Two Analogues of Pascal’s Triangle

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dedicated to

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on the occasion of his retirement
The posets $P_{ib}$

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- There is a unique minimal element $\hat{0}$
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- The Hasse diagram is planar. We draw the Hasse diagram upside-down (with $\hat{0}$ at the top).
Let $i, b \geq 2$. Define the poset (partially ordered set) $P_{ib}$ by

- There is a unique minimal element $\hat{0}$
- Each element is covered by exactly $i$ elements.
- The Hasse diagram is planar. We draw the Hasse diagram upside-down (with $\hat{0}$ at the top).
- Every $\wedge$ extends to a $2b$-gon ($b$ edges on each side)
Construction of $P_{23}$
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Number of elements of rank $n$

$p_{ib}(n)$: number of elements of $P_{ij}$ of rank $n$
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In $P_{ib}$, every element of rank $n - 1$ is covered by $i$ elements, giving a first approximation $p_{ib}(n) \approx ip_{ib}(n - 1)$. Each element of rank $n - b$ is the bottom of $i - 1$ $2b$-gons, so there are $(i - 1)p_{ib}(n - b)$ elements of rank $n$ that cover two elements. The remaining elements of rank $n$ cover one element. Hence

$$p_{ib}(n) = ip_{ib}(n - 1) - (i - 1)p(n - b).$$
**Number of elements of rank \( n \)**

\( p_{ib}(n) \): number of elements of \( P_{ij} \) of rank \( n \)

In \( P_{ib} \), every element of rank \( n - 1 \) is covered by \( i \) elements, giving a first approximation \( p_{ib}(n) \approx ip_{ib}(n - 1) \). Each element of rank \( n - b \) is the bottom of \( i - 1 \) \( 2b \)-gons, so there are \((i - 1)p_{ib}(n - b)\) elements of rank \( n \) that cover two elements. The remaining elements of rank \( n \) cover one element. Hence

\[
p_{ib}(n) = ip_{ib}(n - 1) - (i - 1)p(n - b).
\]

Initial conditions: \( p_{ij}(n) = i^n, \ 0 \leq n \leq b - 1 \)

\[
\Rightarrow \sum_{n \geq 0} p_{ij}(n)x^n = \frac{1}{1 - ix + (i - 1)x^b}.
\]
The numbers $e(t)$

For $t \in P_{ib}$, let $e(t)$ be the number of saturated chains from $\hat{0}$ to $t$. 
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Example. $P_{22}$

```plaintext
0
1
2
3
4
```

```
1
1
1
1
```

```
1
2
```

```
1
```

```
1
```

```
1
```

```
1
```

```
1
```

```
1
```

```
1
```
The numbers $e(t)$

For $t \in P_{ib}$, let $e(t)$ be the number of saturated chains from $\hat{0}$ to $t$.

Example. $P_{22}$

Pascal’s triangle
Pascal’s triangle

rows 0–4:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$k$th entry in row $n$, beginning with $k = 0$: \( \binom{n}{k} \)
Pascal’s triangle

rows 0–4:

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
```

$k$th entry in row $n$, beginning with $k = 0$: \( \binom{n}{k} = \frac{n!}{k! (n-k)!} \)
Pascal’s triangle

rows 0–4:

\[
\begin{array}{cccccc}
& & & 1 & & \\
& & 1 & & 1 & \\
& 1 & & 2 & & 1 \\
1 & & 3 & & 3 & 1 \\
1 & 4 & 6 & 4 & 1 & \\
\end{array}
\]

kth entry in row n, beginning with k = 0: \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)

\[
\sum_k \binom{n}{k} x^k = (1 + x)^n
\]
Sums of powers

\[ \sum_k \binom{n}{k}^2 = \binom{2n}{n} \]
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\[ \sum_k \binom{n}{k}^2 = \binom{2n}{n} \]

\[ \sum_{n \geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1 - 4x}}, \]

not a rational function (quotient of two polynomials)
Sums of powers

\[ \sum_k \binom{n}{k}^2 = \binom{2n}{n} \]

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**not** a rational function (quotient of two polynomials)

\[ \sum_k \binom{n}{k}^3 = ?? \]

Even worse! Generating function is not algebraic.
Stern’s triangle

Similar to Pascal’s triangle, but we also “bring down” (copy) each number from one row to the next.
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```
  1
 1 1 1
 2 2
 1
 1 1
 1 2 2 1
 1
 1
 1
 1
 1
 1
 1
 1
```

