Adjacent transposition:

\[ s_i = (i, i + 1) \in S_n, \ 1 \leq i \leq n - 1 \]
Definitions

Adjacent transposition:

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**reduced decomposition** \((a_1, \ldots, a_p)\) of \(w \in S_n\):

\[ w = s_{a_1} \cdots s_{a_p}, \]

where \(p\) is **minimal**, i.e.,

\[ p = \ell(w) = \# \{(i, j) : i < j, \ w(i) > w(j)\}. \]
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\(p\) is the number of inversions \(\text{inv}(w)\) or length \(\ell(w)\) of \(w\).
An example

\[ 1234 \xrightarrow{S_2} 1324 \xrightarrow{S_3} 1342 \xrightarrow{S_2} 1432 \xrightarrow{S_1} 4132 \]
An example

1234 $\overset{s_2}{\rightarrow}$ 1324 $\overset{s_3}{\rightarrow}$ 1342 $\overset{s_2}{\rightarrow}$ 1432 $\overset{s_1}{\rightarrow}$ 4132

$R(w)$: set of reduced decompositions of $w$
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\( R(w) \): set of reduced decompositions of \( w \)

\[(2, 3, 2, 1) \in R(4132)\]
**Tits’ theorem**

**Theorem.** If \( a = (a_1, a_2, \ldots, a_p) \in R(w) \) then all reduced decompositions of \( w \) can be obtained from \( a \) by applying

\[
    s_is_j = s_js_i, \quad |i - j| \geq 2
\]

\[
    s_is_{i+1}s_i = s_{i+1}s_is_{i+1}.
\]

(We don’t need \( s_i^2 = 1 \).)
Tits’ theorem

**Theorem.** If \( a = (a_1, a_2, \ldots, a_p) \in R(w) \) then all reduced decompositions of \( w \) can be obtained from \( a \) by applying

\[
\begin{align*}
    s_is_j &= s_js_i, \quad |i - j| \geq 2 \\
    s_is_{i+1}s_i &= s_{i+1}s_is_{i+1}.
\end{align*}
\]

(We don’t need \( s_i^2 = 1 \).)

E.g., \( (2, 3, 2, 1) \in R(4132) \Rightarrow \)

\[
R(4132) = \{(2, 3, 2, 1), (3, 2, 3, 1), (3, 2, 1, 3)\}.
\]
\[ r(w) = \#R(w), \text{ the number of reduced decompositions of } w \]

Main question (this lecture): what is \( r(w) \)?
\( r(w) = \#R(w) \), the number of reduced decompositions of \( w \)

**Main question** (this lecture): what is \( r(w) \)?

\[
R(4132) = \{(2, 3, 2, 1), (3, 2, 3, 1), (3, 2, 1, 3)\}
\]

\[
\Rightarrow r(4132) = 3.
\]
Let $w \in S_n$ and $p = \ell(w)$. Define

$$G_w = \sum_{(a_1,\ldots,a_p) \in R(w)} \sum_{1 \leq i_1 \leq \cdots \leq i_p} x_{i_1} \cdots x_{i_p},$$

a power series in $x_1, x_2, \ldots$, homogeneous of degree $p$. 
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a power series in $x_1, x_2, \ldots$, homogeneous of degree $p$.

**Example.** $w = 321 \in S_3$, so $R(w) = \{121, 212\}$.

$$G_{321} = \sum_{1 \leq i < j \leq k} x_i x_j x_k + \sum_{1 \leq i \leq j < k} x_i x_j x_k.$$
Theorem (Billey-Jockusch-S, Fomin-S, Jia-Miller, ...). $G_w$ is a symmetric function of $x_1, x_2, \ldots$. 
Symmetry of $G_w$

**Theorem** (Billey-Jockusch-S, Fomin-S, Jia-Miller, ...). $G_w$ is a symmetric function of $x_1, x_2, \ldots$.

$$G_{321} = \sum_{1 \leq i < j < k} x_i x_j x_k + \sum_{1 \leq i \leq j < k} x_i x_j x_k.$$
Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of $p$ (denoted $\lambda \vdash p$), i.e.,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad \sum \lambda_i = p.$$
Schur functions

Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition of \( p \) (denoted \( \lambda \vdash p \)), i.e.,

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad \sum \lambda_i = p.
\]

\( s_\lambda \): the **Schur function** indexed by \( \lambda \).

