A combinatorial proof of Macdonald positivity

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Macdonald positivity

In 1988, Macdonald defined $\widetilde{H}_\mu(x;q,t)$ and proved it to be symmetric. Writing $\widetilde{H}_\mu(x;q,t) = \sum_\lambda \widetilde{K}_{\lambda,\mu}(q,t)s_\lambda(x)$, he conjectured:

Theorem. (Haiman 2001) *We have* $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.

Original proof realizes $\widetilde{H}_\mu(x;q,t)$ as the Frobenius series of doubly graded $\mathcal{S}_n$-module (Garsia-Haiman module) using algebraic geometry of the Hilbert scheme.

Newer proof due to Grojnowski and Haiman relates $\widetilde{H}_\mu(x;q,t)$ to LLT positivity (Haglund's formula) and then uses Kazhdan-Lusztig theory.

Problem: Find a *combinatorial* proof of positivity.

A monomial expansion for $\widetilde{H}_\mu(x;q,t)$

Theorem. (Haglund, Haiman, Loehr 2005)

$$\begin{aligned}\widetilde{H}_\mu(x;q,t) &= \sum_{S:\mu\rightarrow\mathbb{N}} q^{\mathrm{inv}(S)} t^{\mathrm{maj}(S)} x^S \\ &= \sum_{S:\mu\overset{\sim}{\rightarrow}[n]} q^{\mathrm{inv}(S)} t^{\mathrm{maj}(S)} Q_{\sigma(S)}(x)\end{aligned}$$

Proposition. (Gessel 1984)

$$\begin{aligned}s_\lambda(x) &= \sum_{T\in\mathrm{SSYT}(\lambda)} x^T \\ &= \sum_{T\in\mathrm{SYT}(\lambda)} Q_{\sigma(T)}(x)\end{aligned}$$

Quasi-symmetric functions

For $\sigma \in \{\pm 1\}^{n-1}$, define the *quasi-symmetric function*

$$Q_\sigma(x) \; = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_j = i_{j+1} \Rightarrow \sigma_j = +1}} x_{i_1} \cdots x_{i_n}.$$

Define the *descent signature* $\sigma : \mathrm{SYT} \rightarrow \{\pm 1\}^{n-1}$ by

$$\sigma(T)_i \; = \left\{ \begin{array}{ll} +1 & i \text{ left of } i{+}1 \text{ in } w_T \\ -1 & i{+}1 \text{ left of } i \text{ in } w_T \end{array} \right.$$

$$\sigma\left(\begin{array}{|c|c|c|c} \hline 5 & 7 & 10 & \\ \hline 2 & 6 & 8 & \\ \hline 1 & 3 & 4 & \multicolumn{1}{c|}{9} \\ \hline \end{array} \right) \; = \; - \; + \; + \; - \; + \; - \; + \; + \; -$$

Making a vertex-signed graph

Given a quasi-symmetric expansion

$$G(x) = \sum_{v \in V} Q_{\sigma(v)}(x),$$

define a vertex-signed graph $\mathcal{G}$ with vertex set V and signature function $\sigma : V \to \{\pm 1\}^{n-1}$.

Goal: Give sufficient conditions for a vertex-signed graph $\mathcal{G} = (V, \sigma, E)$ to have connected components which satisfy

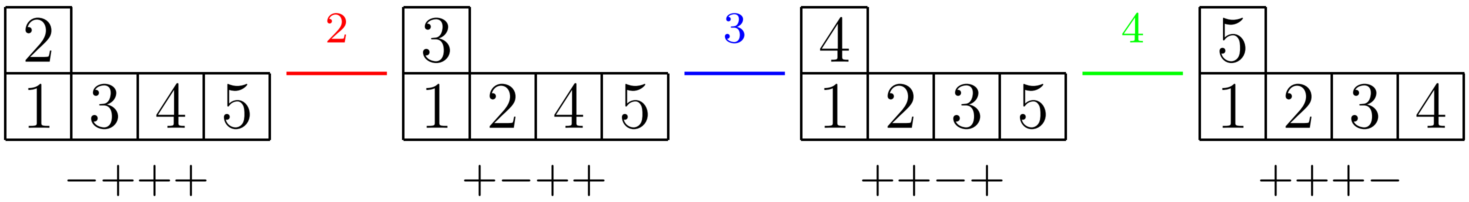
$$\sum_{v \in \mathcal{C}} Q_{\sigma(v)}(x) \ = \ s_{\lambda}(x).$$

Dual equivalence and graphs

An *elementary dual equivalence* for $i-1, i, i+1$ on a standard word is given by

$$\cdots\ i\ \cdots\ i\pm 1\ \cdots\ i\mp 1\ \cdots\ \equiv^{*}\ \cdots\ i\mp 1\ \cdots\ i\pm 1\ \cdots\ i\ \cdots$$

For $T, U \in \mathrm{SYT}$, connect T and U with an i-colored edge whenever w_T and w_U differ by an elementary dual equivalence for $i-1, i, i+1$.



Standard dual equivalence graphs

Proposition. (Haiman 1992) For T, U of partition shape,

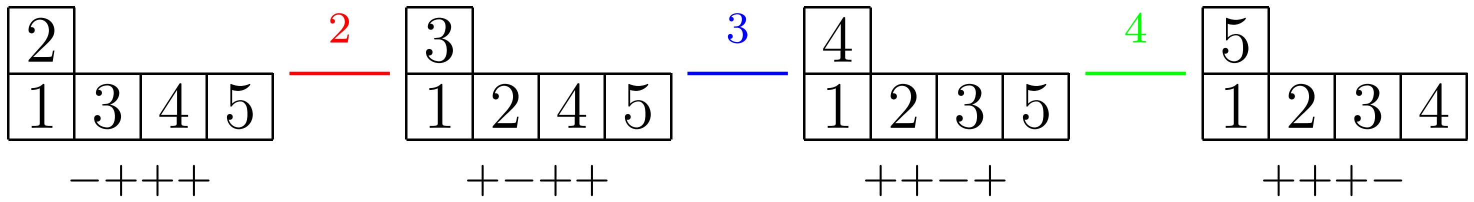
$$T \equiv^* U \;\Leftrightarrow\; \text{shape}(T) = \text{shape}(U).$$

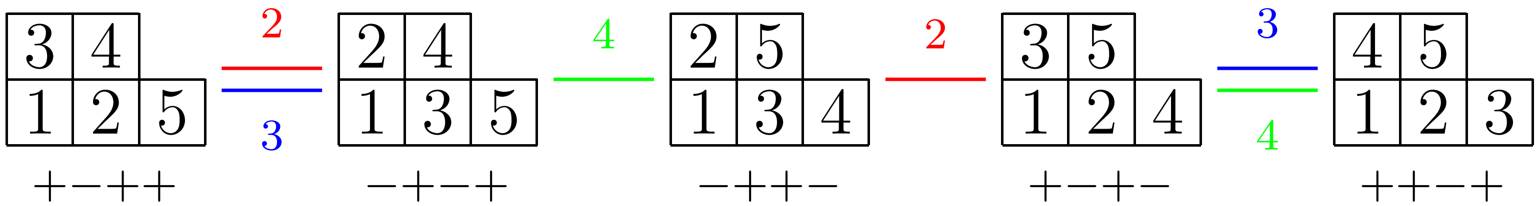
Definition. Let $\mathcal{G}_\lambda$ denote the connected component of the graph containing $\mathrm{SYT}(\lambda)$.

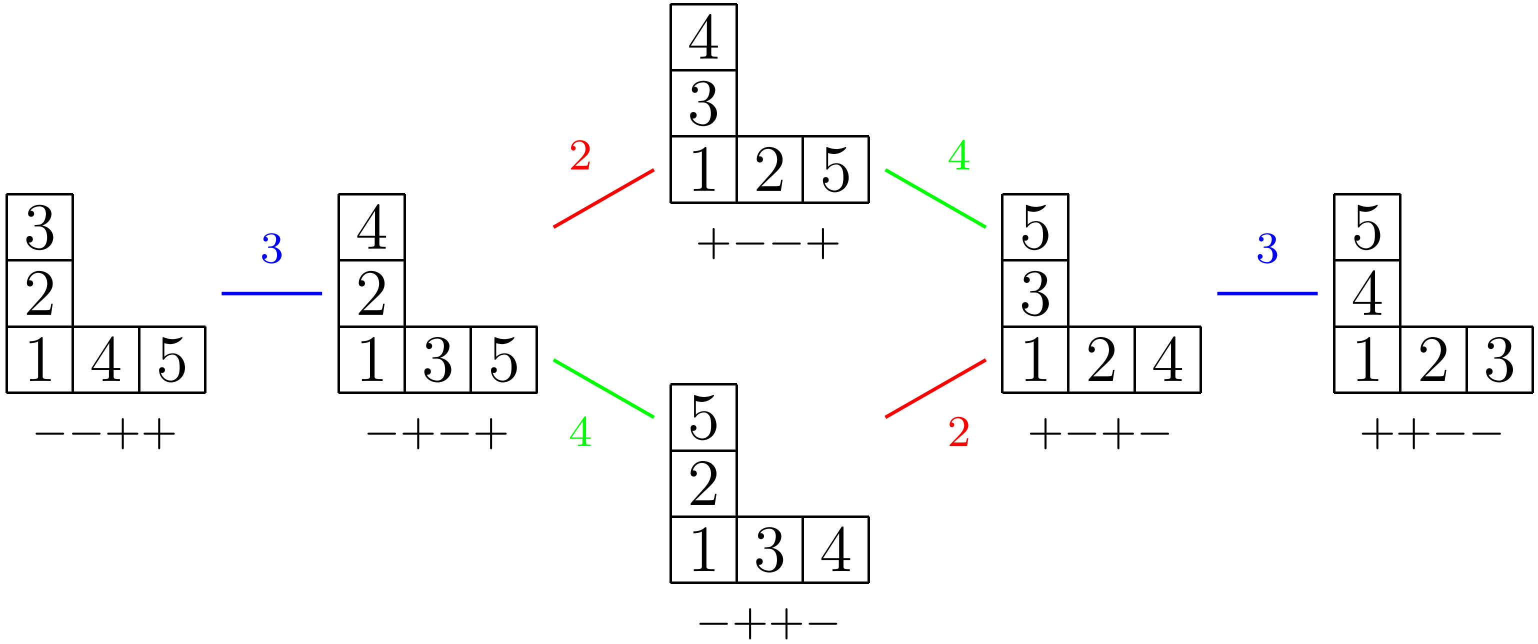
The generating function of $\mathcal{G}_\lambda$ is $\displaystyle\sum_{T \in \mathrm{SYT}(\lambda)} Q_{\sigma(T)}(x) \;=\; s_\lambda(x)$.

Goal: Give sufficient conditions for a vertex-signed, edge-colored graph $\mathcal{G} = (V, \sigma, E)$ to have connected components isomorphic to $\mathcal{G}_\lambda$.

Examples of $\mathcal{G}_\lambda$







Dual equivalence graphs

Definition. A vertex-signed, edge-colored graph $\mathcal{G}$ is a *dual equivalence graph* (DEG) if it satisfies 5 local axioms about signatures and edge colors.