⋮
**Stern’s triangle**

Similar to Pascal’s triangle, but we also “bring down” (copy) each number from one row to the next.

\[
\begin{array}{cccccc}
1 & 1 & 2 & 1 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 & 2 & 1 & 1 \\
\vdots \\
1
\end{array}
\]
Stern’s triangle

Similar to Pascal’s triangle, but we also “bring down” (copy) each number from one row to the next.

```
1
1 1 1
1 1 2 1 2 1 1
1
2 3 3 3 3 2
1
```

⋮
Stern’s triangle

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\[
\begin{array}{ccccccccc}
1 & & & & & & & & \\
1 & 1 & & & & & & & \\
1 & 1 & 2 & 1 & 2 & 1 & 1 & & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
\vdots
\end{array}
\]
Some properties

- Number of entries in row \( n \) (beginning with row 0): \( 2^{n+1} - 1 \)
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- Largest entry in row \( n \): \( F_{n+1} \) (Fibonacci number)
Some properties

- Number of entries in row \( n \) (beginning with row 0): \( 2^{n+1} - 1 \)
- Sum of entries in row \( n \): \( 3^n \)
- Largest entry in row \( n \): \( F_{n+1} \) (Fibonacci number)
- Let \( \binom{n}{k} \) be the \( k \)th entry (beginning with \( k = 0 \)) in row \( n \).

Write
\[
P_n(x) = \sum_{k \geq 0} \binom{n}{k} x^k.
\]

Then
\[
P_{n+1}(x) = (1 + x + x^2)P_n(x^2),
\]
since \( xP_n(x^2) \) corresponds to bringing down the previous row, and
\( (1 + x^2)P_n(x^2) \) to summing two consecutive entries.
Stern analogue of binomial theorem

Corollary. \( P_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2^{2i}}\right) \)
An essentially equivalent array is due to Moritz Abraham Stern around 1858 and is known as Stern’s diatomic array:

\[
\begin{array}{cccccccc}
1 & & & & & & & 1 \\
1 & & & & & & 2 & \\
1 & 3 & & 2 & & 3 & & 1 \\
1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 1 \\
1 & 5 & 4 & 7 & 8 & 5 & 7 & & & \\
1 & & & & & & & & & \\
\end{array}
\]
Sums of squares

\[
\begin{array}{cccccccc}
1 & & & & & & & \\
1 & 1 & & & & & & \\
1 & 1 & 2 & 1 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
\vdots
\end{array}
\]

\[
u_2(n) := \sum_k \binom{n}{k}^2 = 1, 3, 13, 59, 269, 1227, \ldots
\]
Sums of squares

\[
\begin{array}{ccccccccc}
1 & & & & & & & & \\
1 & 1 & & & & & & & \\
1 & 1 & 2 & 1 & 2 & 1 & 1 & & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

\[u_2(n) := \sum_k \binom{n}{k}^2 = 1, 3, 13, 59, 269, 1227, \ldots\]

\[u_2(n + 1) = 5u_2(n) - 2u_2(n - 1), \quad n \geq 1\]
Sums of squares

\[
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 \\
1 & 1 & 2 & 1 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
\vdots
\end{array}
\]

\[u_2(n) := \sum_{k} \binom{n}{k}^2 = 1, 3, 13, 59, 269, 1227, \ldots\]

\[u_2(n + 1) = 5u_2(n) - 2u_2(n - 1), \quad n \geq 1\]