Fact: the Schur functions \( s_\lambda \) for \( \lambda \vdash p \) form a \( \mathbb{Z} \)-basis for all symmetric functions in \( x_1, x_2, \ldots \) over \( \mathbb{Z} \) that are homogeneous of degree \( p \).
The case $p = 3$

Example. $s_3 = \sum_{i \leq j \leq k} x_i x_j x_k$

$s_{21} = \sum_{1 \leq i < j \leq k} x_i x_j x_k + \sum_{1 \leq i \leq j < k} x_i x_j x_k$

$s_{111} = \sum_{i < j < k} x_i x_j x_k$
The case \( p = 3 \)

**Example.** \( s_3 = \sum_{i\leq j \leq k} x_i x_j x_k \)

\[
\begin{align*}
  s_{21} &= \sum_{1 \leq i < j \leq k} x_i x_j x_k + \sum_{1 \leq i \leq j < k} x_i x_j x_k \\
  s_{111} &= \sum_{i < j < k} x_i x_j x_k
\end{align*}
\]

Thus every \( G_w \) can be uniquely written

\[
G_w = \sum_{\lambda \vdash p} \alpha_w \lambda s_{\lambda}.
\]
Recall that

\[ G_{321} = \sum_{1 \leq i < j < k} x_i x_j x_k + \sum_{1 \leq i \leq j < k} x_i x_j x_k \]

\[ s_{21} = \sum_{1 \leq i < j \leq k} x_i x_j x_k + \sum_{1 \leq i \leq j < k} x_i x_j x_k, \]

so

\[ G_{321} = s_{21}. \]
Recall:

\[ G_w = \sum_{(a_1, \ldots, a_p) \in R(w)} \sum_{1 \leq i_1 \leq \cdots \leq i_p} x_{i_1} \cdots x_{i_p}. \]

Note. The monomial \( x_1 \cdots x_p \) occurs once in the inner sum for each \((a_1, \ldots, a_p) \in R(w)\).
Coefficient of $x_1 \cdots x_p$ 

Recall:

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Note. The monomial $x_1 \cdots x_p$ occurs once in the inner sum for each $(a_1, \ldots, a_p) \in R(w)$. 
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Note. The monomial $x_1 \cdots x_p$ occurs once in the inner sum for each $(a_1, \ldots, a_p) \in R(w)$.

$[x_1 \cdots x_p]F$: coefficient of $x_1 \cdots x_p$ in $F$

$$\Rightarrow r(w) = [x_1 \cdots x_p]G_w,$$
A “formula” for $r(w)$

Recall:

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A “formula” for $r(w)$

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What is $[x_1 \cdots x_p] s_{\lambda}$?
Standard Young tableaux

A standard Young tableau (SYT) of shape 4421:

<table>
<thead>
<tr>
<th>1</th>
<th>3</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
standard Young tableau (SYT) of shape 4421:

\[
\begin{array}{cccc}
1 & 3 & 5 & 6 \\
2 & 7 & 9 & 11 \\
4 & 10 & \\
8 & \\
\end{array}
\]

\[f^\lambda: \text{number of SYT of shape } \lambda\]

E.g., \(f^{32} = 5\):

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 1 & 2 & 4 & 1 & 2 & 5 \\
1 & 3 & 4 & 1 & 3 & 5 & 4 & 5 & 3 \\
2 & 5 & 2 & 4 & 2 & 4 & & & \\
& & & & & & & & \\
4 & 5 & & 3 & 5 & & 3 & 4 & 2 & 5 & 2 & 4
\end{array}
\]
What is $[x_1 \cdots x_p] s_\lambda$?

Facts:

- ∃ simple formula for $f^\lambda$ (hook length formula)
- If $\lambda \vdash p$ then $[x_1 \cdots x_p] s_\lambda = f^\lambda$. 
What is $[x_1 \cdots x_p]s_\lambda$?

Facts:

- ∃ simple formula for $f^\lambda$ (hook length formula)
- If $\lambda \vdash p$ then $[x_1 \cdots x_p]s_\lambda = f^\lambda$.