Proposition. *The graph $\mathcal{G}_\lambda$ is a DEG.*

Theorem. *Every connected component of a DEG is isomorphic to $\mathcal{G}_\lambda$ for a unique partition λ.*

Corollary. *If $\mathcal{G}$ is a DEG and α, β are statistics on $V(\mathcal{G})$ which are* constant on connected components, *then*

$$\sum_{v \in V(\mathcal{G})} q^{\alpha(v)} t^{\beta(v)} Q_{\sigma(v)}(x) = \sum_{\lambda} \left(\sum_{\mathcal{C} \cong \mathcal{G}_\lambda} q^{\alpha(\mathcal{C})} t^{\beta(\mathcal{C})} \right) s_\lambda(x).$$

Back to Macdonald polynomials

$$\widetilde{H}_\mu(x;q,t) \;=\; \sum_{S:\mu\tilde{\to}[n]} q^{\mathrm{inv}(S)} t^{\mathrm{maj}(S)} Q_{\sigma(S)}(x)$$

$V_\mu \;=\; \{$standard fillings of $\mu\}$

$\sigma : V_\mu \to \{\pm 1\}^{n-1}$ using row reading word

E must preserve inv and maj

$\mathcal{H}_\mu = (V_\mu, \sigma, E)$ must be a DEG

The inv ***and*** maj ***statistics***

For S a filling of the Young diagram of μ, define

$$\begin{aligned}\mathrm{maj}(S) &= |\mathrm{Des}(S)| + \sum_{c\in\mathrm{Des}(S)} l(c),\\ \mathrm{inv}(S) &= |\mathrm{Inv}(S)| - \sum_{c\in\mathrm{Des}(S)} a(c).\end{aligned}$$

$$\mathrm{Des}\left(\begin{array}{|c|c|c|c}\cline{1-3} 6 & 7 & 5 & \\ \cline{1-3} 4 & 1 & 10 & \\ \hline 3 & 9 & 2 & \multicolumn{1}{c|}{8} \\ \hline\end{array}\right) = \{6, 7, 4, 10\}$$

$$\mathrm{Inv}\left(\begin{array}{|c|c|c|c}\cline{1-3} 6 & 7 & 5 & \\ \cline{1-3} 4 & 1 & 10 & \\ \hline 3 & 9 & 2 & \multicolumn{1}{c|}{8} \\ \hline\end{array}\right) = \left\{\begin{array}{c} (6,5), (7,5), (7,4), (5,4), \\ (5,1), (10,3), (10,9), \\ (3,2), (9,2), (9,8) \end{array}\right\}$$

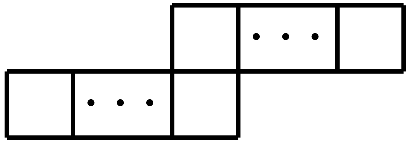
Edges of the graph

Define involutions

$$i \quad i \pm 1 \quad i \mp 1 \quad \xleftrightarrow{d_i} \quad i \mp 1 \quad i \pm 1 \quad i$$

$$i \quad i \pm 1 \quad i \mp 1 \quad \xleftrightarrow{\widetilde{d}_i} \quad i \pm 1 \quad i \mp 1 \quad i$$

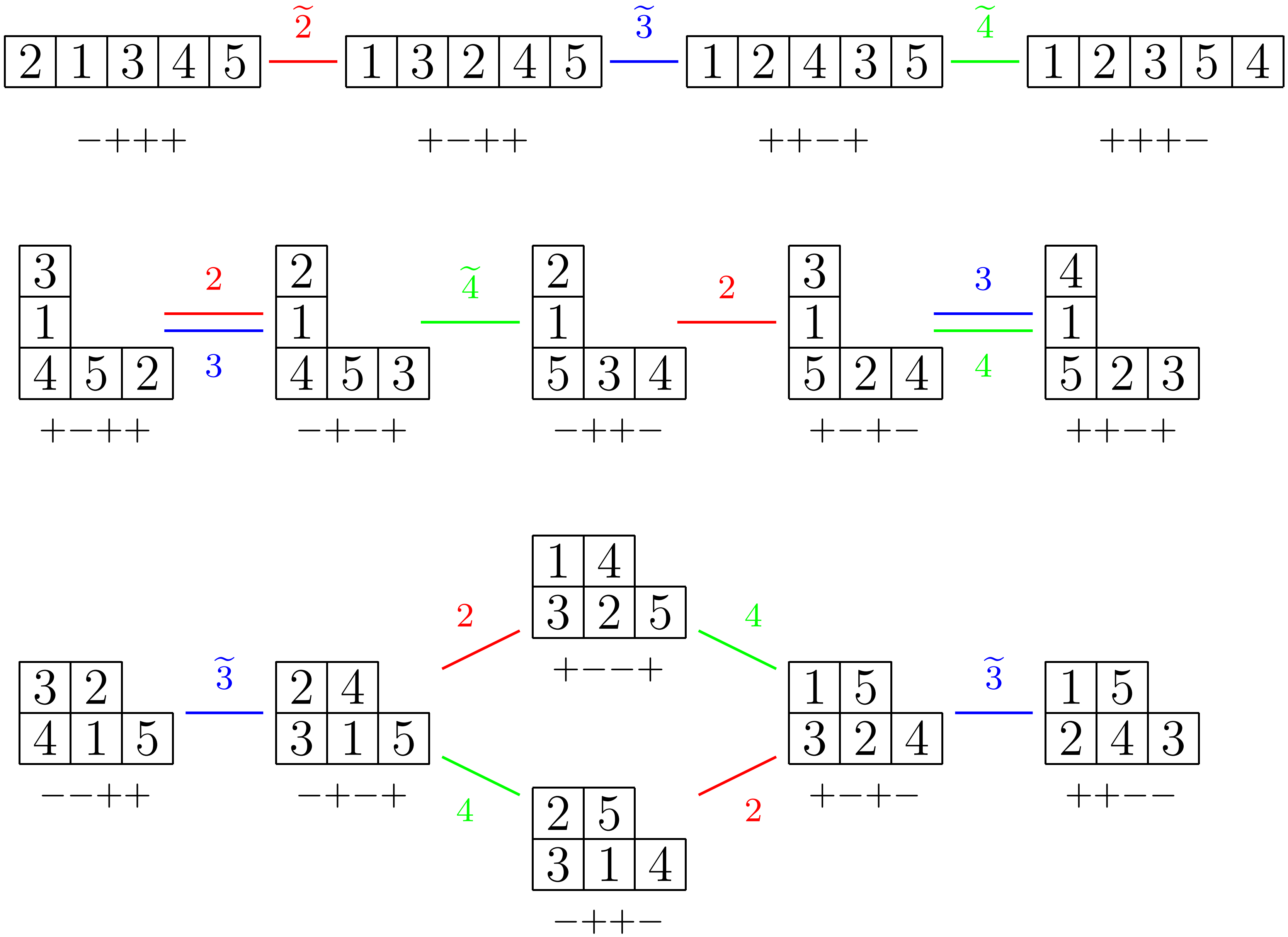
$$D_i(S) = \begin{cases} \widetilde{d}_i(S) & \text{if } i-1, i, i+1 \text{ fit in} \\ d_i(S) & \text{otherwise.} \end{cases}$$



Note: $\mathrm{Des}(S) = \mathrm{Des}\left(D_i(S)\right)$ and $|\mathrm{Inv}(S)| = \left|\mathrm{Inv}\left(D_i(S)\right)\right|$.

Define i-colored edges on V_μ by $S \overset{i}{\rule{2cm}{0.4pt}} D_i(S)$.

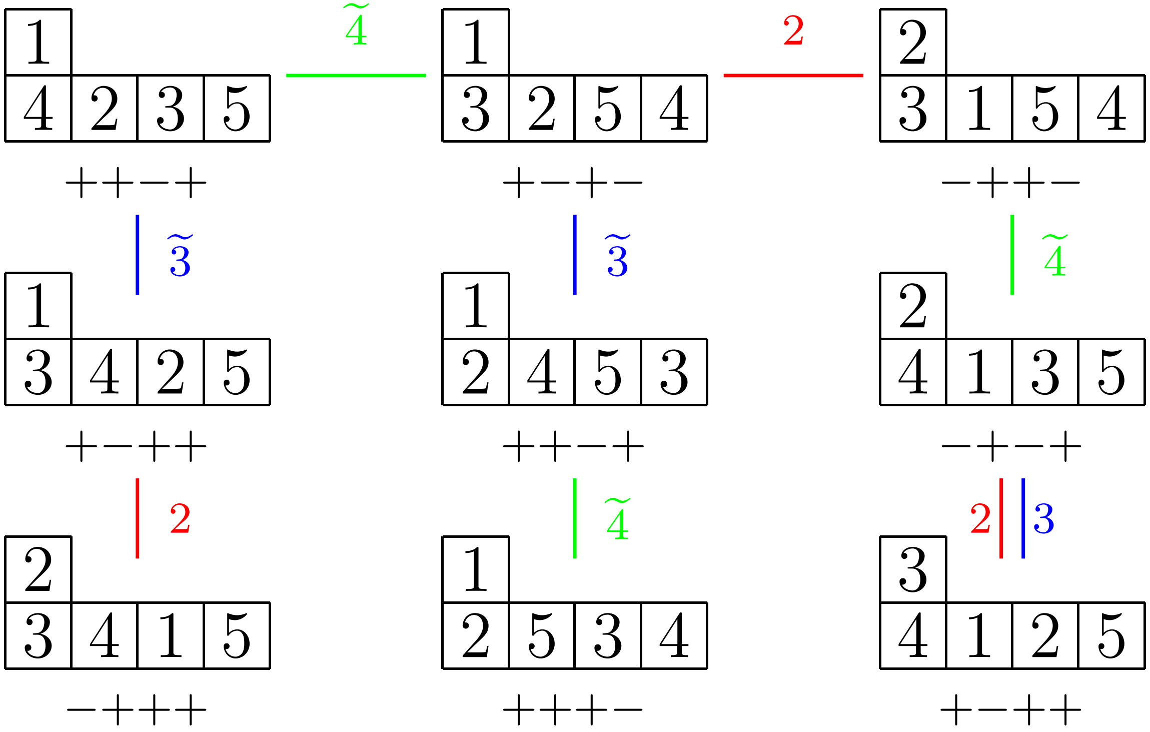
Examples of $\mathcal{H}_\mu$



Almost a DEG

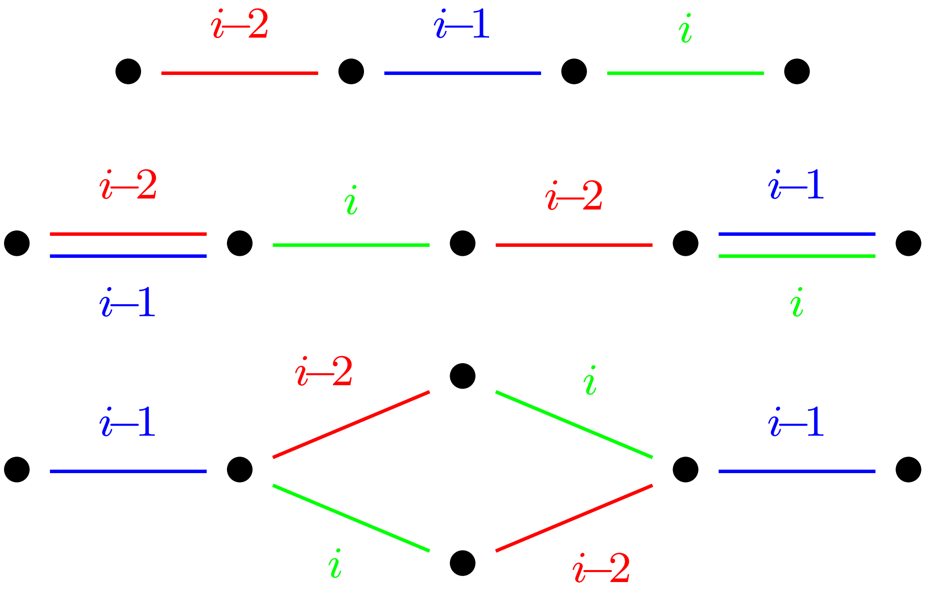
Theorem. *For $\mu_1 \leq 2$, the graph $\mathcal{H}_\mu$ is a DEG for which maj and inv are constant on connected components.*

In general, $\mathcal{H}_\mu$ satisfies DEG axioms 1, 2, 3 and 5.
For $\mu_1 \geq 3$, $\mathcal{H}_\mu$ fails to satisfy axiom 4, so $\mathcal{H}_\mu$ is not a DEG.



Generalizing DEG axiom 4

Axiom 4: Every nontrivial connected component of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$ is one of the following:



Axiom $4'$: Every nontrivial connected component of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$ is in bijection with $\bigcup_{\lambda_j} \mathrm{SYT}(\lambda_j)$ for distinct partitions λ_j of 5, and this bijection, say ϕ, satisfies $\sigma(v)_{i-3,i-2,i-1,i} = \sigma(\phi(v))_{1,2,3,4}$ for all v.

D graphs

Definition. A vertex-signed, edge-colored graph $\mathcal{G}$ is a *D graph* if it satisfies axioms 1, 2, 3, $4'$ and 5.

Theorem. *The graph $\mathcal{H}_\mu$ is a D graph for which maj and inv are constant on connected components.*

Theorem. *Let $\mathcal{G} = (V, \sigma, E)$ be a D graph with statistics α and β which are constant on connected components of $\mathcal{G}$. Then there exists a DEG $\widetilde{\mathcal{G}} = (V, \sigma, \widetilde{E})$ such that α and β are constant on connected components of $\widetilde{\mathcal{G}}$.*

The proof of the theorem is constructive. In fact, the transformation from $\mathcal{G}$ to $\widetilde{\mathcal{G}}$ is canonical.