\[
\sum_{n \geq 0} u_2(n)x^n = \frac{1 - 2x}{1 - 5x + 2x^2}
\]
Proof

\[ u_2(n+1) = \cdots + \langle n \rangle^2 + (\langle n \rangle + \langle n \mid k+1 \rangle)^2 + \langle n \mid k+1 \rangle^2 + \cdots \]

\[ = 3u_2(n) + 2 \sum_{k} \langle n \mid k \rangle \langle n \mid k+1 \rangle. \]
Proof

\[
\begin{align*}
    u_2(n+1) &= \cdots + \left\langle \frac{n}{k} \right\rangle^2 + \left( \left\langle \frac{n}{k} \right\rangle + \left\langle \frac{n}{k+1} \right\rangle \right)^2 + \left\langle \frac{n}{k+1} \right\rangle^2 + \cdots \\
    &= 3u_2(n) + 2 \sum_k \left\langle \frac{n}{k} \right\rangle \left\langle \frac{n}{k+1} \right\rangle.
\end{align*}
\]

Thus define \( u_{1,1}(n) := \sum_k \left\langle \frac{n}{k} \right\rangle \left\langle \frac{n}{k+1} \right\rangle \), so

\[
u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).
\]
What about $u_{1,1}(n)$?

\[
\begin{align*}
\quad u_{1,1}(n+1) & = \cdots + \left( \left\langle \begin{array}{c} n \\ k-1 \end{array} \right\rangle + \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle \right) \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle + \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle \left( \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle + \left\langle \begin{array}{c} n \\ k+1 \end{array} \right\rangle \right) \\
+ \left( \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle + \left\langle \begin{array}{c} n \\ k+1 \end{array} \right\rangle \right) \left\langle \begin{array}{c} n \\ k+1 \end{array} \right\rangle + \cdots \\
= 2u_2(n) + 2u_{1,1}(n)
\end{align*}
\]
What about $u_{1,1}(n)$?

\[ u_{1,1}(n+1) = \cdots + \left( \left\langle \frac{n}{k-1} \right\rangle + \left\langle \frac{n}{k} \right\rangle \right) \left\langle \frac{n}{k} \right\rangle + \left\langle \frac{n}{k} \right\rangle \left( \left\langle \frac{n}{k} \right\rangle + \left\langle \frac{n}{k+1} \right\rangle \right) \\
+ \left( \left\langle \frac{n}{k} \right\rangle + \left\langle \frac{n}{k+1} \right\rangle \right) \left\langle \frac{n}{k+1} \right\rangle + \cdots \\
= 2u_2(n) + 2u_{1,1}(n) \]

Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$. 
Two recurrences in two unknowns

Let \( \mathbf{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \). Then

\[
\mathbf{A} \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n + 1) \\ u_{1,1}(n + 1) \end{bmatrix}.
\]
Two recurrences in two unknowns

Let $A := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$ 

$$\Rightarrow A^n \begin{bmatrix} u_2(1) \\ u_{1,1}(1) \end{bmatrix} = \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix}.$$
Two recurrences in two unknowns

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\[
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\]

Characteristic (or minimum) polynomial of \( A \): \( x^2 - 5x + 2 \)
Two recurrences in two unknowns

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Characteristic (or minimum) polynomial of \( A \): \( x^2 - 5x + 2 \)

\[
(A^2 - 5A + 2I)A^{n-1} = 0_{2 \times 2}
\]

\[\Rightarrow u_2(n + 1) = 5u_2(n) - 2u_2(n - 1)\]
Two recurrences in two unknowns

Let \( A := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \). Then

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A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n + 1) \\ u_{1,1}(n + 1) \end{bmatrix}.
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\[
\Rightarrow u_2(n + 1) = 5u_2(n) - 2u_2(n - 1)
\]

Also \( u_{1,1}(n + 1) = 5u_{1,1}(n) - 2u_{1,1}(n - 1) \).
Sums of cubes

\[ u_3(n) := \sum_k \binom{n}{k}^3 = 1, 3, 21, 147, 1029, 7203, \ldots \]
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\[ u_3(n) = 3 \cdot 7^{n-1}, \quad n \geq 1 \]
Sums of cubes

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\[ u_3(n) = 3 \cdot 7^{n-1}, \quad n \geq 1 \]

Equivalently, if \( \prod_{i=0}^{n-1} \left( 1 + x^{2^i} + x^{2^{2^i}} \right) = \sum a_j x^j \), then

\[ \sum a_j^3 = 3 \cdot 7^{n-1}. \]
Why so simple?