Recall:

$$r(w) = \sum_{\lambda \vdash p} \alpha_{w\lambda}[x_1 \cdots x_p]s_\lambda$$

Thus

$$r(w) = \sum_{\lambda \vdash p} \alpha_{w\lambda}f^\lambda.$$
Vexillary permutations

Nicest situation: $G_w = s_\lambda$ for some $\lambda \vdash p$. Then $r(w) = f^\lambda$. 

Vexillary permutations

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**Definition.** A permutation $w = a_1a_2\cdots a_n \in S_n$ is vexillary or 2143-avoiding if

$$a < b < c < d, \quad w_b < w_a < w_d < w_c.$$
Vexillatory permutations

Nicest situation: $G_w = s_\lambda$ for some $\lambda \vdash p$. Then $r(w) = f^\lambda$.

**Definition.** A permutation $w = a_1a_2 \cdots a_n \in S_n$ is **vexillatory** or **2143-avoiding** if $\not\exists$

$$a < b < c < d, \quad w_b < w_a < w_d < w_c.$$

Named by Lascoux and Schützenberger from *vexillum*, Latin for “flag,” because of a connection with flag Schur functions.
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957281463 not vexillary
Vexillary permutations

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957281463 not vexillary
Vexillary asymptotics

\( v(n) = \) number of vexillary \( w \in S_n \)

**Theorem** (A. Regev, J. West).

\[
v(n) \sim \frac{81}{16} \sqrt{3\pi} \frac{9^n}{n^4}
\]

\[
= 2.791102533 \cdots \frac{9^n}{n^4}.
\]
$\lambda(w)$

\[ w = a_1 \cdots a_n \in S_n \]

\[ c_i = \# \{ j : i < j \leq n, \ a_i > a_j \}, \quad 1 \leq i \leq n - 1 \]

$\lambda(w)$: partition whose parts are the $c_i$’s (sorted into decreasing order).
\[ w = a_1 \cdots a_n \in S_n \]
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**Example.** \( w = 5361472 \in S_7, \)
\[ (c_1, \ldots, c_6) = (4, 2, 3, 0, 1, 1, 0) \]
\[ \Rightarrow \lambda(w) = 43211 \]
\( w = a_1 \cdots a_n \in S_n \)

\[ c_i = \# \{ j : i < j \leq n, \, a_i > a_j \}, \quad 1 \leq i \leq n - 1 \]

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**Example.** \( w = 5361472 \in S_7, \)

\[
(c_1, \ldots, c_6) = (4, 2, 3, 0, 1, 1, 0)
\]

\[
\Rightarrow \lambda(w) = 43211
\]

Clearly \( \lambda(w) \vdash p = \ell(w) \).
Theorem. We have $G_w = s_\lambda$ for some $\lambda$ if and only if $w$ is vexillary. In this case $\lambda = \lambda(w)$, so $r(w) = f^{\lambda(w)}$. 

Example. $w = 5361472$ is vexillary, and $(w) = 43211$. Hence $G_w = s_{43211}$; $r(w) = f_{43211} = 2310$.
Theorem. We have $G_w = s_\lambda$ for some $\lambda$ if and only if $w$ is vexillary. In this case $\lambda = \lambda(w)$, so $r(w) = f^{\lambda(w)}$.

Example. $w = 5361472 \in S_7$ is vexillary, and $\lambda(w) = 43211$. Hence

$$G_w = s_{43211}, \quad r(w) = f^{43211} = 2310.$$
Example. \( w_0 = n, n - 1, \ldots, 1 \in S_n \) is vexillary, and \( \lambda(w_0) = (n - 1, n - 2, \ldots, 1) \). Hence

\[
r(w_0) = f^{(n-1,n-2,\ldots,1)} = \frac{\binom{n}{2}!}{1^{n-1} 3^{n-2} 5^{n-3} \ldots (2n - 1)^{1}}.
\]
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\]

\[
\begin{array}{c|c|c|c|c|c}
  n & 3 & 4 & 5 & 6 & 7 \\
  \hline
  r(w_0) & 2 & 16 & 768 & 292864 & 1100742656 \\
\end{array}
\]
Recall:

\[ G_w = \sum_{\lambda \vdash p} \alpha_{w\lambda} s_{\lambda} \]

\[ \Rightarrow r(w) = \sum_{\lambda \vdash p} \alpha_{w\lambda} f^{\lambda} \]

