1})$ or simply (V, σ, E).

We say that two signed, colored graphs are *isomorphic* if there is a bijection between vertex sets that respects signatures and color-adjacency. Definition 3.1 gives criteria for which signed, colored graphs are isomorphic to $\mathcal{G}_\lambda$.

Definition 3.1. A signed, colored graph $\mathcal{G} = (V, \sigma, E)$ of degree n is a *dual equivalence graph* if the following hold:

- *(ax1)* For $w \in V$ and $1 < i < n$, $\sigma(w)_{i-1} = -\sigma(w)_i$ if and only if there exists $x \in V$ such that $\{w, x\} \in E_i$. Moreover, x is unique when it exists.
- *(ax2)* Whenever $\{w, x\} \in E_i$, $\sigma(w)_j = -\sigma(x)_j$ for $j = i-1, i$, and $\sigma(w)_h = \sigma(x)_h$ for $h < i-2$ and $h > i+1$.
- *(ax3)* Whenever $\{w, x\} \in E_i$, if $\sigma(w)_{i-2} = -\sigma(x)_{i-2}$, then $\sigma(w)_{i-2} = -\sigma(w)_{i-1}$, and if $\sigma(w)_{i+1} = -\sigma(x)_{i+1}$, then $\sigma(w)_{i+1} = -\sigma(w)_i$.
- *(ax4)* Every connected component of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$ appears in Figure 4, and between any two vertices of a connected component of $(V, \sigma, E_2 \cup \cdots \cup E_i)$ is a path containing at most one E_i edge.
- *(ax5)* Whenever $|i - j| \geq 3$, $\{w, x\} \in E_i$ and $\{x, y\} \in E_j$, there exists $v \in V$ such that $\{w, v\} \in E_j$ and $\{v, y\} \in E_i$.

For a dual equivalence graph of degree 3, we must stipulate that every connected component of $(V, \sigma, E_{i-1} \cup E_i)$ appears in Figure 3. However, for graphs of degree > 3, this is implied by axiom 4.

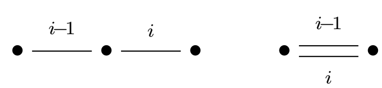


FIGURE 3. Possible 2-color connected components of a dual equivalence graph.

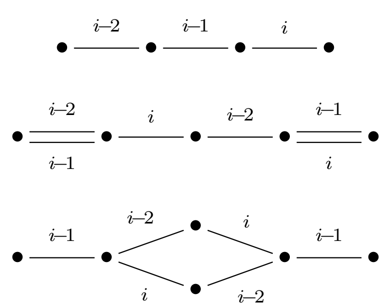


FIGURE 4. Possible 3-color connected components of a dual equivalence graph.

Comparing Figure 1 with Figure 4, connected components of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$ are exactly the graphs for $\mathcal{G}_\lambda$ when λ is a partition of 5. Taking this comparison to its ultimate conclusion, we have the following result.

Theorem 3.2. *For λ a partition of n, $\mathcal{G}_\lambda$ is a dual equivalence graph of degree n. Moreover, every connected component of a dual equivalence graph of degree n is isomorphic to $\mathcal{G}_\lambda$ for a unique partition λ of n.*

The first statement of Theorem 3.2 is a straightforward check of the axioms. The second statement is proved by induction. We assume the existence such an isomorphism for a connected component $\mathcal{C}$ of $(V, \sigma, E_2 \cup \cdots \cup E_{i-1})$ and extend the isomorphism consistently to connected components of $(V, \sigma, E_2 \cup \cdots \cup E_{i-1})$ which lie across an i-edge from $\mathcal{C}$ using the former part of axiom 4. Finally, we use the latter part of axiom 4 together with the fact that the $\mathcal{G}_\lambda$ are pairwise non-isomorphic and contain no nontrivial automorphisms to lift the isomorphism to $(V, \sigma, E_2 \cup \cdots \cup E_i)$.

We conclude this section by interpreting Theorem 3.2 in terms of symmetric functions.

