Let $\boldsymbol{\lambda}, \boldsymbol{\nu} \vdash n$. Let $\boldsymbol{\chi^\lambda(\nu)}$ denote the irreducible character χ^λ of $\mathfrak{S}_n$ evaluated at a permutation $w \in \mathfrak{S}_n$ of cycle type ν.

If $\mu \vdash k \leq n$ let

$$\boldsymbol{(\mu, 1^{n-k})} = (\mu, \underbrace{1, \ldots, 1}_{n-k\ 1'\text{s}}) \vdash n.$$

Normalized character:

$$\widehat{\chi}^\lambda(\mu, 1^{n-k}) = \frac{(n)_k\, \chi^\lambda(\mu, 1^{n-k})}{\chi^\lambda(1^n)},$$

where

$$\boldsymbol{\chi^\lambda(1^n)} = \dim \chi^\lambda = f^\lambda$$

$$\boldsymbol{(n)_k} = n(n-1)\cdots(n-k+1).$$

Let $\boldsymbol{p} \times \boldsymbol{q} = (\underbrace{q, \ldots, q}_{p\ q'\mathrm{s}})$, and let $\boldsymbol{\kappa(w)}$ denote the number of cycles of $w \in \mathfrak{S}_k$.

Theorem. *Let $\mu \vdash k$ and fix a permutation $w_\mu \in \mathfrak{S}_k$ of cycle type μ. Then*

$$\widehat{\chi}^{p\times q}(\mu, 1^{pq-k}) = (-1)^k \sum_{uv=w_\mu} p^{\kappa(u)}(-q)^{\kappa(v)},$$

where the sum ranges over all $k!$ pairs $(u,v) \in \mathfrak{S}_k \times \mathfrak{S}_k$ satisfying $uv = w_\mu$.

Proof based on Murnaghan-Nakayama rule:

$$\chi^{p\times q}(\mu, 1^{pq-k}) = \sum_T (-1)^{\mathrm{ht}(T)},$$

and on

$$\sum_{\lambda\vdash k} H_\lambda\, s_\lambda(x) s_\lambda(y) s_\lambda(z)$$

$$= \frac{1}{k!} \sum_{\substack{uvw=1 \\ \text{in } \mathfrak{S}_k}} p_{\rho(u)}(x) p_{\rho(v)}(y) p_{\rho(w)}(z).$$

Example.

$$\mu = (1): \quad pq$$

$$\mu = (2): \quad -p^2q + pq^2$$

$$\mu = (1,1): \quad pq(pq-1)$$

$$\mu = (3): \quad p^3q - 3p^2q^2 + pq^3 + pq$$

$$\mu = (2,1): \quad (-p^2q + pq^2)(pq-2)$$

$$\mu = (1,1,1): \quad pq(pq-1)(pq-2)$$

$$\mu = (4): \quad -p^4q + 6p^3q^2 - 6p^2q^3 + pq^4 - 5p^2q + 5pq^2$$

$$\mu = (2,2): \quad p^4q^2 - 2p^3q^3 + p^2q^4 - 4p^3q + 10p^2q^2 - 4pq^3 - 2pq$$

Kerov's character polynomial. Let **Par** denote the set of all partitions of all $n \geq 0$. There exist functions

