(2n − 1)!!

April 14, 2020
Semifactorials

\[(2n - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n - 1) = \frac{(2n)!}{2^n n!},\]

called \(2n - 1\) **double factorial** (bad?) or **semifactorial**
(complete) matching on $2n$-element set:
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Theorem. The number of matchings on $[2n]$ is $(2n - 1)!!$. 

(complete) matching on 2n-element set:

\[ \text{Theorem. } \text{The number of matchings on } [2n] \text{ is } (2n - 1)!!. \]

\[ \text{Proof. } \text{Pick } i \in [2n] \text{ and match it in } 2n - 1 \text{ ways. Then pick some unmatched element } j \text{ and match it in } (2n - 3) \text{ ways, etc. } \square \]
Schröder’s third problem

Ernst Schröder, *Vier kombinatorische Probleme*, 1870

**Problem 3 (complete binary partitions).** How many ways to partition an $n$-set ($n > 1$) into two nonempty blocks, then partition each nonsingleton block into two nonempty blocks, etc., until only singletons remain?
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leaf labelled (unordered) binary tree
Bijection with matchings

Label by $n + 1$ the unlabelled vertex with two labelled children, with the least possible label of a child.
Bijection with matchings

Label by $n + 2$ the unlabelled vertex with two labelled children, with the least possible label of a child.

```
  8 6 3
/\ / \ /
1 4 2 9
  \  \ 
   5 7
```
Bijection with matchings

Continue until all nonroot vertices are labelled 1, 2, \ldots, 2n − 2.
Bijection with matchings

Continue until all nonroot vertices are labelled $1, 2, \ldots, 2n - 2$.

Now match the two children of any nonleaf vertex: $5, 7 - 2, 9 - 3, 10 - 1, 4 - 6, 8 - 11, 12$. 
Bijection with matchings

Continue until all nonroot vertices are labelled $1, 2, \ldots, 2n - 2$.

Now match the two children of any nonleaf vertex: $5, 7 - 2, 9 - 3, 10 - 1, 4 - 6, 8 - 11, 12$.

**Theorem.** The number of leaf-labelled binary trees with $n$ leaves is $(2n - 3)!!$. 
Theorem.

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-\frac{1}{2}x^2} \, dx = (2n - 1)!! \]

the \((2n)\)th moment of the standard normal distribution.
An $S_{2n}$ action

$M_n$: set of all matchings on $[2n]$, so $\#M_n = (2n - 1)!!$

$S_{2n}$ acts of $M_n$ by permuting vertices. What is this action? I.e., what is the multiplicity of each irreducible character $\chi^\lambda$, $\lambda \vdash 2n$?
The subgroup $S_2^n$

$S_2^n$: subgroup of $S_{2n}$ generated by $(1, 2), (3, 4), \ldots, (2n - 1, 2n)$, so $S_2^n \equiv (\mathbb{Z}/2\mathbb{Z})^n$ and $\#S_2^n = 2^n$. 
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$N(S_2^n)$: the normalizer of $S_2^n$, i.e., all $w \in S_{2n}$ such that

$v \in S_2^n \Rightarrow wvw^{-1} \in S_2^n$
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$N(S_n^2)$ consists of all $w \in S_{2n}$ that permute the elements in each row and permute the rows among themselves of the array ($n = 5$)

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Action on cosets

Aside. $N(S_2^n)$ is the wreath product $S_n \wr S_2$. 
Aside. $N(\mathcal{S}_2^n)$ is the **wreath product** $\mathcal{S}_n \wr \mathcal{S}_2$.

$\# N(\mathcal{S}_2^n) = 2^n n!$, so $[\mathcal{S}_{2n} : N(\mathcal{S}_2^n)] = (2n - 1)!!$. 
Action on cosets

Aside. $N(\mathfrak{S}_2^n)$ is the **wreath product** $\mathfrak{S}_n \wr \mathfrak{S}_2$.

$\#N(\mathfrak{S}_2^n) = 2^n n!$, so $[\mathfrak{S}_2^n : N(\mathfrak{S}_2^n)] = (2n - 1)!!$.