Constructing a DEG from a D graph

Let $\mathcal{G}$ be a D graph. We construct a sequence of graphs

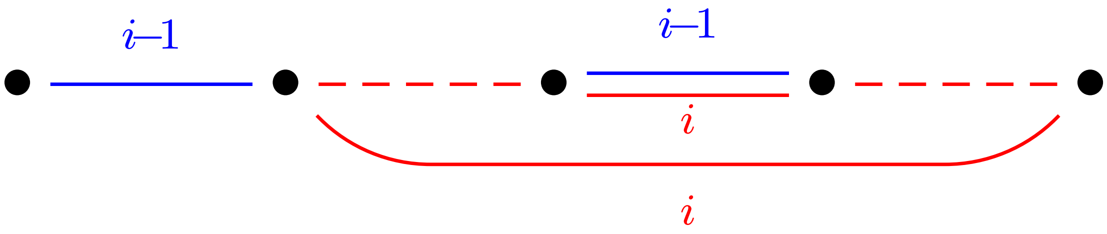
$$\mathcal{G} = \mathcal{G}_2, \mathcal{G}_3, \ldots, \mathcal{G}_{n-1} = \widetilde{\mathcal{G}}$$

with the following properties:

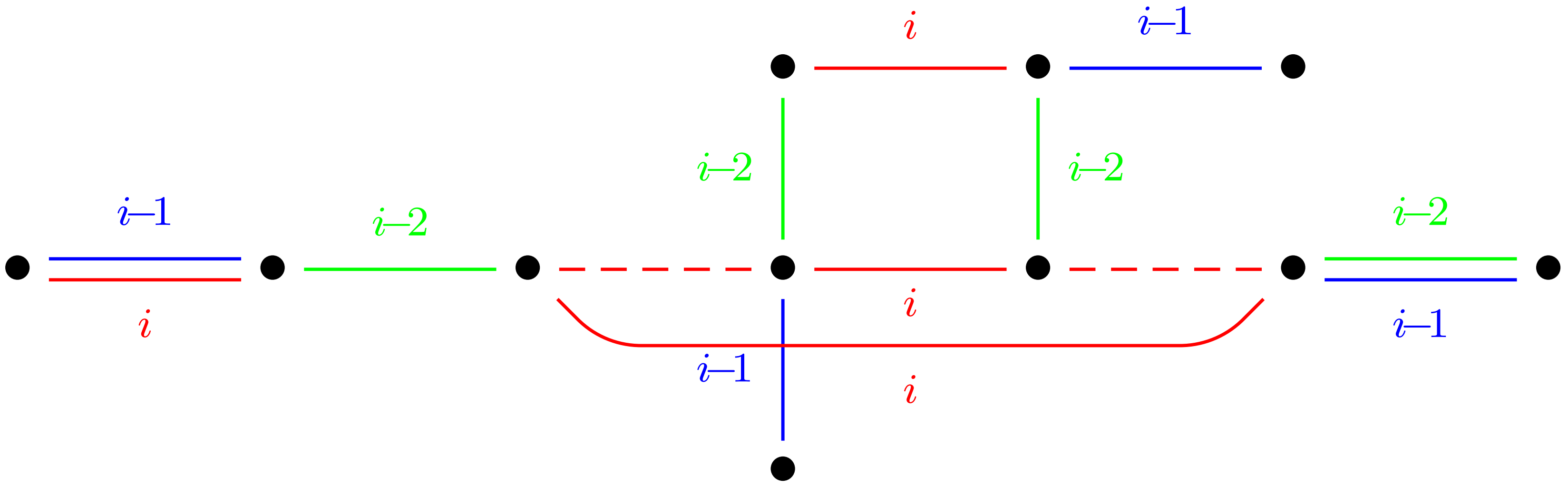
- the vertex set is the same for $\mathcal{G}_i$ and $\mathcal{G}_{i-1}$;
- the signature function is the same for $\mathcal{G}_i$ and $\mathcal{G}_{i-1}$;
- for $j \neq i$, the j-edges are the same for $\mathcal{G}_i$ and $\mathcal{G}_{i-1}$;
- if $w \overset{i}{\text{——}} x$ in $\mathcal{G}_i$, then w and x lie on the same connected component of $(V, \sigma, E_2 \cup \cdots \cup E_i)$ in $\mathcal{G}_{i-1}$;
- each $\mathcal{G}_i$ is a D graph, and its restriction to $(V, \sigma, E_2 \cup \cdots \cup E_i)$ is a DEG.

Constructing $\mathcal{G}_i$ from $\mathcal{G}_{i-1}$

Step 1: Fix connected components of $(V, \sigma, E_{i-1} \cup E_i)$.

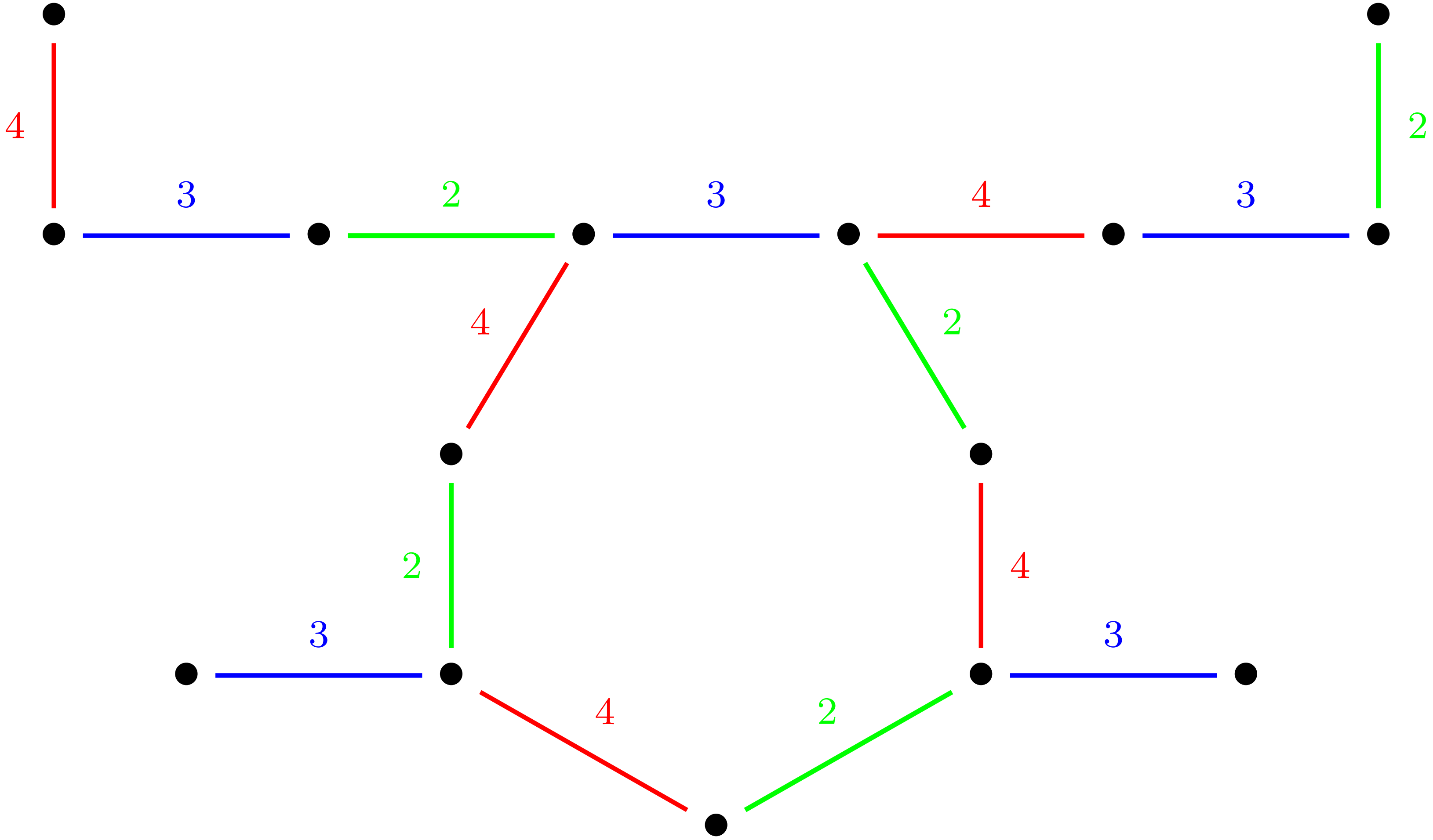


Step 2: Fix connected components of $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$.



A final example: $\mathcal{G} \rightsquigarrow \widetilde{\mathcal{G}}$

$$\mathcal{G} = \mathcal{G}_2 \rightsquigarrow \mathcal{G}_{2.5} \rightsquigarrow \mathcal{G}_3 \rightsquigarrow \mathcal{G}_{3.5} \rightsquigarrow \mathcal{G}_4 = \widetilde{\mathcal{G}}$$



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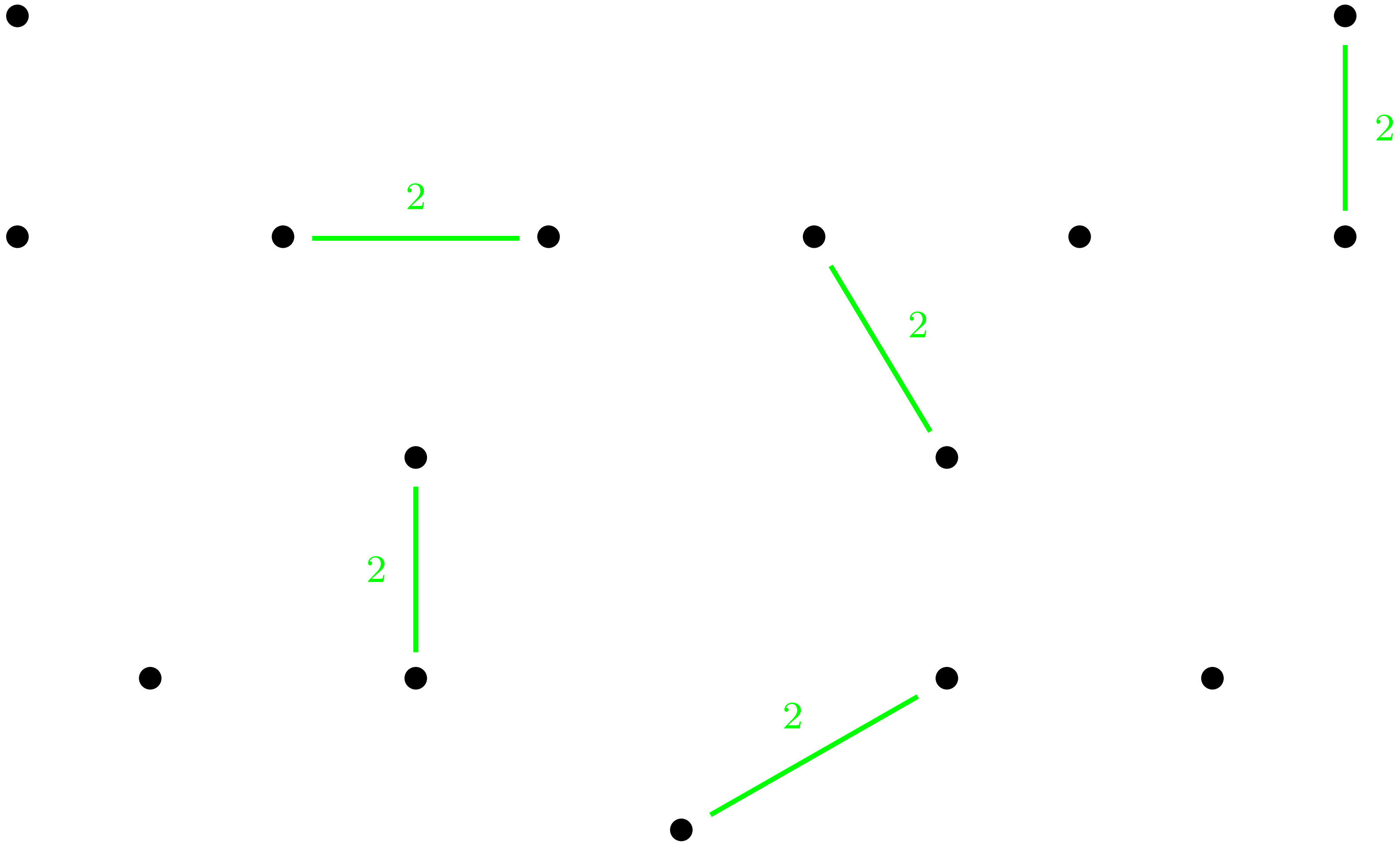

