GENERALIZED RIFFLE SHUFFLES AND QUASISYMMETRIC FUNCTIONS

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Transparencies available at:
Let $\mathbf{x}_i =$ probability of $i \in \mathbb{P} = \{1, 2 \ldots \}$.

Fix $n \in \mathbb{P}$. Define a random $w \in \mathcal{S}_n$ as follows:

For $1 \leq j \leq n$, choose independently an integer $i_j$ from the distribution $x_i$. Then standardize the sequence $\mathbf{i} = i_1 \cdot \cdot \cdot i_n$, i.e., replace the 1’s with 1, 2, \ldots, $a_1$ from left-to-right, then the 2’s with $a_1 + 1$, $a_1 + 2$, \ldots, $a_1 + a_2$ from left-to-right, etc.

$$\mathbf{i} = \underbrace{311431}_{\text{word}}$$

$$w = \underbrace{412653}_{\text{word}}$$

Call this the $\textbf{QS}$-distribution or $\textbf{QS}(x)$-distribution.
Previously studied by

- Diaconis-Fill-Pitman
- Fulman
- Its-Tracy-Widom
- Kuperberg,

at least when $x_i$ has finite support.
**Example.** \( w = 213 \). The sequence 
\( i_1 i_2 i_3 \) has standardization 213 if and only 
if \( i_2 < i_1 \leq i_3 \). Hence 
\[
\text{Prob}(213) = \sum_{a < b \leq c} x_a x_b x_c = L_{(1,2)}(x).
\]

**Theorem.** Let \( w \in \mathfrak{S}_n \). The probability \( \text{Prob}(w) \) that a permutation in 
\( \mathfrak{S}_n \) chosen from the QS-permutation 
is equal to \( w \) is given by 
\[
\text{Prob}(w) = L_{\text{co}(w^{-1})}(x).
\]

**Example.** \( w = 74513826 \)
\[
w^{-1} = 47 \cdot 5 \cdot 238 \cdot 16
\]
\[
\text{co}(w^{-1}) = (2, 1, 3, 2)
\]
\[
L_{(2,1,3,2)}(x) = \sum_{a \leq b < c < d \leq e \leq f < g \leq h} x_a \cdots x_h.
\]
Special cases:

- $x_1 = x_2 = 1/2$: riffle or dovetail shuffle (Bayer-Diaconis), the $U_2$-distribution
- $x_1 = \cdots = x_q = 1/q$: $q$-shuffle (Bayer-Diaconis), the $U_q$-distribution
- $\lim_{q \to \infty} U_q$: the uniform distribution $U$

**Note.** A $q$-shuffle followed by an $r$-shuffle is a $qr$-shuffle, i.e., $U_q * U_r = U_{qr}$.

**Theorem.** Let $y$ have finite support. Then

\[ QS(x) * QS(y) = QS(xy) , \]

where $xy$ denotes the variables $x_i y_j$ in lexicographic order.
The QS-distribution defines a Markov chain (or random walk) on $\mathfrak{S}_n$ by
\[
\text{Prob}(u, uw) = L_{\text{co}(w^{-1})}(x).
\]

**Theorem.** The eigenvalues of $M_n$ are the power sum symmetric functions $p_\lambda(x)$ for $\lambda \vdash n$. The eigenvalue $p_\lambda(x)$ occurs with multiplicity $n!/z_\lambda$, the number of elements in $\mathfrak{S}_n$ of cycle type $\lambda$.

(consequence of Bergeron-Garsia or Bi-digare-Hanlon-Rockmore)
Sample enumerative results. For $w \in \mathfrak{S}_n$ let

$$\text{inv}(w) = \# \{(i, j) : i < j, \ w(i) > w(j)\}$$

$$\text{maj}(w) = \sum_{i: w(i) > w(i+1)} i$$

$$I_n(j) = \text{Prob}(\text{inv}(w) = j)$$

$$M_n(j) = \text{Prob}(\text{maj}(w) = j).$$

Theorem. We have

$$M_n(j) = I_n(j)$$

$$\sum_{n \geq 0} \sum_{j \geq 0} \frac{M_n(j) t^j z^n}{(1 - t)(1 - t^2) \cdots (1 - t^n)}$$

$$= \prod_{i,j \geq 1} \left(1 - t^{i-1} x_j z\right)^{-1}.$$
MacMahon (1913):

$$\# \{ w \in \mathfrak{S}_n : \text{maj}(w) = j \}$$

$$= \# \{ w \in \mathfrak{S}_n : \text{inv}(w) = j \}.$$ 

Since $U = \lim_{q \to \infty} U_q$, the result $M_n(j) = I_n(j)$ is a generalization.
In fact, if
\[ F_\lambda(t) = \sum_v t^{\text{maj}(v)} \]
\[ G_\lambda(t) = \sum_v t^{\text{inv}(v)}, \]
where \( v \) ranges over all permutations of the multiset \( \{1^{\lambda_1}, 2^{\lambda_2}, \ldots\} \), then
\[ \sum_j M_n(j) t^j = \sum_{\lambda \vdash n} F_\lambda(t)m_\lambda(x) \]
\[ \sum_j I_n(j) t^j = \sum_{\lambda \vdash n} G_\lambda(t)m_\lambda(x). \]
Thus \( M_n(j) = I_n(j) \) is equivalent to MacMahon’s result for multisets.
Let
\[ L_n(x) = \frac{1}{n} \sum_{d|n} \mu(d) p^{n/d}_d(x) \]
\[ = \text{ch ind}_{C_n} \mathcal{S}_n e^{2\pi i/n}. \]