$$\boldsymbol{R_i} : \mathrm{Par} \to \mathbb{Z}$$

and polynomials

$$\Sigma_k(R_2, \ldots, R_{k+1}), \quad k \geq 1$$

such that for all partitions $\lambda \vdash n \geq k$,

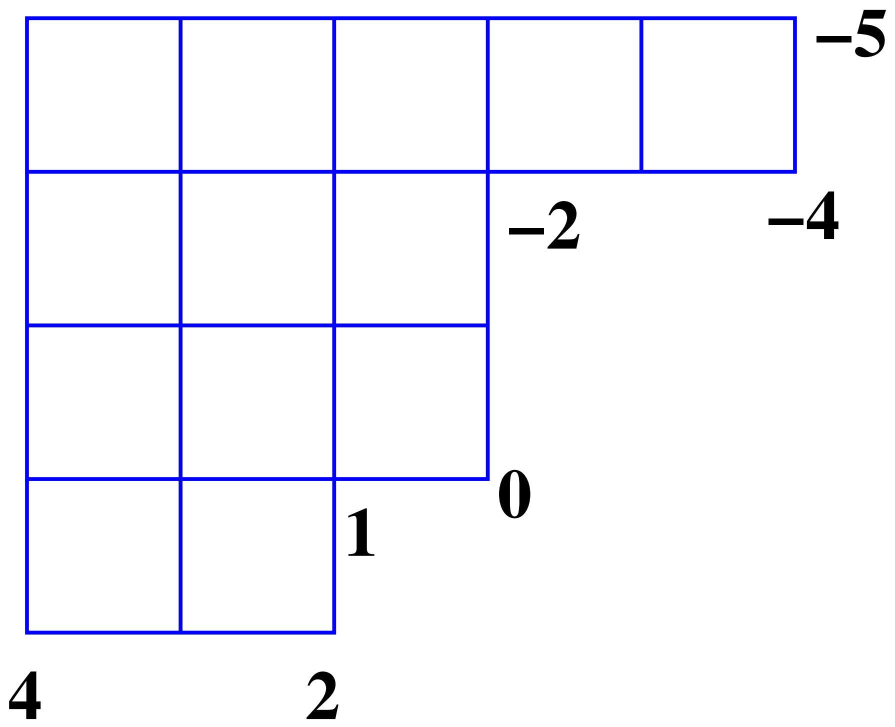
$$\hat{\chi}^\lambda(k, 1^{n-k}) = \Sigma_k(R_2(\lambda), \ldots, R_{k+1}(\lambda)).$$

E.g.,

$$\begin{aligned}
\Sigma_1 &= R_2 \\
\Sigma_2 &= R_3 \\
\Sigma_3 &= R_4 + R_2 \\
\Sigma_4 &= R_5 + 5R_3 \\
\Sigma_5 &= R_6 + 5R_2^2 + 15R_4 + 8R_2 \\
\Sigma_6 &= R_7 + 35R_2R_3 + 35R_5 + 84R_3.
\end{aligned}$$

What is $R_i : \mathrm{Par} \to \mathbb{Z}$? Illustrated for

$$\lambda = (5, 3, 3, 2).$$



$$F(x) = \frac{(1-2x)(1-0x)(1+4x)}{(1-4x)(1-x)(1+2x)(1+5x)}$$
$$= 1 + 13x^2 - 4x^3 + 241x^4 - 280x^5 + \cdots.$$

$$\begin{aligned}\sum_{i\geq 0} \boldsymbol{R_i} x^i &= \frac{x}{(xF(x))^{\langle -1\rangle}} \\ &= \frac{x}{(x+13x^3-4x^4+241x^5+\cdots)^{\langle -1\rangle}} \\ &= \frac{x}{x-13x^3+4x^4+266x^5+\cdots} \\ &= 1+13x^2-4x^3-97x^4+\cdots.\end{aligned}$$

Since e.g. $\Sigma_3 = R_4 + R_2$, we get

$$\hat{\chi}^{(5,3,3,2)}(3,1^{10}) = -97+13 = -84,$$

so

$$\begin{aligned}\chi^{(5,3,3,2)}(3,1^{10}) &= \frac{-84 f^{(5,3,3,2)}}{(13)_3} \\ &= -567.\end{aligned}$$

Properties of Σ_k.