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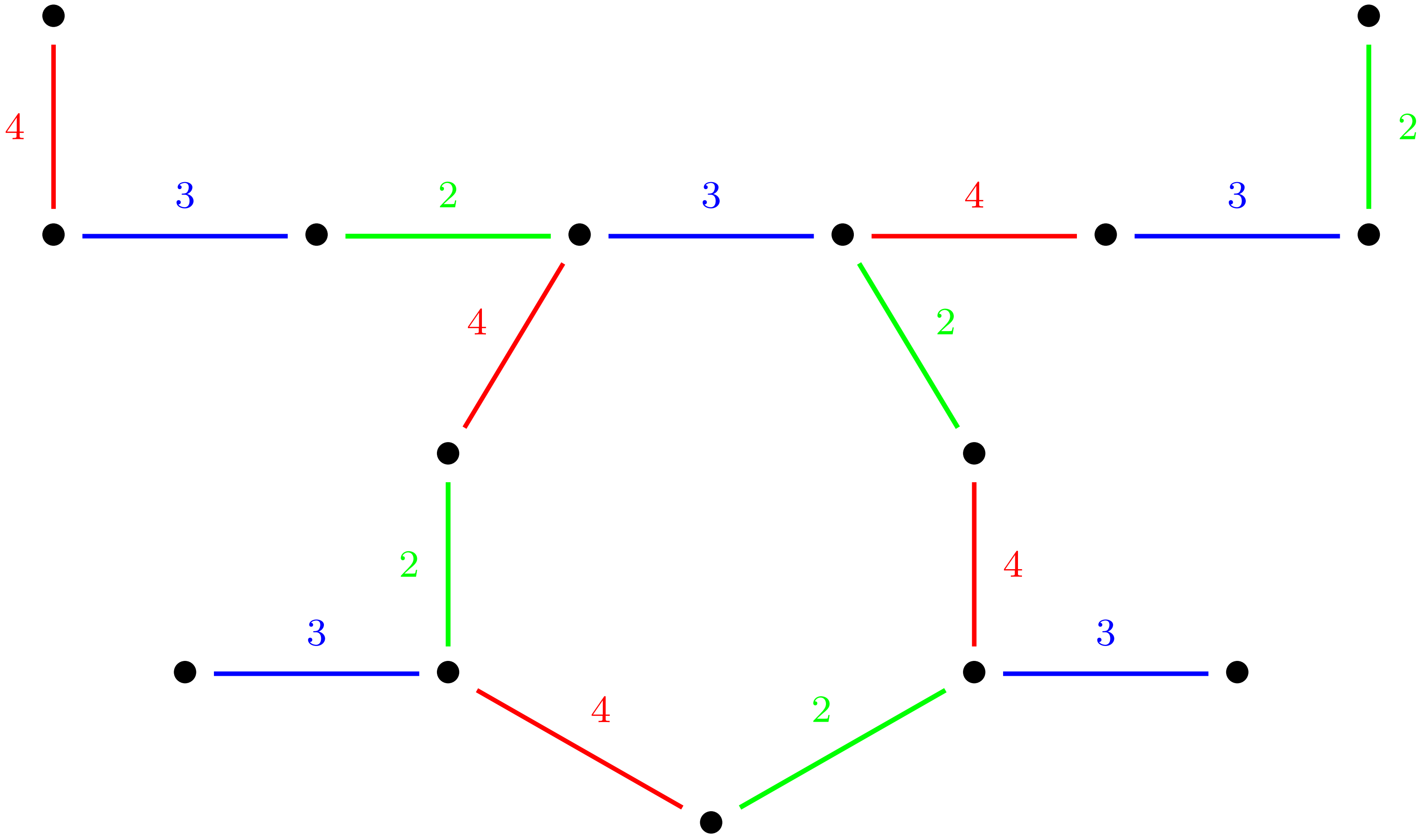

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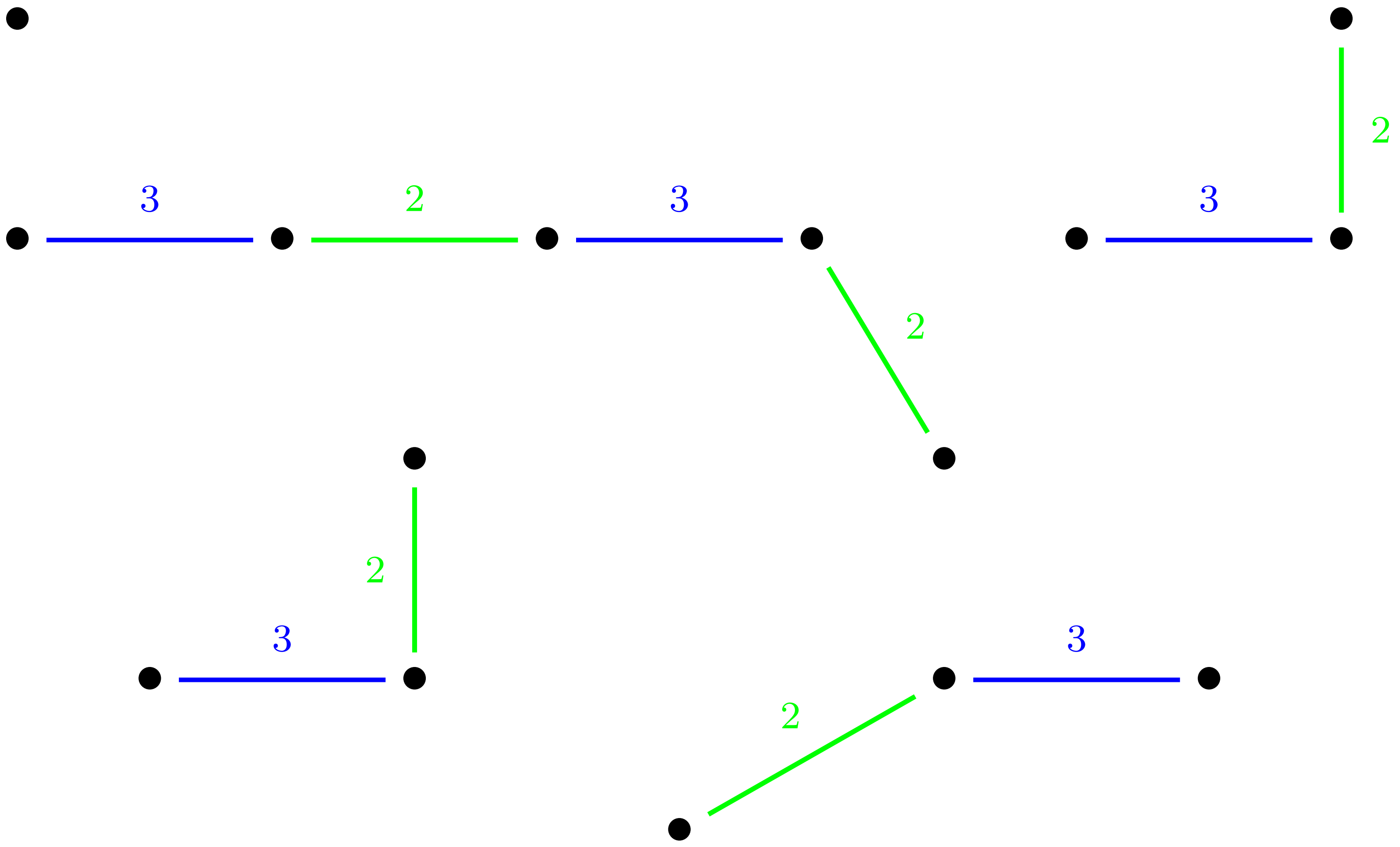

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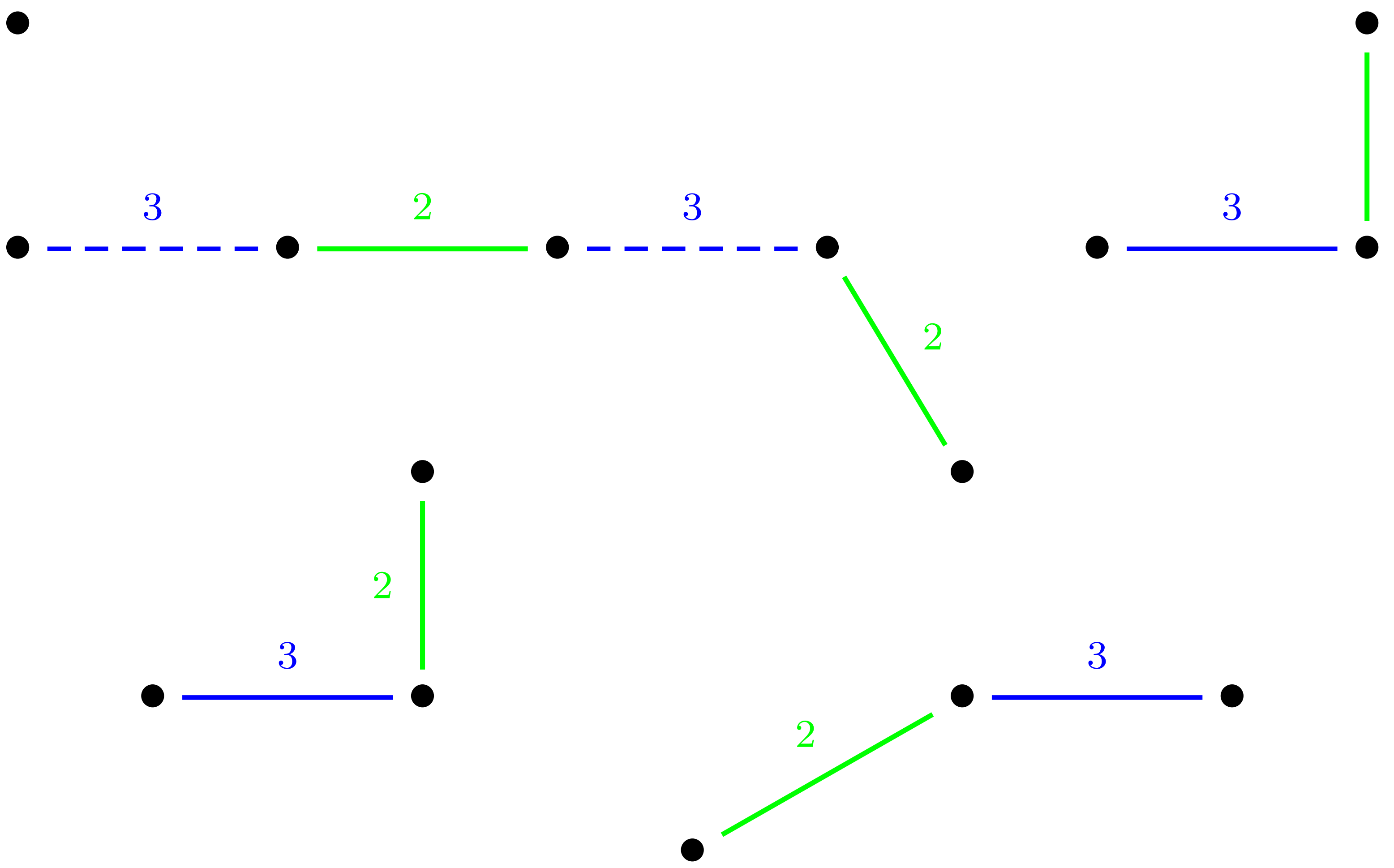