Same method gives the matrix \[
\begin{bmatrix}
3 & 6 \\
2 & 4
\end{bmatrix}.
\]
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Characteristic polynomial: \(x(x - 7)\) (zero eigenvalue!)
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Thus \(u_3(n + 1) = 7u_3(n)\).
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Thus \(u_3(n + 1) = 7u_3(n)\).

Much nicer than \(\sum_k \binom{n}{k}^3\)
What about $u_r(n)$ for general $r \geq 1$?

By the same technique, can show that

$$\sum_{n \geq 0} u_r(n)x^n$$

is rational.
What about \( u_r(n) \) for general \( r \geq 1 \)?

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\[
\sum_{n \geq 0} u_r(n) x^n
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is rational.

**Example.** \[
\sum_{n \geq 0} u_4(n) x^n = \frac{1 - 7x - 2x^2}{1 - 10x - 9x^2 + 2x^3}
\]
What about $u_r(n)$ for general $r \geq 1$?

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**Example.** $\sum_{n \geq 0} u_4(n)x^n = \frac{1 - 7x - 2x^2}{1 - 10x - 9x^2 + 2x^3}$

Much more can be said!
The Stern poset
The Stern poset

\[ P_{32} \]
Label $t$ by $e(t)$. Then the $k$th label (beginning with $k = 0$) at rank $n$ is $\binom{n}{k}$:

$$\sum_k \binom{n}{k} x^k = \prod_{i=0}^{n-1} \left( 1 + x^{2^i} + x^{2^{i+1}} \right).$$
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$$
\sum_{k} \binom{n}{k} x^k = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2^i \cdot 2^i}\right).
$$

Similar product formulas for all $P_{ib}$. 
A Fibonacci product

**Fibonacci numbers**: $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ $(n \geq 3)$
A Fibonacci product

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\[
l_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)
\]
A Fibonacci product

Fibonacci numbers: \( F_1 = F_2 = 1, \ F_n = F_{n-1} + F_{n-2} \ (n \geq 3) \)

\[
I_n(x) = \prod_{i=1}^{n} \left( 1 + x^{F_{i+1}} \right)
\]

\[
I_4(x) = (1 + x)(1 + x^2)(1 + x^3)(1 + x^5)
= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11}
\]
A Fibonacci product

Fibonacci numbers: $F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad (n \geq 3)$

$$I_n(x) = \prod_{i=1}^{n} (1 + x^{F_{i+1}})$$

$I_4(x)$ = \begin{align*}
&= (1 + x)(1 + x^2)(1 + x^3)(1 + x^5) \\
&= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11}
\end{align*}$

$v_r(n)$: sum of $r$th powers of coefficients of $I_n(x)$
The Fibonacci triangle $\mathcal{F}$
The Fibonacci triangle $\mathcal{F}$

- Copy each entry of row $n \geq 1$ to the next row.
- Add two entries if separated by at bullet (and form group of 3).
- Copy once more the middle entry of a group of 3 (group of 2).
- Adjoin 1 at beginning and end of each row after row 0.
“Binomial theorem” for $\mathcal{F}$

$\binom{n}{k}$: $k$th entry (beginning with $k = 0$) in row $n$ (beginning with $n = 0$) in $\mathcal{F}$
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**Theorem.** $\sum_k \binom{n}{k} x^k = l_n(x) := \prod_{i=1}^n (1 + x^{F_{i+1}})$
“Binomial theorem” for $\mathcal{F}$