Corollary 3.3. *Let $\mathcal{G}$ be a dual equivalence graph of degree n for which every vertex is assigned statistics α, β. If α and β are constant on connected components of $\mathcal{G}$, then*

$$\sum_{v \in V(\mathcal{G})} q^{\alpha(v)} t^{\beta(v)} Q_{\sigma(v)}(x) = \sum_{\lambda} \sum_{\mathcal{C} \cong \mathcal{G}_\lambda} q^{\alpha(\mathcal{C})} t^{\beta(\mathcal{C})} s_\lambda(x), \tag{3.3}$$

where the sum of over connected components of $\mathcal{G}$ which are isomorphic to $\mathcal{G}_\lambda$. In particular, the generating function for $\mathcal{G}$ so defined is symmetric and Schur positive.

4. A graph for Macdonald polynomials

Next we construct a signed, colored graph with generating function $\widetilde{H}_\mu(x; q, t)$. While the graph will not be a dual equivalence graph, it motivates a new type of graph, called a D graph, which is a generalization of dual equivalence graphs with a similar positivity result for the generating function.

4.1. Statistic-preserving involutions.

The vertex set V_μ must be standard fillings of the shape μ, and the signature function must be given by equation (2.3). The key to defining the edges is the following two involutions on words which admit an elementary dual equivalence for $i-1, i, i+1$. Define

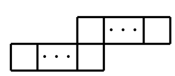
$$\cdots i \cdots i \pm 1 \cdots i \mp 1 \cdots \quad \overset{d_i}{\longleftrightarrow} \quad \cdots i \mp 1 \cdots i \pm 1 \cdots i \cdots\ , \tag{4.1}$$

$$\cdots i \cdots i \pm 1 \cdots i \mp 1 \cdots \quad \overset{\widetilde{d}_i}{\longleftrightarrow} \quad \cdots i \pm 1 \cdots i \mp 1 \cdots i \cdots\ , \tag{4.2}$$

where all other entries remain fixed. As in the case for dual equivalence, we may extend d_i and $\widetilde{d}_i$ to standard fillings of a shape using the reading word.

If the cells of i and the farther away of $i \pm 1$ form a potential descent or inversion, then $\widetilde{d}_i$ will preserve the maj and inv statistics. In the other case, when the cell of i does not form a potential descent or inversion with the cells of $i-1, i+1$, then d_i will preserve the maj and inv statistics. Two cells form a potential descent or inversions precisely when they fit inside the pistol which begins at a cell c and follows the reading word until reaching the cell directly below c. This motivates the following involution:

$$D_i(S) \;=\; \begin{cases} \widetilde{d}_i(S) & \text{if } i-1, i, i+1 \text{ fit in } \\ d_i(S) & \text{otherwise.} \end{cases} \tag{4.3}$$



From the previous discussion, we see that

$$\mathrm{maj}(S) = \mathrm{maj}(D_i(S)) \text{ and } \mathrm{inv}(S) = \mathrm{inv}(D_i(S)). \tag{4.4}$$

For $1 < i < n$, define the i-colored edges E_i to be the set of pairs $\{v, D_i(v)\}$ for each v which admits an elementary dual equivalence for $i-1, i, i+1$. Finally, define $\mathcal{H}_\mu = (V_\mu, \sigma, E)$.

Theorem 4.1. *The graph $\mathcal{H}_\mu$ satisfies axioms 1, 2, 3 and 5. Furthermore, the major index and inversion number are constant on connected components of $\mathcal{H}_\mu$.*

