A Survey of Unimodality and Log-Concavity

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(3) **strongly log-concave** if $\left( \frac{a_i}{\binom{n}{i}} \right)^2 \geq \frac{a_{i-1}}{\binom{n}{i-1}} \frac{a_{i+1}}{\binom{n}{i+1}}$. 


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**Example.** $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ (strongly log-concave)
I. REAL ZEROS
Newton’s theorem

**Theorem (I. Newton).** Let

\[ \gamma_1, \ldots, \gamma_n \in \mathbb{R} \]

and

\[ P(x) = \prod (x + \gamma_i) = \sum a_i \binom{n}{i} x^i = \sum b_i x^i. \]

Then \( a_0, a_1, \ldots, a_n \) is log-concave. Same as \( b_0, \ldots, b_n \) strongly log-concave.
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**Proof.** \( P^{(n-i-1)}(x) \) has real zeros

\[ \Rightarrow Q(x) := x^{i+1} P^{(n-i-1)}(1/x) \text{ has real zeros} \]

\[ \Rightarrow Q^{(i-1)}(x) \text{ has real zeros.} \]

But \( Q^{(i-1)}(x) = \frac{n!}{2} \left( a_{i-1} + 2a_i x + a_{i+1} x^2 \right) \]

\[ \Rightarrow a_i^2 \geq a_{i-1}a_{i+1}. \]
Basic linear algebra

**Theorem.** If $A$ is a (real) symmetric matrix, then every zero of $\det(I + xA)$ is real.
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**Example.** $G$: finite graph with vertex set $V$ and $\mu_{uv}$ edges between vertices $u$ and $v$

$L$: Laplacian matrix of $G$. Rows and columns indexed by $V$, with

$$L_{uv} = \begin{cases} \deg(v), & \text{if } u = v \\ -\mu_{uv}, & \text{if } u \neq v. \end{cases}$$
The Matrix-Tree theorem

Matrix-Tree Theorem (slightly expanded). \( \det(I + xL) = \sum a_i x^i \), where \( a_i \) is the number of rooted spanning forests of \( G \) with \( i \) edges. Thus \( \sum a_i x^i \) has only real zeros, so \( a_0, a_1, \ldots, a_{\#V} \) is strongly log-concave.
What about **unrooted** spanning forests?

$b_i$: number of (unrooted) spanning forests of $G$ with $i$ edges.

More generally, let $X$ be a finite subset of a vector space of dimension $n$, and let $b_i$ be the number of $i$-element linearly independent subsets of $X$. 
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**Theorem** (Lenz, 2013, based on Huh, 2012) \( b_0, b_1, \ldots, b_n \) is log-concave (with no external zeros).

**Proof** of Huh based on Hodge-Riemann relations for the cohomology of certain varieties. Later generalized by Adiprasito, Huh, and Katz to any finite matroid (an abstract generalization of a finite subset of a vector space).
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What about strongly log-concave? To be discussed.
Definition. An $m \times n$ real matrix is **totally nonnegative** if all minors (determinants of square submatrices) are nonnegative.
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**Theorem.** Let \( A \) be an \( n \times n \) totally nonnegative matrix. Then all eigenvalues of \( A \) are real and nonnegative. Hence the characteristic polynomial \( \det(xI - A) \) has only real zeros.
An application

Let $P$ be a finite poset (partially ordered set) with no induced $3+1$ or $2+2$, i.e., there do not exist elements $s < t < u, v$ with no other relations among them, nor elements $s < t, u < v$ with no other relations among them. Let $c_i$ be the number of $i$-element chains of $P$.

\[
\begin{align*}
\text{bad} & \\
c_0 &= 1 \\
c_1 &= 5 \\
c_2 &= 5 \\
c_3 &= 1
\end{align*}
\]

**Theorem.** $\sum c_i x^i$ has only real zeros.
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Proof. Let \( A \) be the matrix with rows and columns indexed by \( P \), with

\[
A_{st} = \begin{cases} 
0, & \text{if } s \leq t \\
1, & \text{otherwise.}
\end{cases}
\]

Then \( A \) is totally nonnegative, and \( \det(I + xA) = \sum c_i x^i \). \( \square \)
Two further remarks

- Can be shown that the $(2+2)$-avoiding hypothesis is unnecessary (using symmetric functions).
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- Multivariate generalizations of real-rooted polynomials: **stable polynomials** (P. Brandén) and **Lorentzian polynomials** (P. Brandén and J. Huh). Sample application:

  **Theorem.** If $l_k$ is the number of $k$-element independent sets of a matroid, then the sequence $l_0, l_1, \ldots$ is strongly log-concave. Conjectured by **Mason** in 1972. Also proved in a similar way by **Anari-Liu-Gharan-Vinzant**. (We mentioned earlier the proof by Lenz of log-concavity.)
II. ANALYTIC METHODS
Let $p(n, k)$ be the number of partitions of $n$ into $k$ parts. E.g., $p(7, 3) = 4$:

$$5 + 1 + 1, \quad 4 + 2 + 1, \quad 3 + 3 + 1, \quad 3 + 2 + 2.$$ 

$$\sum_{n\geq 0} p(n, k)x^n = \frac{x^k}{(1-x)(1-x^2) \cdots (1-x^k)}$$

$$\Rightarrow p(n, k) = \frac{1}{2\pi i} \oint \frac{s^{k-n-1} ds}{(1-s)(1-s^2) \cdots (1-s^k)}.$$
Theorem of Szekeres

**Theorem (G. Szekeres, 1954)** For $n > N_0$, the sequence

$$p(n,1), p(n,2), \ldots, p(n,n)$$

is unimodal, with maximum at

$$k = c\sqrt{n}L + c^2\left(\frac{3}{2} + \frac{3}{2}L - \frac{1}{4}L^2\right) - \frac{1}{2}$$

$$+ O\left(\frac{\log^4 n}{\sqrt{n}}\right)$$

where

$$c = \sqrt{6}/\pi, \quad L = \log c\sqrt{n}.$$
III. ALEXANDROV-FENCHEL INEQUALITIES
Let $K, L$ be convex bodies (nonempty compact convex sets) in $\mathbb{R}^n$, and let $x, y \geq 0$. Define the **Minkowski sum**

$$xK + yL = \{x\alpha + y\beta : \alpha \in K, \beta \in L\}.$$ 

Then there exist $V_i(K, L) \geq 0$, the *(Minkowski) mixed volumes* of $K$ and $L$, satisfying

$$\text{Vol}(xK + yL) = \sum_{i=0}^{n} \binom{n}{i} V_i(K, L)x^{n-i}y^i.$$ 

Note $V_0 = \text{Vol}(K)$, $V_n = \text{Vol}(L)$. 
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Note $V_0 = \text{Vol}(K)$, $V_n = \text{Vol}(L)$.

**Theorem** *(Alexandrov-Fenchel, 1936–38)* $V_i^2 \geq V_{i-1} V_{i+1}$
Corollary. Let $P$ be an $n$-element poset. Fix $x \in P$. Let $N_i$ denote the number of order-preserving bijections (linear extensions)

$$f : P \to \{1, 2, \ldots, n\}$$

such that $f(x) = i$. Then

$$N_i^2 \geq N_{i-1}N_{i+1}.$$ 

Proof. Find $K, L \subset \mathbb{R}^{n-1}$ such that $V_i(K, L) = N_{i+1}$. □
An example

\[(N_1, \ldots, N_5) = (0, 1, 2, 2, 2)\]
Generalizations

There are algebraic and algebraic-geometric generalizations of the Alexandrov-Fenchel inequalities with many applications.
IV. REPRESENTATIONS OF $\text{SL}(2, \mathbb{C})$ AND $\mathfrak{sl}(2, \mathbb{C})$
Representations of $\text{SL}(2, \mathbb{C})$

Let

$$G = \text{SL}(2, \mathbb{C}) = \{2 \times 2 \text{ complex matrices with determinant 1}\}.$$  

Let $A \in G$, with eigenvalues $\theta, \theta^{-1}$. For all $n \geq 0$, there is a unique irreducible (polynomial) representation

$$\varphi_n : G \to \text{GL}(V_{n+1})$$

of dimension $n + 1$, and $\varphi_n(A)$ has eigenvalues

$$\theta^{-n}, \theta^{-n+2}, \theta^{-n+4}, \ldots, \theta^n.$$  

Every (continuous) representation is a direct sum of irreducibles.
Unimodal weight multiplicities

If $\varphi : G \to \text{GL}(V)$ is any (finite-dimensional) representation, then

$$\text{tr} \varphi(A) = \sum_{i \in \mathbb{Z}} a_i \theta^i, \quad a_i = a_{-i}$$

$$= a_0 + a_1(\theta + \theta^{-1}) + \sum_{i \geq 2}(a_i - a_{i-2})(\theta^{-i} + \theta^{-i+2} + \ldots + \theta^i)$$

$$\Rightarrow a_i \geq a_{i-2}$$

$$\Rightarrow \{a_{2i}\}, \{a_{2i+1}\} \text{ are unimodal}$$

(and symmetric)