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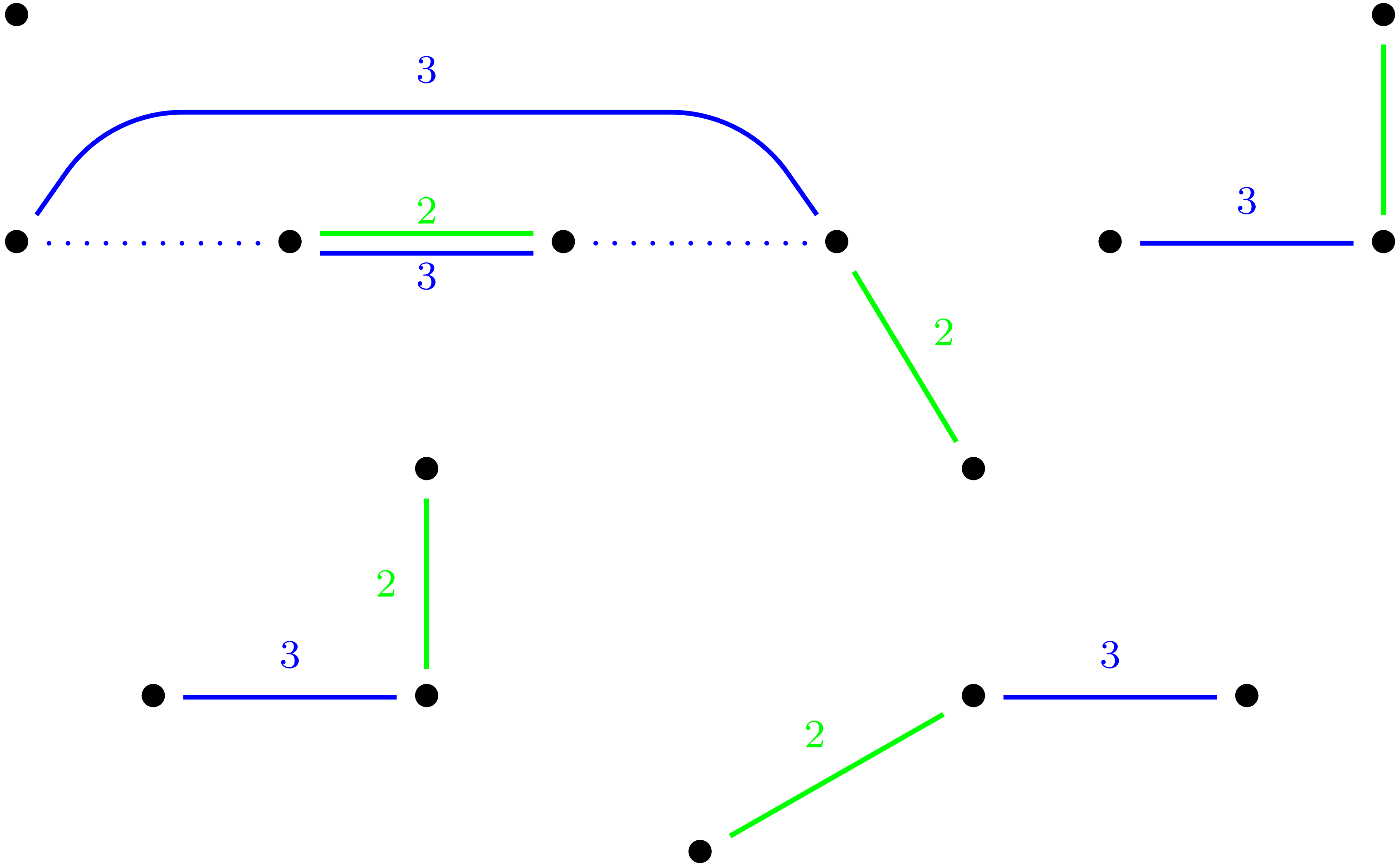

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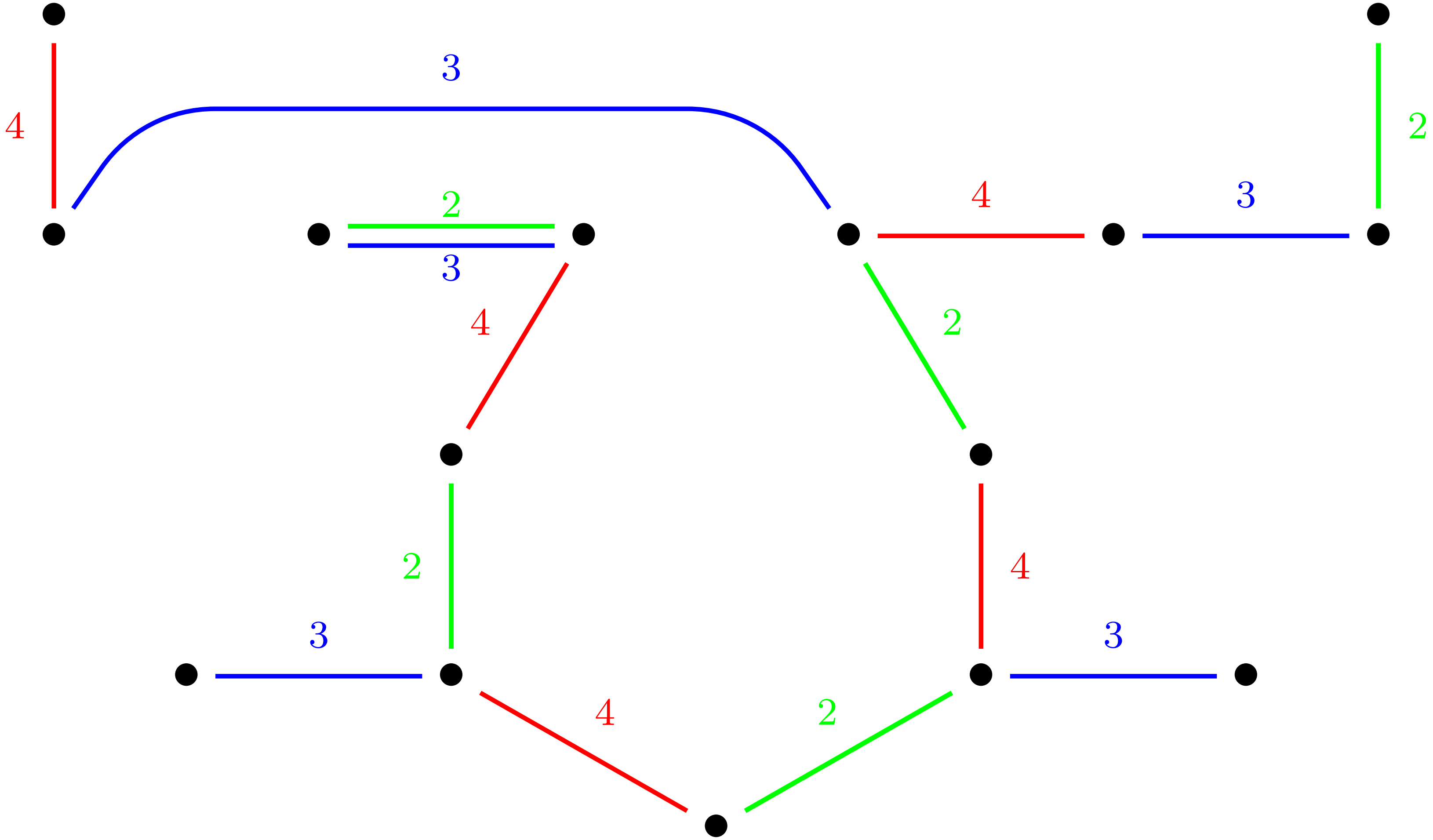

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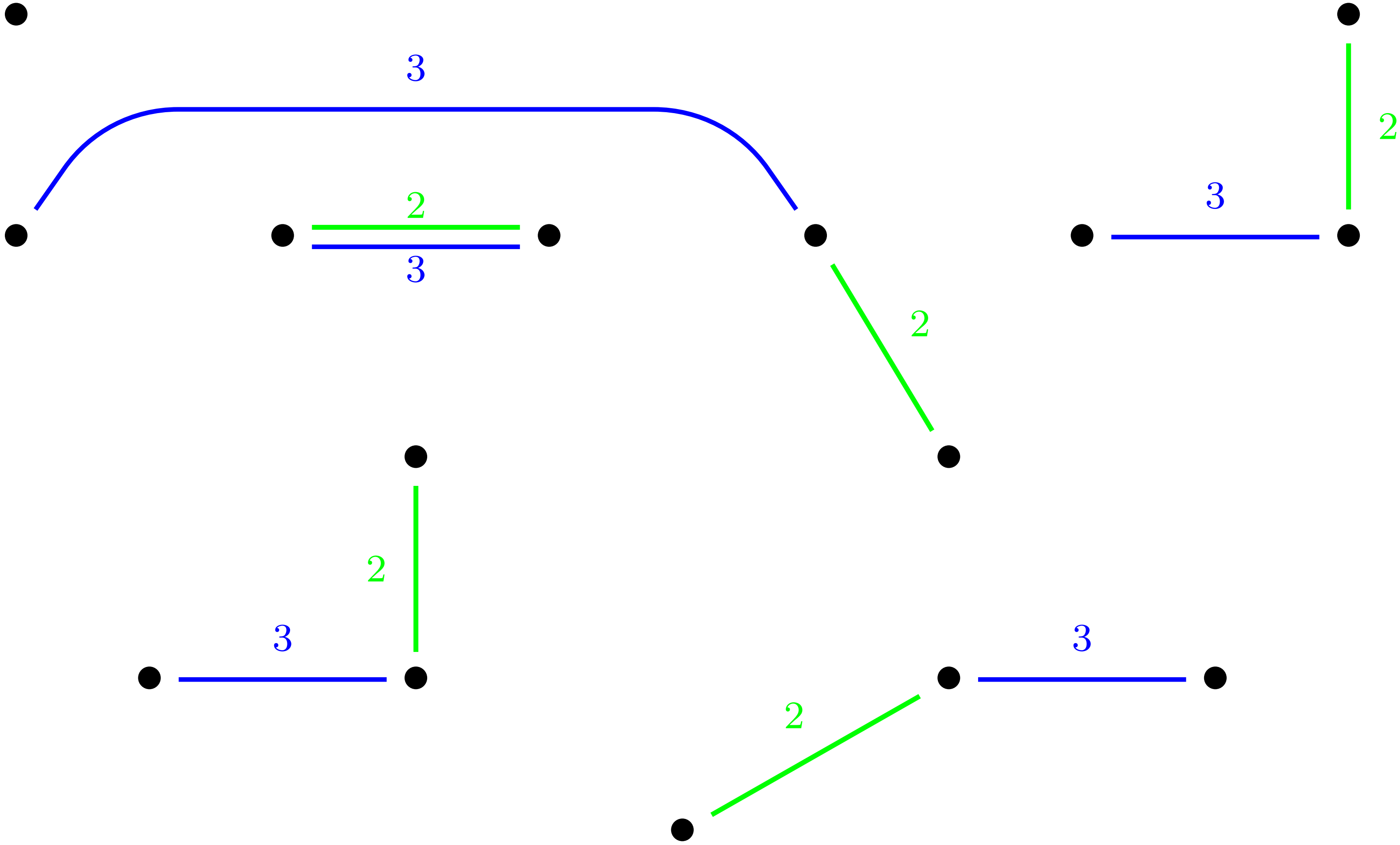