Proof. Equation (4.4) shows that maj and inv are connected components of $\mathcal{H}_\mu$. Axioms 1 and 2 are immediate from the definitions of d_i and $\widetilde{d}_i$. For axiom 3, suppose $\{w, x\} \in E_i$ and $\sigma(w)_{i-2} = -\sigma(x)_{i-2}$. If $x = d_i(w)$, then both $i-2$ and $i+1$ must lie between $i-1$ and i. In particular, $\sigma(w)_{i-2} = -\sigma(w)_{i-1}$. If $x = \widetilde{d}_i(w)$, then $i-2$ must lie between the position of $i-1$ in w and the position of $i-1$ in x. In particular, $i-2$ must lie between $i-1$ and i in both w and x, and so again $\sigma(w)_{i-2} = -\sigma(w)_{i-1}$. The result for $\{w, x\} \in E_i$

with $\sigma(w)_{i+1} = -\sigma(x)_{i+1}$ is analogous. If $|i-j| \geq 3$, then $\{i-1, i, i+1\} \cap \{j-1, j, j+1\} = \emptyset$, and so $D_i D_j(w) = D_j D_i(w)$ when defined, thereby showing axiom 5. □

4.2. **D graphs.** While $\mathcal{H}_\mu$ is not quite a dual equivalence graph, it shares many of the same local properties. By characterizing the local structure of $\mathcal{H}_\mu$, we define a *D graph*. Whereas the generating function of a connected dual equivalence graph is a single Schur function, the generating function of a connected D graph is a sum of Schur functions.

Define a D graph by replacing axiom 4 in Definition 3.1 with a weaker axiom satisfied by $\mathcal{H}_\mu$.

Definition 4.2. A signed, colored graph $\mathcal{G} = (V, \sigma, E)$ of degree n is a *D graph* if axioms 1, 2, 3 and 5 (Definition 3.1) hold for $\mathcal{G}$ and the following also holds:

(ax4′) For $2 < i < n$, for every connected component of $(V, \sigma, E_{i-1} \cup E_i)$, there exists a bijection ϕ from the vertex set to $\mathrm{SYT}(\lambda_1) \cup \cdots \cup \mathrm{SYT}(\lambda_m)$ for distinct partitions λ_p of 4 such that for $j = i-2, i-1, i$, $\sigma(v)_j = \sigma(\phi(v))_{j-(i-3)}$. For $3 < i < n$, for every connected component of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$, there exists a bijection ϕ from the vertex set to $\mathrm{SYT}(\lambda_1) \cup \cdots \cup \mathrm{SYT}(\lambda_m)$ for distinct partitions λ_p of 5 such that for $j = i-3, i-2, i-1, i$, $\sigma(v)_j = \sigma(\phi(v))_{j-(i-4)}$.

Notice that axiom 4 implies axiom $4'$; in particular, a dual equivalence graph is a D graph. More to the point, we have the following.

Theorem 4.3. *For μ a partition of n, $\mathcal{H}_\mu$ is a D graph of degree n for which* maj *and* inv *are constant on connected components.*

Proof. By Theorem 4.1, we need only demonstrate axiom $4'$. Note that it suffices to check standard fillings of skew shapes of size 5. Furthermore, since the positions letters only matter relative to whether or not they fit into a pistol, we may assume the cells fit inside the diagram $(5, 4, 3, 2, 1)$. Now that there are finitely many cases, checking each is a straightforward task best left to a computer. □

D graphs may be defined more generally by removing the uniqueness constraint of axiom $4'$. For full details in that case, see the treatment in [Ass07].

5. Transforming a D graph into a dual equivalence graph

The goal is to modify the edges of a D graph so that it becomes a dual equivalence graph. We do this inductively by constructing a sequence of graphs

$$\mathcal{G} = \mathcal{G}_2, \ldots, \mathcal{G}_{n-1} = \widetilde{\mathcal{G}}$$

with the following properties. For each $i = 2, \ldots, n-1$, $\mathcal{G}_i$ will be a D graph of degree n, and the restriction of $\mathcal{G}_i$ to edge colors $< i$ will be a dual equivalence graph. For each $i \geq 3$, the vertex set, signature function and edges E_j for $j \neq i$ of $\mathcal{G}_i$ will be the same as for $\mathcal{G}_{i-1}$. Furthermore, vertices which are paired in E_i in $\mathcal{G}_i$ will have the property that they lie on the same connected component of the $(V, \sigma, E_2 \cup \cdots \cup E_i)$ in $\mathcal{G}_{i-1}$. Once this is established, we will have proven the following.