The action on $\mathfrak{S}_2^n$ on the left cosets of $N(\mathfrak{S}_2^n)$ is isomorphic to the action of $\mathfrak{S}_2^n$ on $\mathcal{M}_n$. Thus, as $\mathfrak{S}_2^n$-modules,

$$\mathcal{M}_n \cong \mathcal{M}_{N(\mathfrak{S}_2^n)}.$$
Let $\text{ch}$ denote the Frobenius characteristic symmetric function of an $\mathcal{S}_m$ action. By the theory of plethysm,

$$\text{ch} \mathcal{M}_n = (\text{ch} 1_{\mathcal{S}_n})[\text{ch} 1_{\mathcal{S}_2}] = h_n[h_2].$$
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**Theorem.** Let $\lambda \vdash 2n$. The multiplicity of $\chi^\lambda$ in the action of $\mathfrak{S}_{2n}$ on $M_n$ is 1 if $\lambda = 2\mu$, and 0 otherwise.
Zonal polynomials

\[ H_n = N(\mathcal{G}_2^n) \text{ (hyperoctahedral group)} \]

Because \( M_n \) is multiplicity-free as an \( \mathcal{G}_{2n} \)-module, the pair \( (\mathcal{G}_{2n}, H_n) \) is a Gelfand pair.
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Let \( \lambda \vdash n \) and \( \chi^{2\lambda} \) be the irreducible character of \( \mathfrak{S}_{2n} \) indexed by

2\( \lambda \). Let \( s \in \mathfrak{S}_{2n} \) of cycle type \( \rho \vdash 2n \).

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\omega^\lambda_\rho = \frac{1}{2^n n!} \sum_{w \in H} \chi^{2\lambda}(sw)
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\omega_{\lambda}^\rho = \frac{1}{2^nn!} \sum_{w \in H} \chi^{2\lambda}(sw)
\]

Define the **zonal polynomial**

\[
Z_\lambda = 2^nn! \sum_{\rho \vdash n} z_{2\rho}^{-1} \omega_{\rho}^\lambda p_\rho,
\]

a homogeneous symmetric function of degree \( n \).
Some properties of zonal polynomials

- \( \{Z_\lambda\}_{\lambda \vdash n} \) is a \( \mathbb{Q} \)-basis for \( \Lambda_\mathbb{Q} \) (symmetric functions over \( \mathbb{Q} \)).
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  \prod_{u \in \lambda} (2a(u) + l(u) + 1).
- Z_\lambda = J_\alpha^{(2)}, \text{ where } J_\alpha^\lambda \text{ (} \alpha \in \mathbb{R} \text{) is a } \textbf{Jack symmetric function}\text{ (a limiting case of Macdonald polynomials)}
(2n − 1)!! is not the order of an “interesting” finite group. However, it is the dimension of a natural “orthogonal analogue” of the group algebra of $\mathfrak{S}_n$. 
The Brauer algebra

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Let $\dim_{\mathbb{C}} V = k$. The general linear group $\text{GL}(V)$ acts diagonally on $V \otimes^n$. The linear transformations $V \otimes^n \rightarrow V \otimes^n$ commuting with this action are generated by the $n!$ permutations of tensor coordinates. For $k \geq n$ these linear transformations form the algebra $\mathbb{C}[\mathfrak{S}_n]$ (the group algebra of $\mathfrak{S}_n$).
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Let $\dim_{\mathbb{C}} V = k$. The general linear group $\text{GL}(V)$ acts diagonally on $V^\otimes n$. The linear transformations $V^\otimes n \to V^\otimes n$ commuting with this action are generated by the $n!$ permutations of tensor coordinates. For $k \geq n$ these linear transformations form the algebra $\mathbb{C}[\mathfrak{S}_n]$ (the group algebra of $\mathfrak{S}_n$).

Let $\dim_{\mathbb{C}} V = k$. The orthogonal group $O(V)$ (i.e., $A(A^*)^t = I$) acts diagonally on $V^\otimes n$. For $k \geq n$, the linear transformations $V^\otimes n \to V^\otimes n$ commuting with this action form an algebra $\mathfrak{B}_n$ of dimension $(2n − 1)!!$ (the **Brauer algebra**).
Brauer algebra multiplication