$\binom{n}{k}$: $k$th entry (beginning with $k = 0$) in row $n$ (beginning with $n = 0$) in $\mathcal{F}$

**Theorem.** $\sum_k \binom{n}{k} x^k = l_n(x) := \prod_{i=1}^{n} (1 + x^{F_{i+1}})$

Proof omitted.
\[ \sum_k \left[ \begin{array}{c} n \\ k \end{array} \right]^2 \]

Can obtain a system of recurrences analogous to

\[
\begin{align*}
  u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\
  u_{1,1}(n+1) &= 2u_2(n) + 2u_{1,1}(n)
\end{align*}
\]

for Stern's triangle.
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Quite a bit more complicated (automated by D. Zeilberger).
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\]

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Quite a bit more complicated (automated by D. Zeilberger).

**Theorem.** \( \sum_{n \geq 0} v_2(n)x^n = \frac{1 - 2x^2}{1 - 2x - 2x^2 + 2x^3}, \) and similarly for higher powers.
A diagram (poset) associated with $\mathcal{F}$
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$P_{23}$
Label \( t \) by \( e(t) \). Then the \( k \)th label (beginning with \( k = 0 \)) at rank \( n \) is \( \binom{n}{k} \):

\[
\sum_k \binom{n}{k} x^k = I_n(x) = \prod_{i=1}^{n} \left( 1 + x F_{i+1} \right).
\]
Strings of size two and three
Strings of size two and three
Strings of size two and three

What is the sequence of string sizes on each level? E.g., on level 5, the sequence 2, 3, 2, 3, 3, 2, 3, 2.
The limiting sequence

As \( n \to \infty \), we get a “limiting sequence”

\[ 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots \]
The limiting sequence

As \( n \to \infty \), we get a “limiting sequence”

\[
2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, \ldots
\]

Let \( \phi = (1 + \sqrt{5})/2 \), the golden mean.
The limiting sequence

As $n \to \infty$, we get a “limiting sequence”

$$2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, \ldots.$$ 

Let $\phi = (1 + \sqrt{5})/2$, the golden mean.

**Theorem.** The limiting sequence $(c_1, c_2, \ldots)$ is given by

$$c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n - 1)\phi \rfloor.$$
Properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n - 1)\phi \rfloor$

$2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, \ldots$

$\gamma = (c_2, c_3, \ldots)$ characterized by invariance under $2 \to 3,$ $3 \to 32$ (Fibonacci word in the letters 2,3).
Properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n - 1)\phi \rfloor$

$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, \ldots$

- $\gamma = (c_2, c_3, \ldots)$ characterized by invariance under $2 \rightarrow 3$, $3 \rightarrow 32$ (Fibonacci word in the letters 2,3).
- $\gamma = z_1z_2\ldots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2}z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \ldots$$
Properties of \( c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n - 1)\phi \rfloor \)

\[
2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, \ldots .
\]

- \( \gamma = (c_2, c_3, \ldots) \) characterized by invariance under \( 2 \rightarrow 3, \ 3 \rightarrow 32 \) (Fibonacci word in the letters 2,3).
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  \[
 3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \ldots
  \]
- Sequence of number of 3’s between consecutive 2’s is the original sequence with 1 subtracted from each term.

\[
\begin{array}{cccccccc}
2 & 3 & 2 & 33 & 2 & 3 & 2 & 33 \\
1 & 2 & 1 & 2 & 2 & 1 & 2 & 2
\end{array}
\]
Coefficients of $I_n(x)$

$$I_n(x) = \prod_{i=1}^{n} (1 + x^{F_{i+1}})$$

Coefficient of $x^m$: number of ways to write $m$ as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$. 
Coefficients of $I_n(x)$

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Coefficient of $x^m$: number of ways to write $m$ as a sum of distinct Fibonacci numbers from \{ $F_2, F_3, \ldots, F_{n+1}$ \}.