Theorem 5.1. *Let $\mathcal{G} = (V, \sigma, E)$ be a D graph of degree n, and let α, β be vertex statistics which are constant on connected components. Then there exists a dual equivalence graph $\widetilde{\mathcal{G}} = (V, \sigma, \widetilde{E})$ of degree n such that α, β are constant on connected components of $\widetilde{\mathcal{G}}$.*

In particular, by Corollary 3.3, the generating function of a D graph, defined analogously to equation (3.3), is symmetric and Schur positive. Since the proof of Theorem 5.1 is constructive, we also get an explicit combinatorial formula for the Schur coefficients.

The base case for the proof of Theorem 5.1 follows from the observation that a D graph of degree 3 is always a dual equivalence graph. The inductive step is demonstrated by explicitly constructing $\mathcal{G}_i$ from $\mathcal{G}_{i-1}$ in three stages. We begin by altering the i-edges of connected components of $(V, \sigma, E_{i-1} \cup E_i)$ which do not satisfy the first part of axiom 4, and then we alter the i-edges of connected components of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$ which still do not satisfy the second part of axiom 4. Finally we alter certain i-edges to ensure the remaining condition of axiom 4 is met.

5.1. **Step 1: 2-color components.** Let $\mathcal{G} = (V, \sigma, E)$ be a D graph of degree n, and suppose that the restriction of $\mathcal{G}$ to edge colors $< i$ is a dual equivalence graph. Then each connected component of $(V, \sigma, E_{i-1} \cup E_i)$ is depicted either in Figure 3 or in Figure 5. We define an involution φ which we can use to reconstruct i-edges so that this extra case can no longer occur.

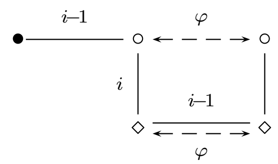


FIGURE 5. The involution of φ on $W(\diamond)$ and $X(\circ)$.

Define sets W and X as indicated in Figure 5. Then define the involution $\varphi : W \cup X \to V$ by

$$\varphi(v) \;=\; \left\{ \begin{array}{rl} E_{i-1}(v) & \text{if } v \in W \\ E_i E_{i-1} E_i(v) & \text{if } v \in X \end{array} \right. . \tag{5.1}$$

In order to ensure that reconnecting the i-edges emanating from vertices of $W \cup X$ still satisfies axiom 5, we must extend φ to i-packages. The *i-package* of a vertex is the connected component of $(V, \sigma, E_2 \cup \cdots \cup E_{i-3})$ which contains it.

Lemma 5.2. *For each $v \in W \cup X$, there exists an isomorphism ϕ_v from the i-package of v to the i-package of $\varphi(v)$ which maps v to $\varphi(v)$. Moreover, if $u \in W \cup X$ lies on the i-package of v, then $\phi_v(u) = \varphi(u)$.*

This is easy to show for vertices of W, and then extending the result to X follows from the fact for every $x \in X$, $x = E_i(w)$ for some $w \in W$. Lemma 5.2 allows us to extend φ consistently as follows.

$$\varphi(u) = \left\{ \begin{array}{l} \phi_v(u) \text{ for } u \text{ on the } i\text{-package of } v \in W \cup X \\ E_i(u) \text{ otherwise} \end{array} \right. \tag{5.2}$$

Define E_i' to be the set of pairs $\{v, \varphi(v)\}$ for each v which admits an i-neighbor. Define a signed, colored graph $\mathcal{G}'$ of type (n, N) by $\mathcal{G}' = (V, \sigma, E_2 \cup \cdots \cup E_{i-1} \cup E_i' \cup E_{i+1} \cup \cdots \cup E_{n-1})$.