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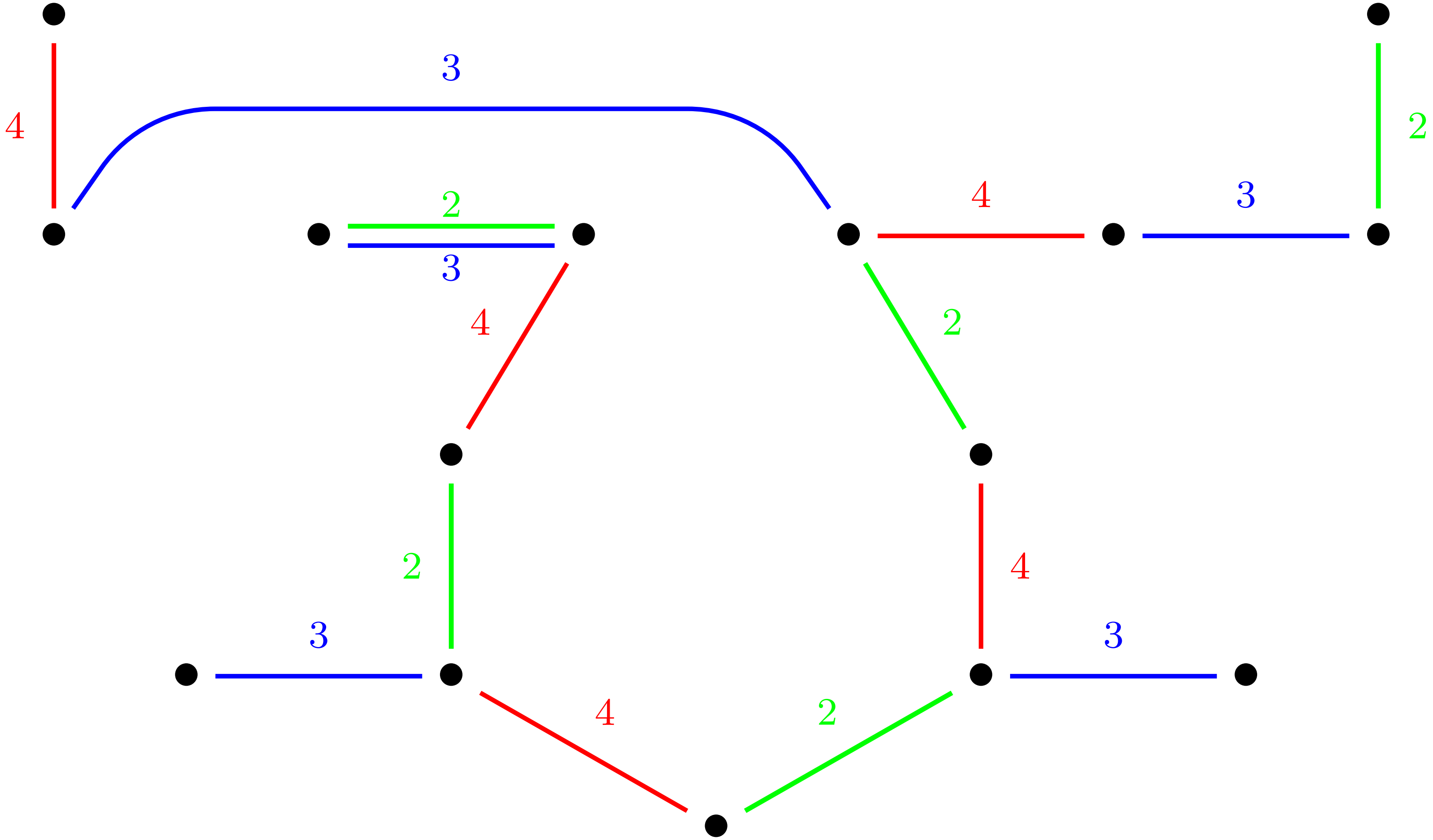

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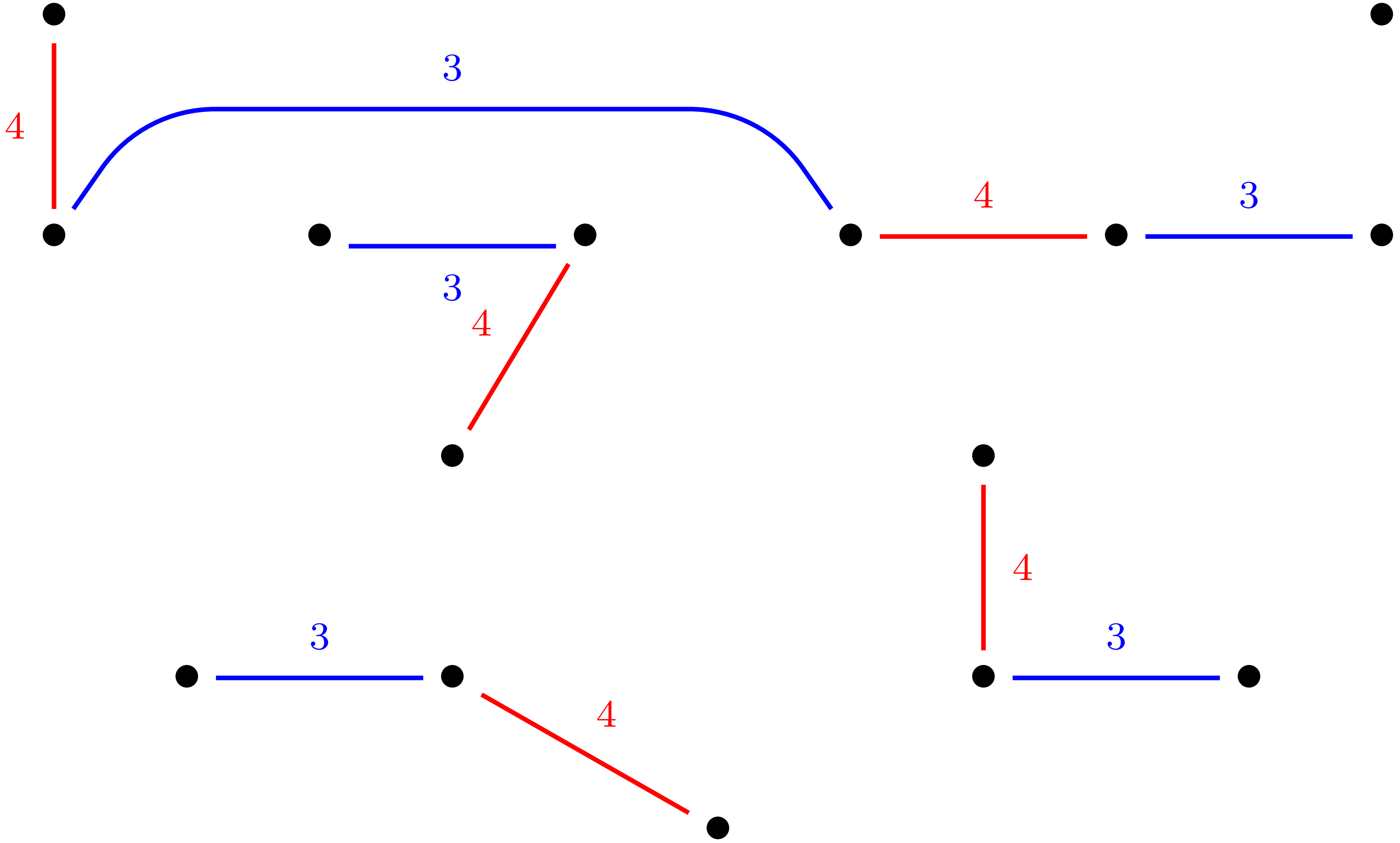

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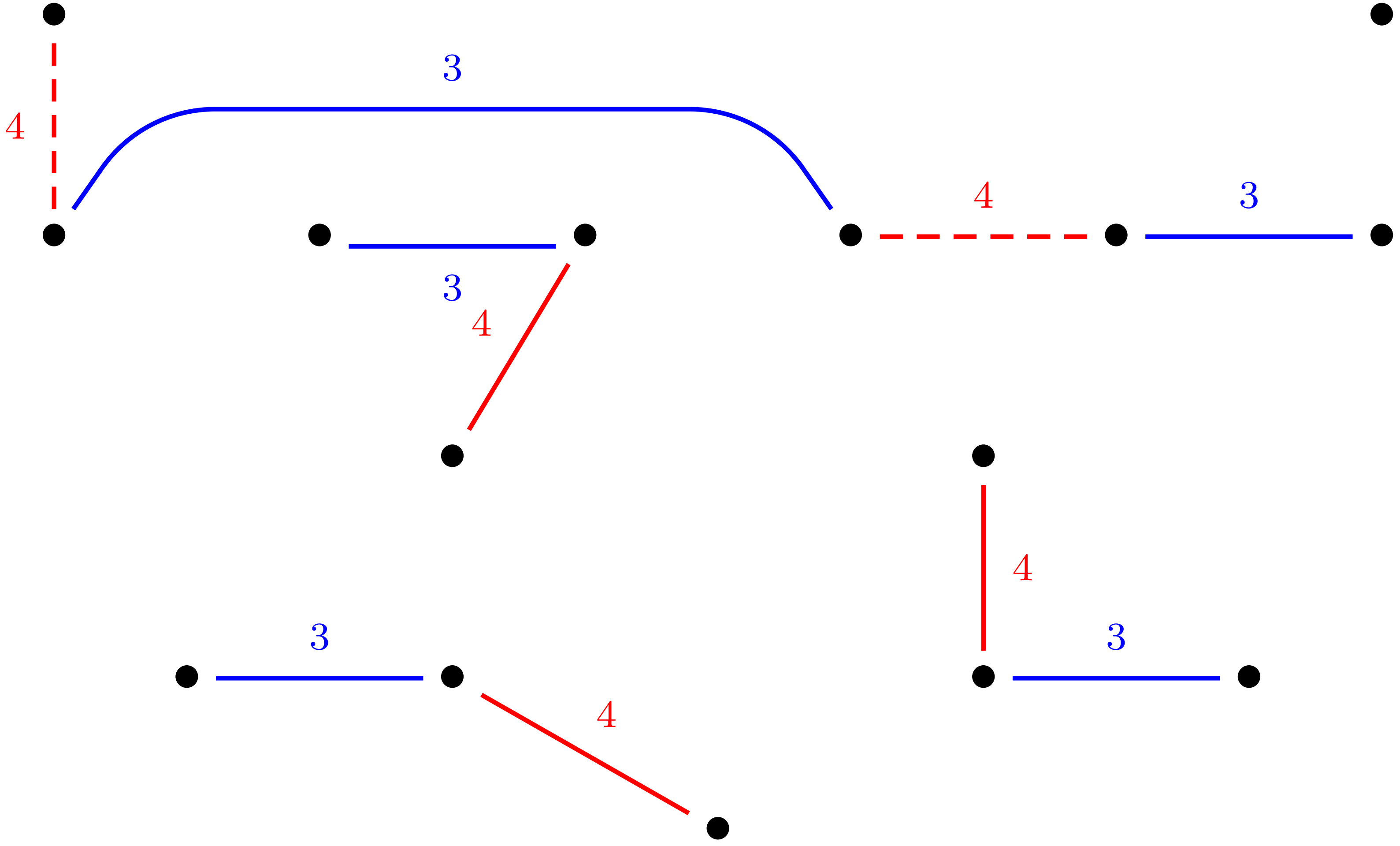

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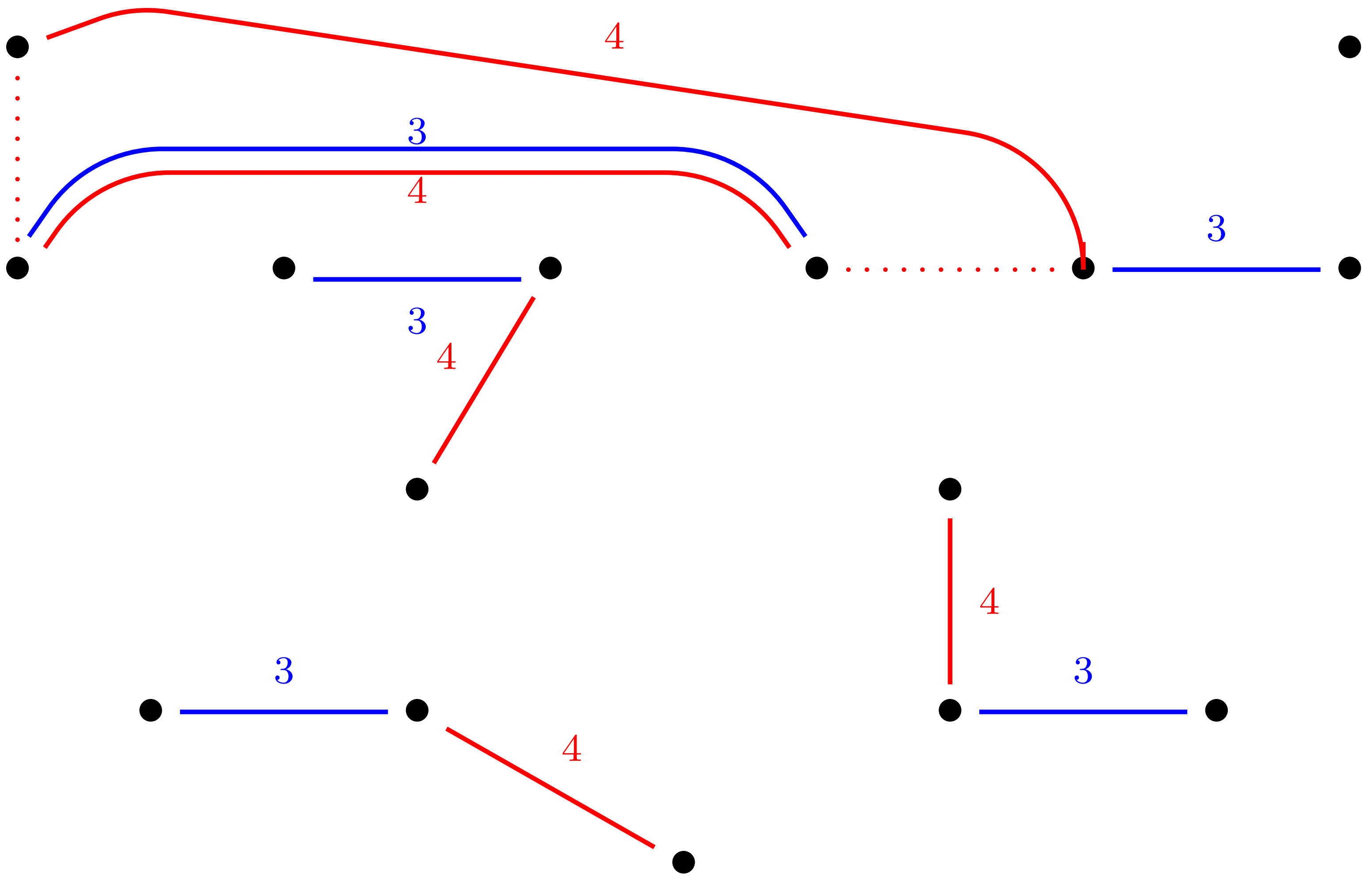