What can we say about \( \alpha_{w\lambda} \)?
A semistandard (Young) tableau (SSYT) $T$ of shape $\lambda = (4, 3, 3, 1, 1)$:

\[
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 3 & 3 & \\
4 & 4 & 6 & \\
5 & \\
7 & \\
\end{array}
\]
A semistandard (Young) tableau (SSYT) $T$ of shape $\lambda = (4, 3, 3, 1, 1)$:

\[
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 3 & 3 & \text{\textbackslash} \\
4 & 4 & 6 & \text{\textbackslash} \\
5 & \text{\textbackslash} & \text{\textbackslash} & \text{\textbackslash} \\
7 & \text{\textbackslash} & \text{\textbackslash} & \text{\textbackslash}
\end{array}
\]

Reading word of $T$: 421133264457
Theorem (S. Fomin and C. Greene). Let $w \in S_n$, $\ell(w) = p$, and $\lambda \vdash p$. The coefficient $\alpha_{w\lambda}$ is equal to the number of SSYT of shape $\lambda$ whose row reading word is a reduced decomposition of $w$. 
Example of Fomin-Greene theorem

Example. \( w = 4152736 \in S_7 \)

\[
\begin{array}{cccc}
1 & 2 & 3 & 1 2 3 6 \\
3 & 4 & 3 & 4 6 3 4 \\
5 & 6 & 5 & 5 \\
\end{array}
\]

\[ 3214365, 3216435, 6321435 \in R(w) \]
Example. \( w = 4152736 \in S_7 \)

\[
\begin{array}{cccccc}
1 & 2 & 3 & 1 & 2 & 3 \\
3 & 4 & 3 & 4 & 6 & 3 \\
5 & 6 & 5 & 5 & \\
\end{array}
\]

\[
3214365, 3216435, 6321435 \in R(w)
\]

\[ r(w) = f^{322} + f^{331} + f^{421} = 21 + 21 + 35 = 77. \]
Recall: \( w_0 = n, n - 1, \ldots, 1 \in S_n, \)

\[
r(w_0) = f^{n-1,n-2,\ldots,1}.
\]

Is there a bijective proof?
Edelman-Greene bijection

Reduced Decompositions – p. 25
The inverse to the previous bijection is given by a version of RSK algorithm (discussed in first lecture).
Irreducible representations \( \varphi^\lambda : S_n \rightarrow \text{GL}(N, \mathbb{C}) \) are indexed by partitions \( \lambda \vdash n \).

\[
N = \dim \varphi^\lambda = f^\lambda
\]
Irreducible representations $\varphi^\lambda : S_n \rightarrow GL(N, \mathbb{C})$ are indexed by partitions $\lambda \vdash n$. 

$$N = \dim \varphi^\lambda = f^\lambda$$

**Specht module** $M_\lambda$: an $S_n$-module constructed from the (Young) diagram of $\lambda$ (using row-symmetrizers and column anti-symmetrizers) that affords the representation $\varphi^\lambda$. 
For any diagram $D$ (finite subset of a square grid) we can carry out the Specht module construction, obtaining an $\mathcal{S}_n$-module $M_D$ (in general reducible).
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In general, $M_D$ is not well-understood.
Example. $w = 361524$; diagram $D_w$:
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Example. $\omega = 361524$; diagram $D_\omega$: 

![Diagram of $D_\omega$ for the permutation $\omega = 361524$.]
Example. \( w = 361524 \); diagram \( D_w \):
Example. $w = 361524$; diagram $D_w$: 

![Diagram $D_w$ of a permutation $w$](image-url)
Example. \( \omega = 361524 \); diagram \( D_\omega \):
Example. \( w = 361524 \); diagram \( D_w \):

Number of squares of \( D_w = \ell(w) \).
Theorem (Kraśkiewicz-Pragacz, 1986, 2004). Let $w = S_n$, $p = \ell(w)$. For $\lambda \vdash p$, the multiplicity of $\varphi^\lambda$ in $M_{D_w}$ is $\alpha_{w\lambda}$. 
Theorem (Kraśkiewicz-Pragacz, 1986, 2004). Let $w = S_n$, $p = \ell(w)$. For $\lambda \vdash p$, the multiplicity of $\varphi^\lambda$ in $M_{Dw}$ is $\alpha_{w\lambda}$.