Proposition 5.3. *The graph $\mathcal{G}'$ is a D graph of degree n, the restriction to edge colors $< i$ is a dual equivalence graph, and $(V, \sigma, E_{i-1} \cup E_i')$ satisfies axiom 4.*

Proof. Showing that $\mathcal{G}'$ satisfies axioms 1, 2 and 3 is straightforward, and axiom 5 follows immediately from the extension of φ in equation (5.2). Showing that the new i-edges satisfy axiom $4'$ is somewhat subtle. The key here is to analyze the finite possibilities for connected components containing vertices of W and X, and to use axiom 5 to extend the result down i-packages. □

5.2. **Step 2: 3-color components.** With the first step done, we observe that the only possibilities for connected components of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i')$ are those in Figure 4 and the one in Figure 6.

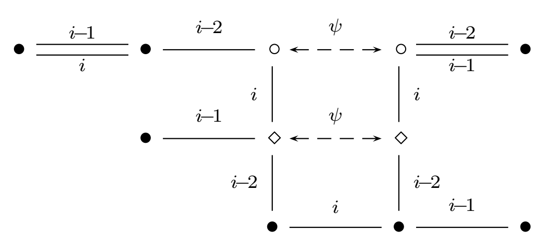


FIGURE 6. The involution of ψ on $Y(\diamond)$ and $Z(\circ)$.

As in the previous step, define sets $Y, Z \subset V$ as indicated in Figure 6, and define an involution $\psi : Y \cup Z \to V$ by

$$\psi(v) \;=\; \left\{ \begin{array}{rl} E_{i-2} E_i' E_{i-2}(v) & \text{if } v \in Y \\ E_i' E_{i-2} E_i' E_{i-2} E_i'(v) & \text{if } v \in Z \end{array} \right. \tag{5.3}$$

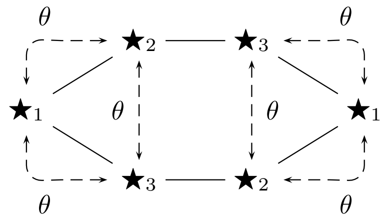


FIGURE 7. An illustration of how to reconnect components of $(V,\sigma,E_2\cup\cdots\cup E_{i-1})$ in $\widehat{\mathcal{G}}$ using the involution θ. Here $\bigstar_1$ is the chosen component.

As with φ, we will use ψ to redefine i-edges so as to remove this extra case. Exactly as before, we show the following.

Lemma 5.4. *For each $v \in Y\cup Z$, there exists an isomorphism ϕ_v from the i-package of v to the i-package of $\psi(v)$ which maps v to $\psi(v)$. Moreover, if $u\in Y\cup Z$ lies on the i-package of v, then $\phi_v(u)=\psi(u)$.*

This is done first for vertices of Y, and then extended to Z using the observation that for $z\in Z$, $z=E_i'(y)$ for some $y\in Y$. This allows us to extend ψ consistently to all vertices of V which admit an i-neighbor as follows.

$$\psi(u) = \begin{cases} \phi_v(u) \text{ for } u \text{ on the } i\text{-package of } v\in Y\cup Z \\ E_i'(u) \text{ otherwise} \end{cases} \tag{5.4}$$

Now define $\widehat{E}_i$ to be the set of pairs $\{v,\psi(v)\}$ for each v which admits an i-neighbor. Define a signed, colored graph $\widehat{\mathcal{G}}$ of degree n by $\widehat{\mathcal{G}}=(V,\sigma,E_2\cup\cdots\cup E_{i-1}\cup\widehat{E}_i\cup E_{i+1}\cup\cdots\cup E_{n-1})$.

Proposition 5.5. *The graph $\widehat{\mathcal{G}}$ is a D graph of degree n, the restriction to edge colors $< i$ is a dual equivalence graph and $(V,\sigma,E_{i-2}\cup E_{i-1}\cup E_i)$ satisfy axiom 4.*

The proof of Proposition 5.5 exactly parallels the proof of Proposition 5.3.

5.3. **Step 3: Resolving axiom** 4. Finally, we must ensure that any two vertices on a connected component of $(V,\sigma,E_2\cup\cdots\cup E_{i-1}\cup\widehat{E}_i)$ are connected by a sequence of edges containing at most one i-edge.