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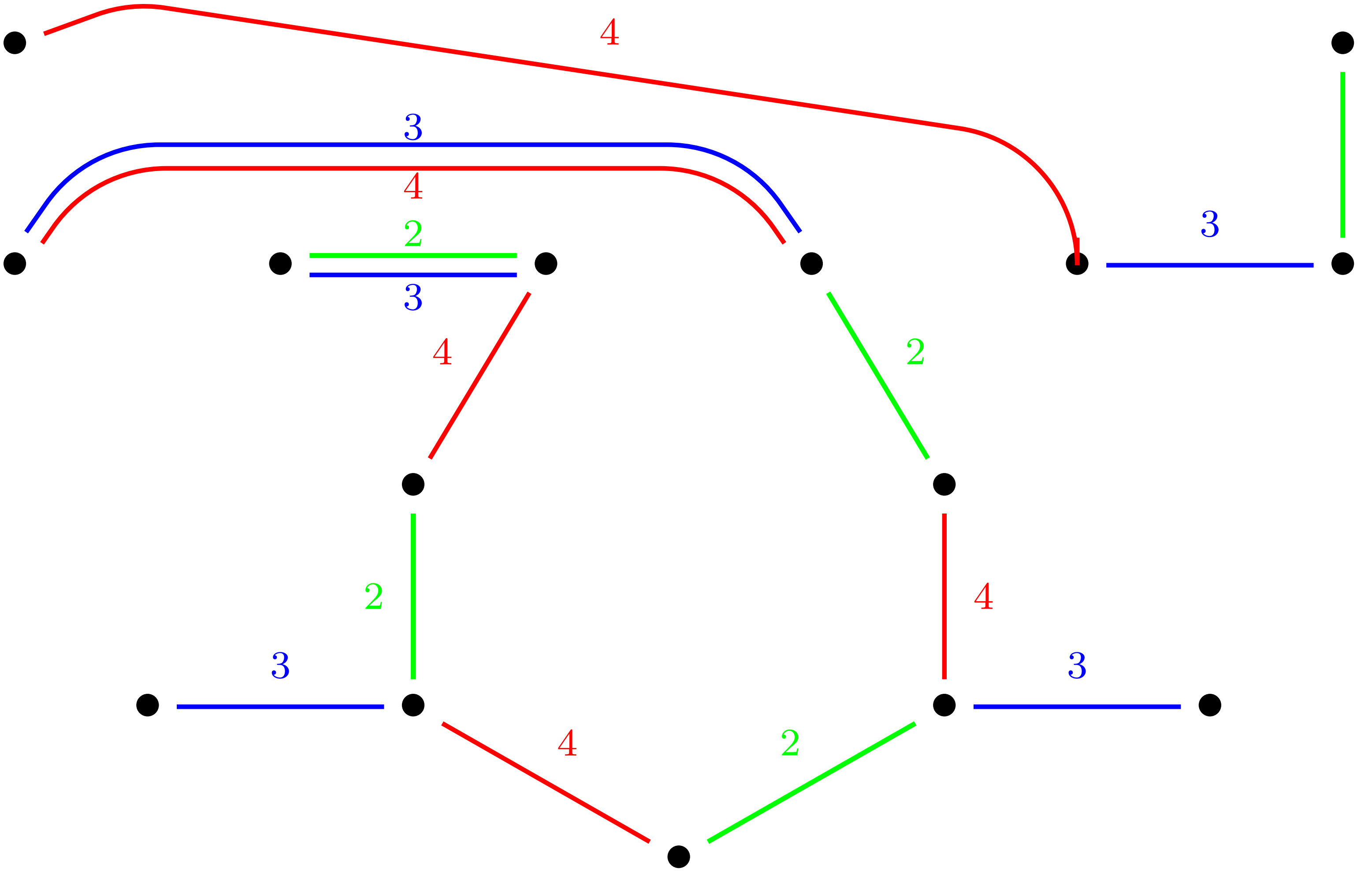

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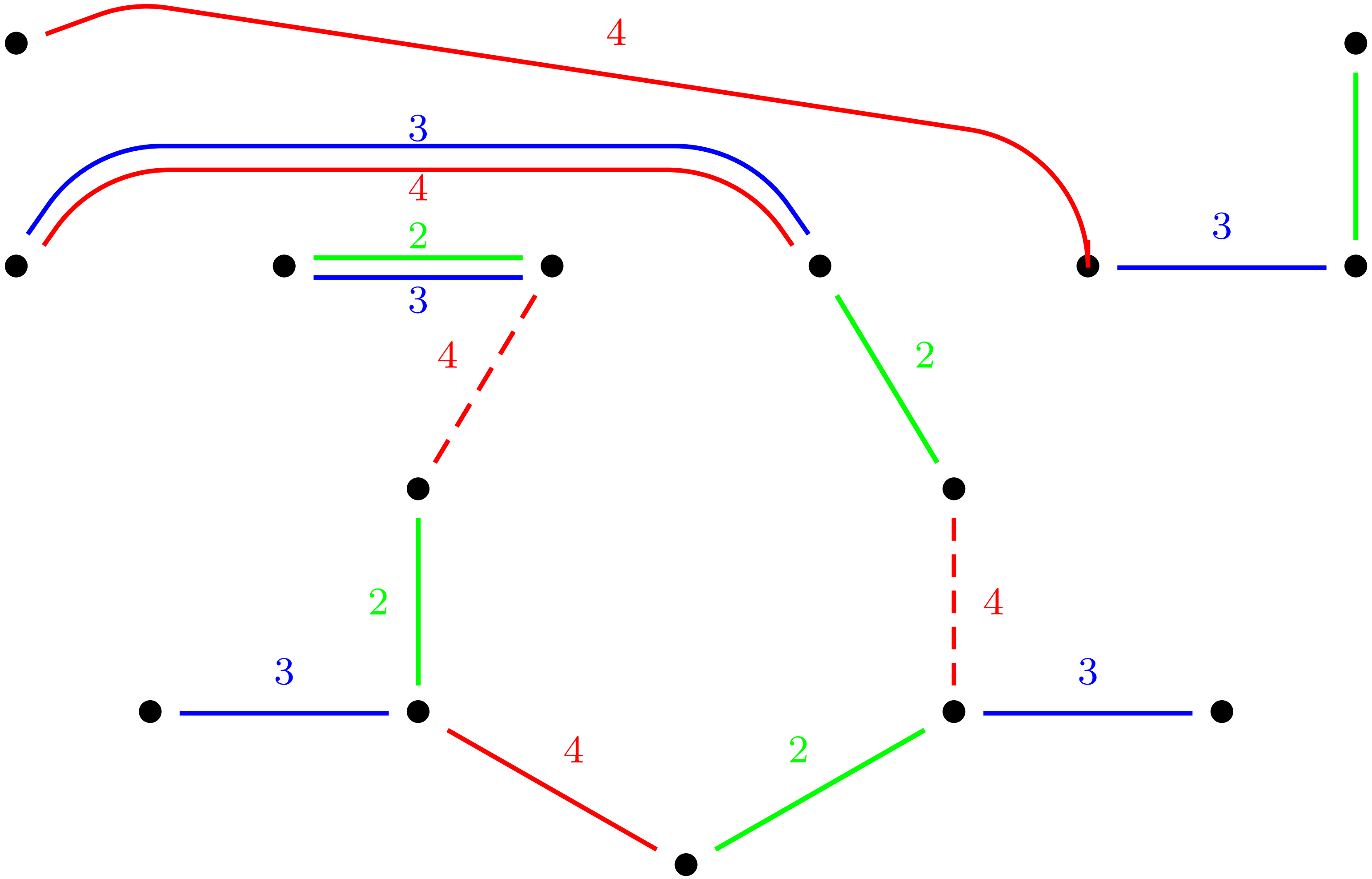

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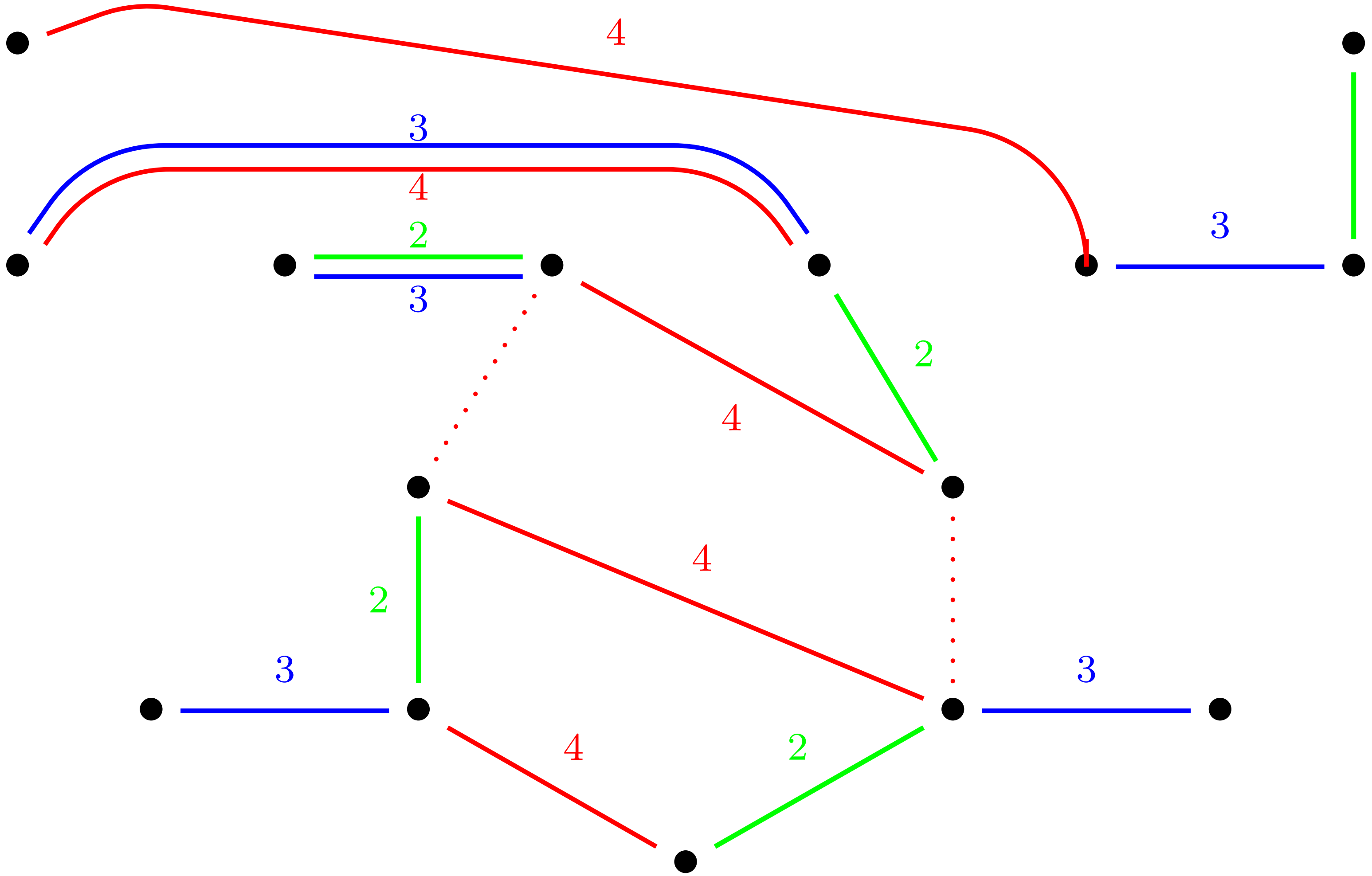