Example. Coefficient of $x^8$ in
\[(1 + x)(1 + x^2)(1 + x^3)(1 + x^5)(1 + x^8) \text{ is 3:} \]
\[8 = 5 + 3 = 5 + 2 + 1.\]
Coefﬁcients of \( I_n(x) \)

\[
I_n(x) = \prod_{i=1}^{n} (1 + x^{F_{i+1}})
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Coefficient of \( x^m \): number of ways to write \( m \) as a sum of distinct Fibonacci numbers from \( \{F_2, F_3, \ldots, F_{n+1}\} \).

**Example.** Coeﬁcient of \( x^8 \) in 
\[(1 + x)(1 + x^2)(1 + x^3)(1 + x^5)(1 + x^8)\]
is 3:

\[
8 = 5 + 3 = 5 + 2 + 1.
\]

Can we see these sums from \( \mathfrak{F} \)? Each path from the top to a point \( t \in \mathfrak{F} \) should correspond to a sum.
An edge labeling of $\mathcal{G}$

The edges between ranks $2k$ and $2k + 1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.
An edge labeling of $\mathcal{F}$

The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

The edges between ranks $2k-1$ and $2k$ are labelled alternately $F_{2k+1}, 0, F_{2k+1}, 0, \ldots$ from left to right.
Diagram of the edge labeling
Connection with sums of Fibonacci numbers

Let $t \in \mathcal{F}$. All paths (saturated chains) from the top to $t$ have the same sum of their elements $\sigma(t)$. 
Let $t \in \mathcal{T}$. All paths (saturated chains) from the top to $t$ have the same sum of their elements $\sigma(t)$.

If $\text{rank}(t) = n$, this gives all ways to write $\sigma(t)$ as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$. 
An example

\[ 2 + 3 = F_3 + F_4 \]
An example

$$5 = F_5$$
In the limit as rank $\to \infty$, get an interesting linear ordering of $\mathbb{N}$. 
Second proof: factorization in a free monoid

\[ I_n(x) := \prod_{i=1}^{n} \left(1 + x^{F_i+1}\right) \]

\[ = \sum_k \binom{n}{k} x^k \]
Second proof: factorization in a free monoid

\[ l_n(x) := \prod_{i=1}^{n} (1 + x^{F_{i+1}}) = \sum_k \binom{n}{k} x^k \]

\[ \binom{n}{k} = \# \left\{ (a_1, \ldots, a_n) \in \{0, 1\}^n : \sum_i a_i F_{i+1} = k \right\} \]
Second proof: factorization in a free monoid

\[ l_n(x) \ := \ \prod_{i=1}^{n} (1 + x^{F_{i+1}}) \]

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\[ \binom{n}{k} = \# \left\{ (a_1, \ldots, a_n) \in \{0, 1\}^n : \sum_{i} a_i F_{i+1} = k \right\} \]

\[ v_2(n) \ := \ \sum_{k} \binom{n}{k}^2 \]

\[ = \ \# \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\} \]
A concatenation product

\[ \mathcal{M}_n := \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\} \]
A concatenation product

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Let

\[ \alpha = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix} \in \mathcal{M}_n, \quad \beta = \begin{pmatrix} c_1 & \cdots & c_m \\ d_1 & \cdots & d_m \end{pmatrix} \in \mathcal{M}_m. \]

Define

\[ \alpha \beta = \begin{pmatrix} a_1 & \cdots & a_n & c_1 & \cdots & c_m \\ b_1 & \cdots & b_n & d_1 & \cdots & d_m \end{pmatrix}, \]
A concatenation product

\[ \mathcal{M}_n := \left\{ \left( \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{array} \right) : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\} \]

Let

\[ \alpha = \left( \begin{array}{cccc} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{array} \right) \in \mathcal{M}_n, \quad \beta = \left( \begin{array}{cccc} c_1 & \cdots & c_m \\ d_1 & \cdots & d_m \end{array} \right) \in \mathcal{M}_m. \]