Let \( z \) be a parameter. Take \( \mathcal{M}_n \) as a basis for an algebra \( \mathcal{B}_n(z) \), where \( \mathcal{B}_n(1) = \mathcal{B}_n \) (not semisimple). For “generic” \( z \) (e.g., \( z \notin \mathbb{Z} \)), \( \mathcal{B}_n(z) \) is semisimple.
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An oscillating tableau $T$ of shape $\lambda$ and length $n$ is a sequence

$$\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^m = \lambda$$

of partitions (identified with their Young diagrams) such that $\lambda^i$ is obtained from $\lambda^{i-1}$ by adding a box or removing a box.
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**Note.** If we only add boxes, then we get a standard Young tableau.
Oscillating tableaux

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**Example.** Shape \( \lambda = (2, 1) \), length \( n = 7 \):

\[
\emptyset \quad \square \quad \square \quad \square \quad \square \quad \square \quad \square \quad \square
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$$\emptyset \quad \square \quad || \quad || | \quad || || \quad || || || \quad || || ||$$

$o^{\lambda,n}$: number of oscillating tableau of shape $\lambda$ and length $n$
**Theorem.** Fix $n \geq 1$. Irreps of $\mathcal{B}_n(z)$ ($z$ generic) are indexed by partitions $\lambda \vdash m$, where $m \leq n$, $n \equiv m \pmod{2}$. The dimension of the irrep indexed by such $\lambda$ is $o^{\lambda,n}$.

**Corollary.** $\sum_\lambda (o^{\lambda,n})^2 = (2n - 1)!!$

Equivalently, number of oscillating tableaux of shape $\emptyset$ and length $2n$ is $(2n - 1)!!$.
Dimension of $\mathcal{B}_n$ irreps

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First combinatorial proof (bijection with $\mathcal{M}_n$) by RS and S. Sundaram.
Sundaram’s bijection
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Crossings and nestings

crossing:

nesting:
**k-crossings and k-nestings**

\[ M = \text{matching} \]
\[ \text{cr}(M) = \max\{ k : \exists k \text{-crossing} \} \]
\[ \text{ne}(M) = \max\{ k : \exists k \text{-nesting} \} . \]
Theorem (Bill Yongchuan Chen (陈永川), Eva Yuping Deng (邓玉平), Rosena Ruoxia Du (杜若霞), Catherine Huafei Yan (颜华菲), RS) Let $M \mapsto (\emptyset = T_0, T_1, \ldots, T_{2n} = \emptyset)$ in the bijection from matchings to oscillating tableau of shape $\emptyset$. Then $\text{cr}(M)$ is equal to the most number of rows of any $T_i$, and $\text{ne}(M)$ is equal to the most number of columns of any $T_i$. 
Some consequences

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**Corollary.** Let \( f_n(i, j) = \# \text{ matchings } M \text{ on } [2n] \text{ with } \text{cr}(M) = i \) and \( \text{ne}(M) = j \). Then \( f_n(i, j) = f_n(j, i) \).

**Corollary.** \# matchings \( M \) on \([2n]\) with \( \text{cr}(M) = k \) equals \# matchings \( M \) on \([2n]\) with \( \text{ne}(M) = k \).
An enumerative consequence

**Theorem** (Grabiner-Magyar, essentially) Let $f_k(n)$ be the number of matchings $M \in \mathcal{M}_n$ satisfying $\text{cr}(M) \leq k$. Define

$$F_k(x) = \sum_n f_k(n) \frac{x^{2n}}{(2n)!}.$$  

Then

$$F_k(x) = \det [l_{i-j}(2x) - l_{i+j}(2x)]_{i,j=1}^k$$

where

$$l_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$$

(hyperbolic Bessel function of the first kind of order $m$).
A probabilistic consequence

**Note.** \( cr(M) \) is the matching analogue of the length of the longest increasing subsequence of \( w \in S_n \), and \( ne(M) \) is the analogue of the length of the longest decreasing subsequence.
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Painlevé II equation:

\[
u''(x) = 2u(x)^3 + xu(x).
\]

Tracy-Widom distribution:

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F(t) = \exp \left( - \int_t^\infty (x - t)u(x)^2 \, dx \right)
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**Painlevé II equation:**

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**Theorem.**

\[ \lim_{n \to \infty} \text{Prob} \left( \frac{cr_n(M) - \sqrt{2n}}{(2n)^{1/6}} \leq \frac{t}{2} \right) = F(t)^{1/2} \exp \left( \frac{1}{2} \int_t^\infty u(s) \, ds \right) \]
The final slide
The final slide

Hope you enjoyed the lectures!

Thanks for listening!