Recall that for a connected dual equivalence graph of degree $i+1$, the connected components of $(V,\sigma,E_2\cup\cdots\cup E_{i-1})$ are pairwise non-isomorphic. The proof of Theorem 3.2 shows that the connected components of $(V,\sigma,E_2\cup\cdots\cup E_{i-1})$ from a given connected component of $(V,\sigma,E_2\cup\cdots\cup E_{i-1}\cup\widehat{E}_i)$ each occur with the same multiplicity, and each component is connected by an $\widehat{E}_i$ edge to exactly one copy of each non-isomorphic component; e.g. see Figure 7.

Select a component $\mathcal{C}$, and for each copy of $\mathcal{C}$, define an involution θ between each pair of components which are both connected to the same copy of $\mathcal{C}$ by an i-edge. Then redefine i-edges using θ similar to before by letting $\widetilde{E}_i$ be the set of pairs $\{v,\theta(v)\}$. Here there is no need for i-packages. Finally, define $\widetilde{\mathcal{G}}=(V,\sigma,E_2\cup\cdots\cup E_{i-1}\cup\widetilde{E}_i\cup E_{i+1}\cup\cdots\cup E_{n-1})$.

Theorem 5.6. *The graph $\widetilde{\mathcal{G}}$ is a D graph of degree n and the restriction to edge colors $\leq i$ is a dual equivalence graph.*

This completes the construction necessary to establish Theorem 5.1.

5.4. **An example.** The left-hand side of Figure 8 is the connected component of $\mathcal{H}_{(5)}$ which contains the tableau with reading word 35412. The signatures have been omitted in the interest of space. On the right is the result of transforming the D graph into a dual equivalence graph. The positions of the vertices have been fixed in order to show how different the resulting edges may be.

6. MACDONALD POSITIVITY

The results of this paper can be summarized as follows.

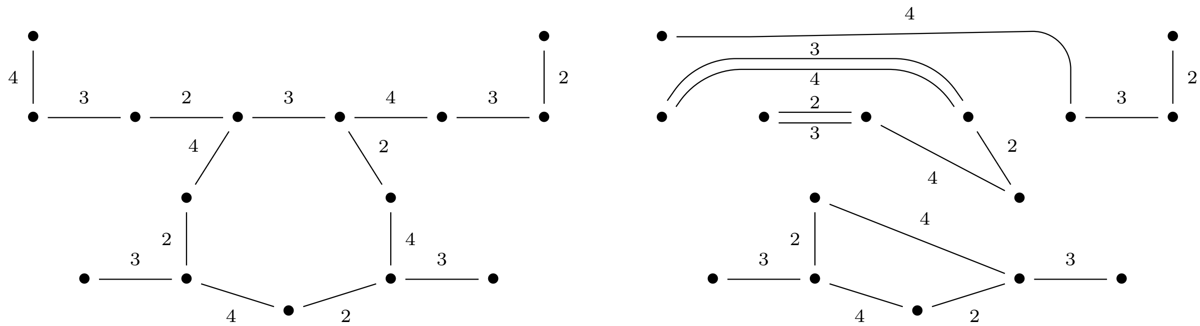


FIGURE 8. A connected component of $\mathcal{H}_{(5)}$ and the corresponding connected components of $\widetilde{\mathcal{H}}_{(5)}$.

Corollary 6.1. *Let $\widetilde{\mathcal{H}}_\mu$ be the dual equivalence graph on standard fillings of μ, which exists by Theorems 4.3 and 5.1. Then*

$$\widetilde{H}_\mu(x;q,t) \;=\; \sum_\lambda \left(\sum_{\mathcal{C}\cong\mathcal{G}_\lambda} q^{\mathrm{inv}(\mathcal{C})} t^{\mathrm{maj}(\mathcal{C})} \right) s_\lambda(x), \tag{6.1}$$

where the sum is over connected components of $\widetilde{\mathcal{H}}_\mu$ which are isomorphic to $\mathcal{G}_\lambda$.

In particular, $\widetilde{H}_\mu(x;q,t)$ is symmetric and Schur positive.

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