(Completely analogous construction for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.)
**q-binomial coefficient**

For $k, n \geq 0$ define

\[
\binom{n + k}{k} = \frac{(1 - q^{n+k})(1 - q^{n+k-1}) \cdots (1 - q^{n+1})}{(1 - q^k)(1 - q^{k-1}) \cdots (1 - q)},
\]

a polynomial in $q$ with nonnegative integer coefficients.
The $k$th symmetric power

**Example.** $S^k(\varphi_n)$, eigenvalues

$$(\theta^{-n})^{t_0} (\theta^{-n+2})^{t_1} \cdots (\theta^n)^{t_n},$$

$$t_0 + t_1 + \cdots + t_n = k, \quad t_i \geq 0$$

$$\Rightarrow \text{tr} \: \varphi (A) =$$

$$\sum_{t_0 + \cdots + t_n = k} \theta^{t_0(-n)+t_1(-n+2)+\cdots+t_n n}$$

$$= \theta^{-nk} \left[ \begin{array}{c} n + k \\ k \end{array} \right] \theta^2$$

$$= \theta^{-nk} \sum_{i \geq 0} P_i(n, k) \theta^{2i},$$

where $P_i(n, k)$ is the number of partitions of $i$ with $\leq k$ parts, largest part $\leq n$. 
Sylvester’s theorem

\[ \Rightarrow P_0(n, k), \ldots, P_{nk}(n, k) \]

is unimodal (Sylvester, 1878).


\[ \sum_{i} P_i(3, 2) q^i = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6 \]

\[ = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{(1 - q^5)(1 - q^4)}{(1 - q^2)(1 - q)} \]
Principal $\mathfrak{sl}(2, \mathbb{C})$

**Example.** Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Then there exists a **principal** $\mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{g}$. A representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ restricts to

$$\varphi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V).$$

**Example.** $\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C})$, $\varphi = \text{spin representation}$:

$$\Rightarrow (1 + q)(1 + q^2)\cdots(1 + q^n)$$

has unimodal coefficients (**Dynkin** 1950, **Hughes** 1977). (No combinatorial proof known.)
Higher dimensional partitions

**Recall:** $P_i(n, k)$: number of partitions of $i$ with $\leq k$ parts, largest part $\leq n$, i.e., number of 1-dimensional integer arrays (sequences) $a_1, a_2, \ldots, a_k$ such that

$$n \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0, \quad \sum a_j = i.$$
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$$n \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0, \quad \sum a_j = i.$$ 

Generalize to $P_i(n_1, n_2, \ldots, n_{d+1})$: number of $d$-dimensional arrays $\left(a_{j_1, j_2, \ldots, j_d}\right)_{1 \leq j_r \leq n_r}$ of nonnegative integers, weakly decreasing in each coordinate, maximum entry $\leq n_{d+1}$, sum of entries $= i$. 
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$P_i(n_1, n_2, \ldots, n_{d+1})$ is symmetric in $n_1, \ldots, n_{d+1}$.
Higher dimensional partitions

Recall: \( P_i(n, k) \): number of partitions of \( i \) with \( \leq k \) parts, largest part \( \leq n \), i.e., number of 1-dimensional integer arrays (sequences) \( a_1, a_2, \ldots, a_k \) such that

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Generalize to \( P_i(n_1, n_2, \ldots, n_{d+1}) \): number of \( d \)-dimensional arrays \( \left( a_{j_1, j_2, \ldots, j_d} \right)_{1 \leq j_r \leq n_r} \) of nonnegative integers, weakly decreasing in each coordinate, maximum entry \( \leq n_{d+1} \), sum of entries = \( i \).

\( P_i(n_1, n_2, \ldots, n_{d+1}) \) is symmetric in \( n_1, \ldots, n_{d+1} \).