Since $r(w) = \sum_{\lambda \vdash p} \alpha_{w\lambda} f^\lambda$ and $\dim \varphi^\lambda = f^\lambda$, we get:

**Corollary.** $\dim M_{Dw} = r(w)$
Flag varieties

$\text{Fl}_n$: set of complete flags

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$$

of subspaces in $\mathbb{C}^n$ (so $\dim V_i = i$)
Flag varieties

\( \text{Fl}_n : \) set of **complete flags**

\[
0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n
\]

of subspaces in \( \mathbb{C}^n \) (so \( \dim V_i = i \))

\[
\text{Fl}_n \cong \text{GL}(n, \mathbb{C})/B,
\]

where \( B \) is the Borel subgroup of invertible upper triangular matrices.
Cohomology of $\text{Fl}_n$

For each $w \in S_n$ there is a projective subvariety $\Omega_w$ of (complex) dimension $\ell(w)$, the **Schubert variety** corresponding to $w$, defined by simple geometric conditions.
For each \( w \in \mathcal{S}_n \) there is a projective subvariety \( \Omega_w \) of (complex) dimension \( \ell(w) \), the **Schubert variety** corresponding to \( w \), defined by simple geometric conditions.

\[ \sigma_w \textbf{ (Schubert cycle): cohomology class Poincaré dual to the fundamental cycle of } \Omega_w \]
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\[ \sigma_w \text{ (Schubert cycle): cohomology class} \]

Poincaré dual to the fundamental cycle of \( \Omega_w \)

\[
\Rightarrow \sigma_w \in H^2\left(\binom{n}{2} - \ell(w)\right) (\text{Fl}_n; \mathbb{C})
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For each \( w \in S_n \) there is a projective subvariety \( \Omega_w \) of (complex) dimension \( \ell(w) \), the **Schubert variety** corresponding to \( w \), defined by simple geometric conditions.

\( \sigma_w \) (**Schubert cycle**): cohomology class Poincaré dual to the fundamental cycle of \( \Omega_w \)

\[
\Rightarrow \sigma_w \in H^2\left(\binom{n}{2} - \ell(w)\right) (\text{Fl}_n; \mathbb{C})
\]

**Standard result from Schubert calculus**: the Schubert cycles \( \sigma_w, w \in S_n \), form a basis of \( H^*(\text{Fl}_n; \mathbb{C}) \)
Schubert polynomial $\mathcal{S}_w = \mathcal{S}_w(x_1, \ldots, x_{n-1})$, $w \in S_n$:

$$\mathcal{S}_w = \sum_{(a_1, \ldots, a_p) \in R(w)} \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_p \\ i_j < i_{j+1} \text{ if } a_j < a_{j+1} \\ i_j \leq j}} x_{i_1} \cdots x_{i_p}.$$
Schubert polynomials

Schubert polynomial $S_w = S_w(x_1, \ldots, x_{n-1})$, $w \in S_n$:

$$S_w = \sum_{(a_1, \ldots, a_p) \in R(w)} \sum_{\begin{array}{c} 1 \leq i_1 \leq \ldots \leq i_p \\ i_j < i_{j+1} \text{ if } a_j < a_{j+1} \end{array}} x_{i_1} \cdots x_{i_p}.$$ 

Compare

$$G_w = \sum_{(a_1, \ldots, a_p) \in R(w)} \sum_{\begin{array}{c} 1 \leq i_1 \leq \ldots \leq i_p \\ i_j < i_{j+1} \text{ if } a_j < a_{j+1} \end{array}} x_{i_1} \cdots x_{i_p}.$$
\( G_w \) is sometimes called a \textbf{stable Schubert polynomial} (a certain limit of Schubert polynomials).
$G_w$ is sometimes called a **stable Schubert polynomial** (a certain limit of Schubert polynomials).

**Example.** $G_{4213}, G_{15324}, G_{126435}, \ldots \rightarrow G_{4213}$
Ring structure of $H^*(\text{Fl}_n; \mathbb{C})$

$$R_n = \mathbb{C}[x_1, x_2, \ldots, x_n]/I_n,$$
where $I_n$ is generated by the elementary symmetric functions $e_1, \ldots, e_n$. 

Theorem. There is an algebra isomorphism $'$ such that for $w \in S_n$ we have $' (S_w 0 w) = w 1 1 \ldots 1$.
Ring structure of $H^*(\mathrm{Fl}_n; \mathbb{C})$

$R_n = \mathbb{C}[x_1, x_2, \ldots, x_n]/I_n$, where $I_n$ is generated by the elementary symmetric functions $e_1, \ldots, e_n$.