Example:

$$\Sigma_6 = R_7 + 35R_2R_3 + 35R_5 + 84R_3$$

- Let $\deg R_i = i$. Then every term of Σ_k has degree $\equiv k+1 \pmod 2$ (easy parity argument).
- $\Sigma_k = R_{k+1} +$ terms of lower order (follows from known character asymptotics).
- **Conjecture.** All coefficients of Σ_k are nonnegative.
- **Conjecture** (Biane). Let $2j_2 + 3j_3 + \cdots = k-1$. The coefficient of $R_2^{j_2}R_3^{j_3}\cdots$ in Σ_k is equal to

$$\frac{1}{4}\binom{k+1}{3}\binom{j_2+j_2+\cdots}{j_2,\, j_3,\, \ldots}\prod_{i\geq 2}(i-1)^{j_i}.$$

Note (added 8/11/03): Biane's conjecture was proved by Piotr Sniady, math.CO/0304275.

Theorem (Biane, RS). *The coefficient of R_j in Σ_k is equal to the number of k-cycles $w \in \mathfrak{S}_k$ such that $(1,2,\ldots,k)w$ has $j-1$ cycles.*

Proof. If $\lambda = p \times q$, then one computes directly that

$$R_j(p \times q) = \sum_{i=1}^{j} (-1)^{i-1} N(j-1,i) p^{j-i} q^i,$$

where

$$\boldsymbol{N(j-1,i)} = \frac{1}{j-1}\binom{j-1}{i-1}\binom{j-1}{i},$$

a **Narayana number**.

Now compare with

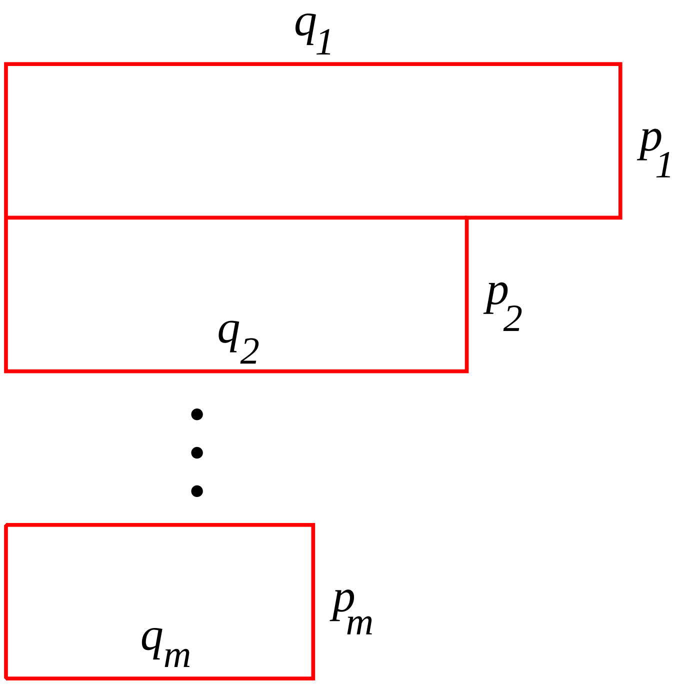
$$\widehat{\chi}^{p\times q}(\mu, 1^{pq-k}) = (-1)^k \sum_{uv=w_\mu} p^{\kappa(u)}(-q)^{\kappa(v)}$$

and

$$\hat{\chi}^{\lambda}(k, 1^{n-k}) = \Sigma_k(R_2(\lambda), \ldots, R_{k+1}(\lambda)). \ \square$$

Generalizations of rectangular shape.

Define the shape $\boldsymbol{\sigma}$ by



Theorem (Katriel & RS). *Fix* $\mu \vdash k$. *Set* $n = |\sigma|$ *and*

$$\boldsymbol{F_\mu(p_1, \ldots, p_m; q_1, \ldots, q_m)} = \widehat{\chi}^\sigma(\mu, 1^{n-k}).$$

Then $F_\mu(p_1, \ldots, p_m; q_1, \ldots, q_m)$ *is a polynomial function of the* p_i*'s and* q_i*'s with integer coefficients, satisfying*

$$(-1)^k F_\mu(1, \ldots, 1; -1, \ldots, -1) = (k+m-1)_k.$$

Proof based on:

Lemma (Frobenius). *Let*

$$\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n, \text{ and}$$

$$\mu = (\mu_1, \ldots, \mu_r) = (\lambda_1 + r - 1, \lambda_2 + r - 2, \ldots, \lambda_r).$$

Define $\varphi(x) = \prod_{i=1}^{r}(x - \mu_i)$. *Then*

$$\widehat{\chi}^{\lambda}(k, 1^{n-k}) = -\frac{1}{k}[x^{-1}]_{\infty} \frac{(x)_k\, \varphi(x-k)}{\varphi(x)},$$

where $[x^{-1}]_{\infty} f(x)$ *denotes the coefficient of* x^{-1} *in the expansion of* $f(x)$ *in* descending *powers of* x *(i.e., as a Taylor series at* $x = \infty$*).*

Example ($m = 2$, $p_1 = p$, $q_1 = q$, $p_2 = a$, $q_2 = b$):

$$-F_1(a, p; -b, -q) = ab + pq$$

$$F_2(a, p; -b, -q) = a^2 b + ab^2 + 2apq + p^2 q + pq^2$$

$$-F_3(a, p; -b, -q) = a^3 b + 3a^2 b^2 + 3a^2 pq + ab^3 + 3abpq + 3ap^2 q + 3apq^2 + p^3 q + 3p^2 q^2 + pq^3 + ab + pq.$$

Conjecture. *The coefficients of the polynomial $F_\mu(p_1, \ldots, p_m; q_1, \ldots, q_m)$ are* ***non-negative***.

Let $\boldsymbol{G_k(p_1,\ldots,p_m;q_1,\ldots,q_m)}$ denote the leading terms of $F_k(p_1,\ldots,p_m;q_1,\ldots,q_m)$, i.e., the terms of degree m.

Theorem (Biane, RS). *We have*

$$\frac{1}{x}+\sum_{k\geq 0} G_k(p_1,\ldots,p_m;q_1,\ldots,q_m)x^k =$$

$$\frac{1}{\left(\frac{x\prod_{i=1}^{m}(1-(q_i+p_{i+1}+p_{i+2}+\cdots+p_m)x)}{\prod_{i=1}^{m}(1-(q_i+p_i+p_{i+1}+\cdots+p_m)x)}\right)^{\langle -1\rangle}},$$

where $^{\langle -1\rangle}$ *denotes compositional inverse with respect to* x. *In particular, the generating function* $\sum G_k x^k$ *is algebraic over* $\mathbb{Q}(p_1,\ldots,p_m,q_1,\ldots,q_m,x)$.

Moreover, if

$$S_k := (-1)^k G_k(1, \ldots, 1; -1, \ldots, -1),$$

then

$$-\frac{1}{x} + \sum_{k \geq 0} S_k x^k = \frac{-1}{\left(\frac{x(1-x)}{1-(m-1)x}\right)^{\langle -1 \rangle}},$$

an algebraic function of degree two.

E.g., $m = 1 \Rightarrow S_k = C_k$ (Catalan number) and

$$(-1)^k G_k(p; -q) = \sum_{i=1}^{k} N(k,i) p^{k+1-i} q^i.$$

If $m = 2$ then $S_k = r_k$ (big Schröder number).

Theorem (Elizalde). *The coefficients of S_k are nonnegative and are given by an explicit expression of the form*

$$\frac{1}{k}\clubsuit\clubsuit\prod_{i+2}^{m}\left(\sum\clubsuit\clubsuit\clubsuit\right),$$

where $\clubsuit$ denotes a binomial coefficient.

Computation of Σ_k. Let

$$F(x) = \frac{x}{1 + \sum_{i \geq 1} R_i x^i}$$

$$G(x) = \frac{1}{F^{\langle -1 \rangle}(1/x)}.$$

Then

$$\Sigma_k(R_2, \ldots, R_{k+1}) =$$

$$-\frac{1}{k+1}[1/x]_{\infty} G(x)G(x-1)\cdots G(x-k)$$

(deformation of Lagrange inversion).