The case \( d = 2 \): plane partitions (MacMahon)
Example: \( n_1 = n_2 = n_3 = 2 \)

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
00 & 10 & 11 & 10 & 20 & 11 & 21 & 20 & \ldots & 22 \\
00 & 00 & 00 & 10 & 00 & 10 & 00 & 10 & \ldots & 22 \\
\end{array}
\]
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\((P_0, \ldots, P_8) = (1, 1, 3, 3, 4, 3, 3, 1, 1)\)

(symmetric, unimodal, not log-concave)
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**Theorem.** *For fixed $(n_1, n_2, n_3)$, the sequence $P_0, P_1, \ldots$ is symmetric (easy) and unimodal.*
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**Theorem.** *For fixed \((n_1, n_2, n_3)\), the sequence \(P_0, P_1, \ldots\) is symmetric (easy) and unimodal.*

**Proof** follows from principal \(\mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{sl}(N, \mathbb{C})\), \(N = 1 + \max n_j\), and choosing a certain irrep of \(\mathfrak{sl}(N, \mathbb{C})\).
A conjecture

**Conjecture.** For fixed \( n_1, \ldots, n_{d+1} \), the sequence \( P_0, P_1, \ldots \) is unimodal.
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Open for $d = 3$. Also open for $n_1 = n_2 = \cdots = n_{d+1} = 2$. In these cases, no nice way to compute $P_i$ or $\sum P_i$.

For $n_1 = n_2 = \cdots = n_{d+1} = 2$, $\sum P_i$ is the order of the **free distributive lattice** on $d + 1$ generators (Dedekind’s problem).
Let $X$ be an irreducible $n$-dimensional complex projective variety with finite quotient singularities (e.g., smooth).

$$\beta_i = \dim_{\mathbb{C}} H^i(X; \mathbb{C})$$

$\mathfrak{sl}(2, \mathbb{C})$ acts on $H^*(X; \mathbb{C})$, and $H^i(X; \mathbb{C})$ is a weight space with weight $i - N$

$$\Rightarrow \{\beta_{2i}\}, \{\beta_{2i+1}\} \text{ are unimodal.}$$
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$$\Rightarrow \{\beta_{2i}\}, \{\beta_{2i+1}\} \text{ are unimodal}.$$

Follows from hard Lefschetz theorem.
Two examples

Example. \( X = G_k(\mathbb{C}^{n+k}) \) (Grassmannian). Then

\[
\sum \beta_i q^i = \left[ \begin{array}{c} n + k \\ k \end{array} \right]_{q^2}.
\]
Two examples

Example. $X = G_k(\mathbb{C}^{n+k})$ (Grassmannian). Then

$$\sum \beta_i q^i = \binom{n+k}{k}_{q^2}.$$ 

Example. (Hessenberg varieties.) Fix $1 \leq p \leq n - 1$. For $w = w_1 \cdots w_n \in \mathfrak{S}_n$, let

$$d_p(w) = \# \{(i, j) : w_i > w_j, \ 1 \leq j - i \leq p\}.$$ 

$$d_1(w) = \#\text{descents of } w$$
$$d_{p-1}(w) = \#\text{inversions of } w.$$

Let $A_p(n, k) = \# \{ w \in \mathfrak{S}_n : d_p(w) = k \}$. 
de Mari-Shayman theorem
Theorem (de Mari-Shayman, 1987). The sequence

\[ A_p(n, 0), A_p(n, 1), \ldots, A_p(n, p(2n - p - 1)/2) \]

is unimodal.
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is unimodal.

Proof. Construct a “generalized Hessenberg variety” $X_{np}$ satisfying $\beta_{2k}(X_{np}) = A_p(n, k)$. □
Polytope definitions

(Convex) polytope: the convex hull $\mathcal{P}$ of a finite set $S \subset \mathbb{R}^n$

$\text{dim } \mathcal{P}$: dimension of affine span of $\mathcal{P}$ (so $\mathcal{P}$ is homeomorphic to a $d$-dimensional ball)

Face of $\mathcal{P}$: the intersection of $\mathcal{P}$ with a supporting hyperplane $H$ (so $\mathcal{P}$ lies on one side of $H$)
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Simplicial polytopes and $f$-vectors

$i$-dimensional simplex: convex hull of $i + 1$ affinely independent points in $\mathbb{R}^n$

simplicial polytope: every proper face is a simplex

E.g, the tetrahedron, octahedron, and icosahedron are simplicial, but not the cube or dodecahedron

Let $\mathcal{P}$ be a simplicial polytope, with $f_i$ $i$-dimensional faces (with $f_{-1} = 0$). E.g., for the octahedron,

$$f_0 = 6, \quad f_1 = 12, \quad f_2 = 8.$$
The *h*-vector

\( \mathcal{P} \): a simplicial polytope of dimension \( d \)

Define the *h*-vector \( h(\mathcal{P}) = (h_0, h_1, \ldots, h_d) \) of \( \mathcal{P} \) by

\[
\sum_{i=0}^{d} f_{i-1}(x - 1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.
\]