**Theorem.** There is an algebra isomorphism

$$\varphi: R_n \rightarrow H^*(\mathrm{Fl}_n; \mathbb{C}),$$

such that for $w \in S_n$ we have

$$\varphi(\sigma_{w_0}w) = \sigma_w,$$

where $w_0 = n, n - 1, \ldots, 1$. 

Let $w \in S_n$ and $\ell(w) = p$. Then

$$
\sum_{(a_1,a_2,\ldots,a_p) \in R(w)} a_1 a_2 \cdots a_p = p! \mathcal{S}_w(1, 1, \ldots, 1).
$$
A curious identity


Let \( w \in S_n \) and \( \ell(w) = p \). Then

\[
\sum_{(a_1, a_2, \ldots, a_p) \in R(w)} a_1 a_2 \cdots a_p = p! \mathcal{G}_w(1, 1, \ldots, 1).
\]

**Theorem.** \( \mathcal{G}_w(1, 1, \ldots, 1) = 1 \) if and only if \( w \) is **132-avoiding**, i.e., there does not exist \( i < j < k \) such that \( a_i < a_k < a_j \).
A curious identity

Let $w \in S_n$ and $\ell(w) = p$. Then

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Theorem. $\mathcal{S}_w(1,1,\ldots,1) = 1$ if and only if $w$ is 132-avoiding, i.e., there does not exist $i < j < k$ such that $a_i < a_k < a_j$.

Number of 132-avoiding $w \in S_n$: the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.
The special case $w_0$

For $w_0 = n, n - 1, \ldots, 1 \in S_n$ we have

$$\sum_{(a_1, a_2, \ldots, a_p) \in R(w_0)} a_1 a_2 \cdots a_p = \binom{n}{2}!.$$
The special case $w_0$

For $w_0 = n, n - 1, \ldots, 1 \in S_n$ we have

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**Example.** For $n = 3$ we have

$$R(w_0) = \{ (1, 2, 1), (2, 1, 2) \}.$$ 

Thus

$$1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2 = 6 = \binom{3}{2}!.$$
Is that a good thing or a bad thing?
An analogue for any transpositions

\((i, j) \in S_n: \text{transposition interchanging } i \text{ and } j\)

For \(w \in S_n, \ell(w) = p\), define

\[
T(w) = \{(((i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)) : \\
w = (i_1, j_1)(i_2, j_2) \cdots (i_p, j_p) \text{ and } \ell(((i_1, j_1) \cdots (i_k, j_k)) = k \text{ for all } 1 \leq k \leq p\}.
\]
An example

Let \( w = w_0 = 321 \in S_3 \).

\[
321 = (1,2)(2,3)(1,2) = (2,3)(1,2)(2,3) = (1,2)(1,3)(2,3) = (2,3)(1,3)(1,2),
\]

so (abbreviating \((i, j)\) as \(ij\))

\[
T_{321} = \{(12, 23, 12), (23, 12, 23), (12, 13, 23), (23, 13, 12)\}.
\]
Theorem (Chevalley ~1958, Stembridge 2002). For $w = w_0 \in S_n$ (so $p = \binom{n}{2}$) we have

$$\sum_{((i_1,j_1),(i_2,j_2),\ldots,(i_p,j_p)) \in T(w_0)} (j_1-i_1)(j_2-i_2) \cdots (j_p-i_p) = p!.$$
Example. Recall

\[ T(321) = \{ (12, 23, 12), (23, 12, 23), (12, 13, 23), (23, 13, 12) \}. \]

Hence

\[ 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 + 1 \cdot 2 \cdot 1 = \binom{3}{2}!. \]
An open problem

\[ \sum_{(a_1, a_2, \ldots, a_p) \in R(w_0)} a_1 a_2 \cdots a_p = p!. \]

\[ \sum_{((i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)) \in T(w_0)} (j_1 - i_1)(j_2 - i_2) \cdots (j_p - i_p) = p!. \]
An open problem

\[ \sum_{(a_1,a_2,\ldots,a_p) \in R(w_0)} a_1 a_2 \cdots a_p = p!. \]

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- Is this similarity just a “coincidence”?
- Is there a common generalization?