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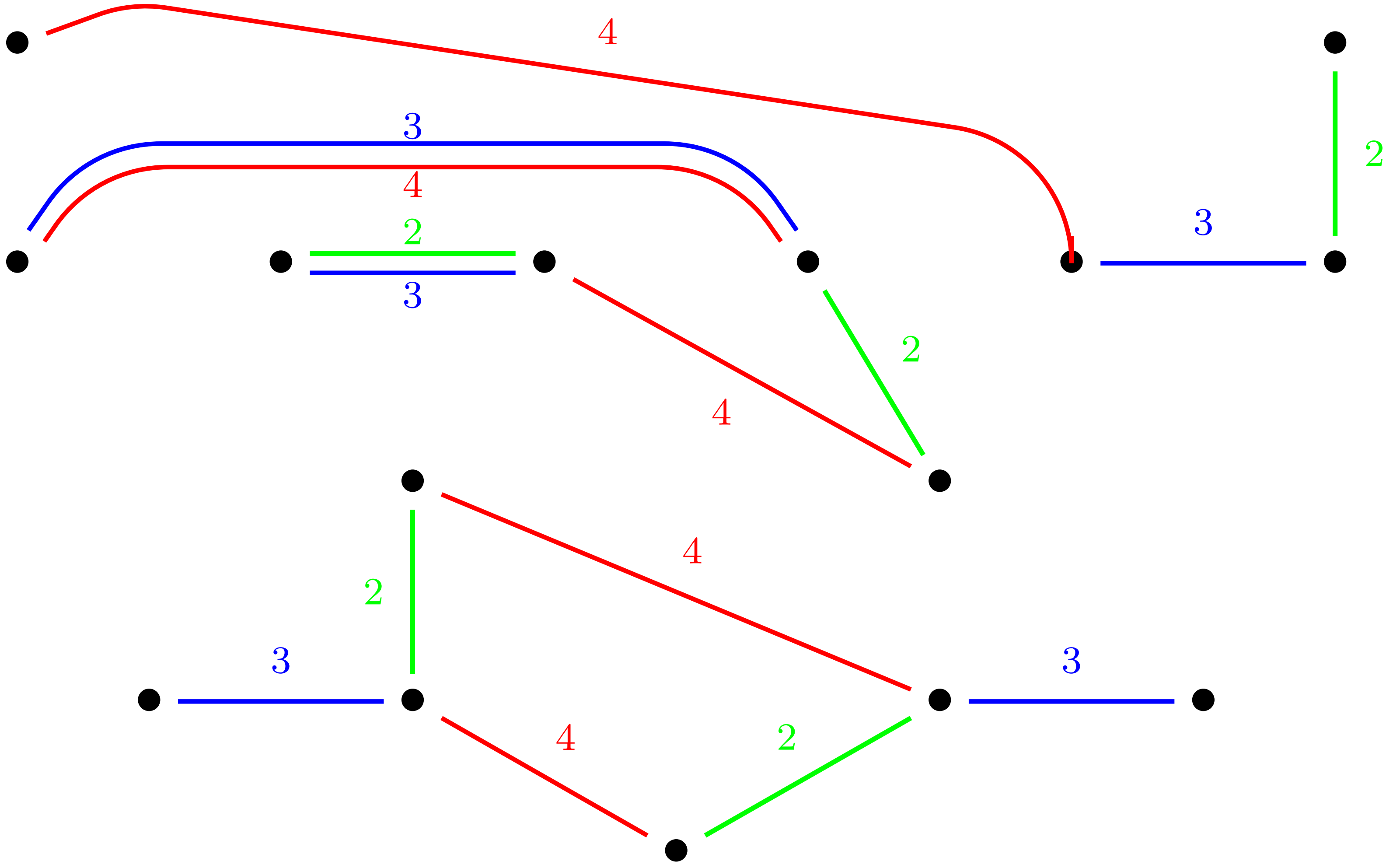

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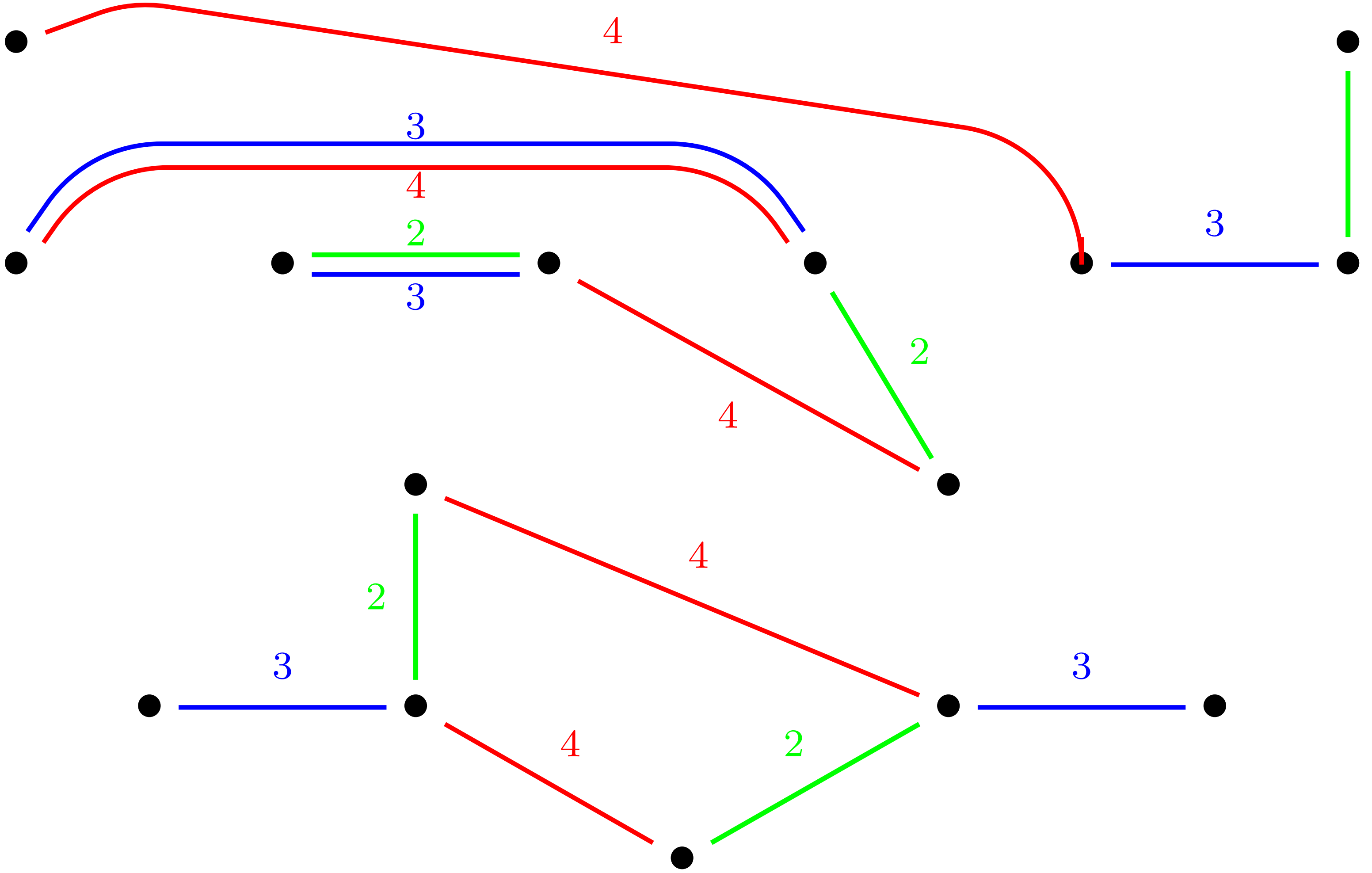

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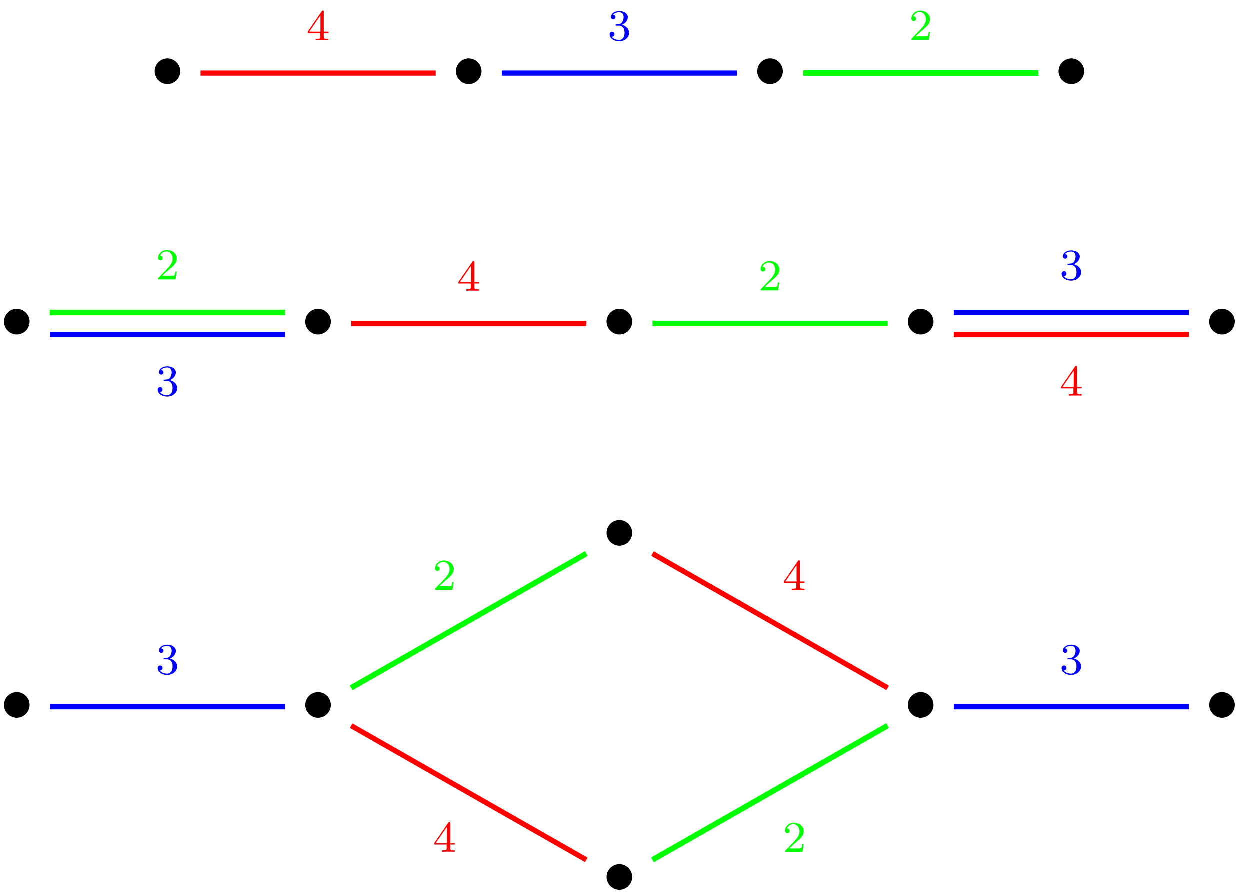

Macdonald positivity

Corollary. *Let* $\widetilde{\mathcal{H}}_\mu$ *be the canonical DEG obtained from the D graph* $\mathcal{H}_\mu$. *Then*

$$\widetilde{K}_{\lambda,\mu}(q,t) = \sum_{\mathcal{C} \cong \mathcal{G}_\lambda} q^{\mathrm{inv}(\mathcal{C})} t^{\mathrm{maj}(\mathcal{C})},$$

where the sum is over connected components of $\widetilde{\mathcal{H}}_\mu$ *which are isomorphic to the standard DEG* $\mathcal{G}_\lambda$.

In particular, $\widetilde{H}_\mu(x;q,t)$ *is symmetric and Schur positive*.

Preprint available online at

http://www.math.upenn.edu/∼sassaf/PDFs/positivity.pdf