E.g., for the octahedron \( \mathcal{O} \),

\[
(x - 1)^3 + 6(x - 1)^2 + 12(x - 1) + 8 = x^3 + 3x^2 + 3x + 1,
\]

so \( h(\mathcal{O}) = (1, 3, 3, 1) \).
Conditions on $h_i$

Dehn-Sommerville equations (1905,1927): $h_i = h_{d-i}$

GLBC (McMullen-Walkup, 1971):

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor},$$

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Even stronger condition (the \textit{g-conjecture for simplicial polytopes}) conjectured by McMullen in 1971. Gave a conjectured complete characterization of $f$-vectors of simplicial polytopes.
Toric varieties

**Note.** Every simplicial polytope in $\mathbb{R}^n$ can be slightly perturbed to have rational vertices without affecting the combinatorial type.

Let $X(\mathcal{P})$ be the **toric variety** corresponding to a rational simplicial polytope $\mathcal{P}$. Then $\mathcal{P}$ is an irreducible complex projective variety with finite quotient singularities. Let

$$H(\mathcal{P}) = H^0 \oplus H^2 \oplus H^4 \oplus \ldots \oplus H^{2d}$$

be its cohomology ring (over $\mathbb{C}$), so $\beta_{2i} := \dim_{\mathbb{C}} H^{2i} < \infty$.

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Fact. $\beta_{2i} = h_i$  

$\Rightarrow$ GLBC.

Also, $H(\mathcal{P})$ is generated as a $\mathbb{C}$-algebra by $H^2$. This and hard Lefschetz imply the $g$-conjecture for simplicial polytopes.
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If $\Delta$ triangulates a $(d-1)$-sphere, then $(h_0, h_1, \ldots, h_d)$ is defined as before, and $h_i = h_{d-i}$. 
Theorem (K. Adiprasito, 2018). The g-conjecture for spheres is true. In particular, if $\Delta$ triangulates a $(d - 1)$-sphere then $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$ (and $h_i = h_{d-i}$).
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Idea of proof. There is a ring $H(\Delta)$ (the face ring modulo a linear system of parameters) which for a certain l.s.o.p is isomorphic to $H(\mathcal{P})$ when $\Delta$ is the boundary complex of a rational simplicial polytope. Then prove a hard Lefschetz theorem for $H(\Delta)$. 


V. SOME OPEN PROBLEMS
$P$: a $p$-element fence, i.e., a poset such as

**order ideal**: $I \subseteq P$ such that $t \in I, s \leq t \Rightarrow s \in I$

$c_i$: number of $i$-element order ideals of $P$
Conjecture of Morier-Genoud and Ovsienko

\[
\emptyset, a, b, ab, bc, abc, abd, abcd
\]

\[
(c_0, \ldots, c_4) = (1, 2, 2, 2, 1)
\]
Conjecture of Morier-Genoud and Ovsienko

\[ \emptyset, a, b, ab, bc, abc, abd, abcd \]
\[ (c_0, \ldots, c_4) = (1, 2, 2, 2, 1) \]

**Conjecture.** For any \( p \)-element fence, the sequence \( c_0, c_1, \ldots, c_p \) is unimodal.
**Knots**

*\( K \): a knot in \( \mathbb{R}^3 \)

*\( \Delta_K(t) \in \mathbb{Z}[t, t^{-1}] \): the **Alexander polynomial** of \( K \) (a famous knot invariant).

**Fact.** A polynomial \( \Gamma(t) \in \mathbb{Z}[t, t^{-1}] \) is the Alexander polynomial of some knot if and only if \( \Gamma(1) = 1 \) and \( \Gamma(1/t) = \Gamma(t) \).
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**Conjecture** (A. Stoimenow, 2014) If \( K \) is alternating, then \( \Delta_K(t) \) has log-concave coefficients. (Unimodality for \( \Delta_K(-t) \) conjectured by R. H. Fox in 1962)
Genus distribution of graphs

$G$: finite connected graph

g_i(G):$ number of combinatorially distinct cellular embeddings (i.e., every face is homeomorphic to an open disk) of $G$ in an orientable surface of genus $i$

**Fact.** The sequence $g_0(G), g_1(G), g_2(G), \ldots$ (the genus distribution of $G$) has finitely many positive terms and no internal zeros.

**Conjecture** (Gross-Robbins-Tucker, 1989) The genus distribution of $G$ is log-concave. (Known that $\sum g_i(G)t^i$ need not have only real zeros.)
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The last slide