Define

\[ \alpha \beta = \left( \begin{array}{cccc} a_1 & \cdots & a_n & c_1 & \cdots & c_m \\ b_1 & \cdots & b_n & d_1 & \cdots & d_m \end{array} \right), \]

**Easy to check:** \( \alpha \beta \in \mathcal{M}_{n+m} \)
The monoid $\mathcal{M}$

$$\mathcal{M} := \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots,$$

a monoid (semigroup with identity) under concatenation. The identity element is $\emptyset \in \mathcal{M}_0$. 
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**Definition.** A subset $G \subset \mathcal{M}$ freely generates $\mathcal{M}$ if every $\alpha \in \mathcal{M}$ can be written uniquely as a product of elements of $G$. (We then call $\mathcal{M}$ a free monoid.)
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**Definition.** A subset $G \subset \mathcal{M}$ freely generates $\mathcal{M}$ if every $\alpha \in \mathcal{M}$ can be written uniquely as a product of elements of $G$. (We then call $\mathcal{M}$ a free monoid.)

Suppose $G$ freely generates $\mathcal{M}$, and let

$G(x) = \sum_{n \geq 1} \#(\mathcal{M}_n \cap G) x^n$. Then

$$\sum_n v_2(n) x^n = \sum_n \#\mathcal{M}_n \cdot x^n$$

$$= 1 + G(x) + G(x)^2 + \ldots$$

$$= \frac{1}{1 - G(x)}.$$
**Theorem.** $\mathcal{M}$ is freely generated by the following elements:

$$
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
11 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \\
00 & * & 0 & * & 0 & * & \cdots & * & 0 & 1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
00 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \\
11 & * & 1 & * & 1 & * & \cdots & * & 1 & 0
\end{pmatrix},
$$

where each * can be 0 or 1, but two *’s in the same column must be equal.
Free generators of $\mathcal{M}$

**Theorem.** $\mathcal{M}$ is freely generated by the following elements:

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0
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1
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$$

$$
= 
\begin{pmatrix}
11 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \\
00 & * & 0 & * & 0 & * & \cdots & * & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
00 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \\
11 & * & 1 & * & 1 & * & \cdots & * & 1 & 0
\end{pmatrix},
$$

where each $*$ can be 0 or 1, but two $*$'s in the same column must be equal.

**Example.**

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix}: 1 + 2 + 3 + 5 = 3 + 8$$
\( G(x) \)

\[
\begin{pmatrix}
0 \\
0 \\
11 \ast 1 \ast 1 \ast 1 \ast \ldots \ast 1 \ast 0 \\
00 \ast 0 \ast 0 \ast 0 \ast \ldots \ast 0 \ast 1 \\
00 \ast 0 \ast 0 \ast 0 \ast \ldots \ast 0 \ast 1 \\
11 \ast 1 \ast 1 \ast 1 \ast \ldots \ast 1 \ast 0
\end{pmatrix}
\]

Two elements of length one: \( G(x) = 2x + \ldots \)
Two elements of length one: $G(x) = 2x + \cdots$

Let $k$ be the number of columns of *’s. Length is $2k + 3$. Thus

$$G(x) = 2x + 2 \sum_{k \geq 0} 2^k x^{2k+3} = 2x + \frac{2x^3}{1 - 2x^2}.$$
Completion of proof

\[ \sum_n v_2(n)x^n = \frac{1}{1 - G(x)} \]

\[ = \frac{1}{1 - \left(2x + \frac{2x^3}{1-2x^2}\right)} \]

\[ = \frac{1 - 2x^2}{1 - 2x - 2x^2 + 2x^3} \]
Further vistas?

What more can be said about $P_{ij}$?
References

These slides: www-math.mit.edu/~rstan/transparencies/yehfest.pdf


The Fibonacci triangle (and much more): arXiv:2101.02131


Factorization in free monoids: EC1, second ed., §4.7.4
The final slide
That's all Folks!