**Theorem.** Let \( w \) be a random permutation in \( \mathcal{S}_n \), chosen from the QS-distribution. The probability \( \text{Prob}(\rho(w) = \lambda) \) that \( w \) has cycle type \( \lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle \)
\( \vdash n \) (i.e., \( m_i \) cycles of length \( i \)) is given by
\[ \text{Prob}(\rho(w) = \lambda) = \prod_{i \geq 1} h_{m_i}[L_i], \]
where brackets denote plethysm.
Connections with the RSK algorithm

Let $w \in \mathfrak{S}_n$, and let $w \xrightarrow{\text{RSK}} (P, Q)$ denote the RSK algorithm, so $P$ and $Q$ are SYT of the same shape $\lambda \vdash n$. Write

$$\text{sh}(w) = \lambda.$$

**Theorem.** Choose $w \in \mathfrak{S}_n$ from the QS-distribution, and let $w \xrightarrow{\text{RSK}} (P, Q)$. Let $T$ be a SYT of shape $\lambda \vdash n$. Then

$$\text{Prob}(P = T) = s_\lambda(x),$$

where $s_\lambda(x)$ denotes a Schur function.
Corollary. Choose $w \in \mathfrak{S}_n$ from the QS-distribution, and let $\lambda \vdash n$. Then

$$\text{Prob}(\text{sh}(w) = \lambda) = f^\lambda s_\lambda(x),$$

where $f^\lambda$ denotes the number of SYT of shape $\lambda$ (given explicitly by the Frame-Robinson-Thrall hook-length formula).
Longest increasing subsequences

Let $\text{is}(w)$ be the length of the longest increasing subsequence of $w = w_1 \cdots w_n$.

**Theorem** (Schensted). If

$$\text{sh}(w) = (\lambda_1, \lambda_2, \ldots),$$

then $\lambda_1 = \text{is}(w)$. Hence

$$E_U(\text{is}(w)) = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left( f^{\lambda} \right)^2.$$  

**Theorem** (Vershik-Kerov):

$$E_U(\text{is}(w)) \sim 2\sqrt{n}.$$
For the QS-distribution,
\[
E(is(w)) = \sum_{\lambda \vdash n} \lambda_1 f^\lambda s_\lambda(x).
\]
\[
E_{U_q}(is(w)) = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left( f^\lambda \right)^2 \prod_{u \in \lambda} \left( 1 + q^{-1} c(u) \right)
= E_U(is(w))
+ \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left( f^\lambda \right)^2 \left( \sum_{u \in \lambda} c(u) \right) \frac{1}{q} + \cdots.
\]

Let
\[
F_1(n) = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left( f^\lambda \right)^2 \left( \sum_{u \in \lambda} c(u) \right).
\]

Numerical evidence suggests that \( F_1(n)/n \) is slowly increasing, possibly to a finite limit. We computed \( F_1(47)/47 \approx 0.6991 \).
Logan-Shepp, Vershik-Kerov: “asymptotic shape” of a “typical” $w \in \mathfrak{S}_n$ (uniform distribution) as $n \to \infty$.

Baik-Deift-Johansson: Asymptotic distribution of $\operatorname{sh}(w)$ for $w \in \mathfrak{S}_n$ (uniform distribution) as $n \to \infty$.

**Theorem.** For each $n \in \mathbb{P}$ let $w^{(n)} \in \mathfrak{S}_n$ be chosen from the QS-distribution. Let $\operatorname{sh}(w^{(n)}) = (\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots)$, and let $y_1 \geq y_2 \geq \cdots$ be the decreasing rearrangement of $x_1, x_2, \ldots$. Then almost surely (i.e., with probability 1) for all $i$ there holds

$$
\lim_{n \to \infty} \frac{\lambda_i^{(n)}}{n} = y_i.
$$
Corollary. \( \text{Fix } x = (x_1, x_2, \ldots), \) with \( x_i \geq 0 \) and \( \sum x_i = 1 \) as usual. Let \( \mu^{(n)} \) be a partition \( \nu \vdash n \) that maximizes \( f^\nu s_\nu(x) \). Then
\[
\lim_{n \to \infty} \frac{\mu_i^{(n)}}{n} = y_i.
\]
**Theorem (Its-Tracy-Widom)** Let

\[ x_1 > x_2 > \cdots. \]

Then

\[ E(is(w)) = x_1 n + \sum_{j>1} \frac{p_j}{p_1 - p_j} + O \left( \frac{1}{\sqrt{n}} \right). \]
**Open:** Find an asymptotic formula for the expected value of $\lambda_1$ (where $\text{sh}(w) = \lambda$ under the $QS(x)$-distribution) that specializes to both the Vershik-Kerov result (uniform distribution) and the case $x$ fixed, $n \to \infty$. 