Catalan Numbers

Richard P. Stanley

March 25, 2020

A000108: 1, 1, 2, 5, 14, 42, 132, 429, . . .

$C_0 = 1, \quad C_1 = 2, \quad C_2 = 3, \quad C_3 = 5, \quad C_4 = 14, \ldots$

$C_n$ is a Catalan number.

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\( C_n \) is a **Catalan number**.

Comments. . . . This is probably the longest entry in OEIS, and rightly so.
Catalan monograph

Catalan monograph


Includes 214 combinatorial interpretations of $C_n$ and 68 additional problems.
History

Sharabiin Myangat, also known as Minggatu, Ming’antu (明安图), and Jing An (c. 1692–c. 1763): a Mongolian astronomer, mathematician, and topographic scientist who worked at the Qing court in China.
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Typical result (1730’s):

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\sin(2\alpha) = 2 \sin \alpha - \sum_{n=1}^{\infty} \frac{C_{n-1}}{4^{n-1}} \sin^{2n+1} \alpha
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First example of an infinite trigonometric series.

No combinatorics, no further work in China.
Ming’antu
Manuscript of Ming’antu
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Euler (1751): conjectured formula for the number of triangulations of a convex \((n + 2)\)-gon. In other words, draw \(n - 1\) noncrossing diagonals of a convex polygon with \(n + 2\) sides.
More history, via Igor Pak

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1, 2, 5, 14, ... 

We define these numbers to be the Catalan numbers \(C_n\).
Completion of proof

- **Goldbach and Segner** (1758–1759): helped Euler complete the proof, in pieces.
- **Lamé** (1838): first self-contained, complete proof.
Eugène Charles Catalan (1838): wrote $C_n$ in the form
\[
\frac{(2n)!}{n!(n+1)!}
\]
and showed it counted (nonassociative) bracketings (or parenthesizations) of a string of $n+1$ letters.
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Born in 1814 in Bruges (now in Belgium, then under Dutch rule). Studied in France and worked in France and Liège, Belgium. Died in Liège in 1894.
Why “Catalan numbers”? 

- **John Riordan** (1948): introduced the term “Catalan number” in *Math Reviews.*
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- **Martin Gardner** (1976): used the term in his Mathematical Games column in *Scientific American*. Real popularity began.
The primary recurrence

\[ C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1 \]
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Solving the recurrence

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Let \( y = \sum_{n \geq 0} C_n x^n \) (generating function).

\[ \Rightarrow \frac{y - 1}{x} = y^2 \]

\[ \Rightarrow y = \frac{1 - \sqrt{1 - 4x}}{2x} \]

\[ = -\frac{1}{2} \sum_{n \geq 1} (-4)^n \binom{-1/2}{n} x^{n-1} \]
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\[ C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} \]
Other combinatorial interpretations

\[ \mathcal{P}_n := \{ \text{triangulations of convex } (n + 2)\text{-gon} \} \]
\[ \Rightarrow \#\mathcal{P}_n = C_n \] (where \#S = number of elements of S)

We want other combinatorial interpretations of \( C_n \), i.e., other sets \( S_n \) for which \( C_n = \#S_n \).
“Transparent” interpretations

4. **Binary trees** with $n$ vertices (each vertex has a left subtree and a right subtree, which may be empty)
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3. Binary *parenthesizations* or *bracketings* of a string of $n + 1$ letters

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$
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\[
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\[
((x(xx))x)(x((xx)(xx)))
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The ballot problem

**Bertrand’s ballot problem:** first published by W. A. Whitworth in 1878 but named after Joseph Louis François Bertrand who rediscovered it in 1887 (one of the first results in probability theory).
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Special case: there are two candidates $A$ and $B$ in an election. Each receives $n$ votes. What is the probability that $A$ will never trail $B$ during the count of votes?

Example. $AABABBBBAAB$ is bad, since after seven votes, $A$ receives 3 while $B$ receives 4.
Definition of ballot sequence

Encode a vote for $A$ by 1, and a vote for $B$ by $-1$ (abbreviated $-$). Clearly a sequence $a_1 a_2 \cdots a_{2n}$ of $n$ each of 1 and $-1$ is allowed if and only if $\sum_{i=1}^{k} a_i \geq 0$ for all $1 \leq k \leq 2n$. Such a sequence is called a ballot sequence.
77. Ballot sequences, i.e., sequences of $n$ 1’s and $n - 1$’s such that every partial sum is nonnegative (with $-1$ denoted simply as $-$ below)

$$111 - - - 11 - 1 - - 11 - - 1 - 11 - - 1 - 1 - 1 -$$
Ballot sequences

77. Ballot sequences, i.e., sequences of \( n \) 1’s and \( n - 1 \)'s such that every partial sum is nonnegative (with \(-1\) denoted simply as \(-\) below)

\[
111 - - - 11 - 1 - - 11 - - 1 - 11 - - 1 - 1 - 1 -
\]

**Note.** Answer to original problem (probability that a sequence of \( n \) each of 1’s and \(-1\)’s is a ballot sequence) is therefore

\[
\frac{C_n}{\binom{2n}{n}} = \frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1}.
\]
The ballot recurrence

\[ 11 - 11 - 1 - - - 1 - 11 - 1 - - \]
The ballot recurrence

\[ \begin{array}{ccccccc}
1 & 1 & 1 & - & - & - & 1 - 1 & 1
\end{array} \]

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1 & 1 & 1 & - & - & - & 1 - 1 & 1
\end{array} \]
The ballot recurrence

\[
11 - 11 - 1 - - - 1 - 11 - 1 - - \\
11 - 11 - 1 - - - | 1 - 11 - 1 - - \\
1 - 11 - 1 - - | 1 - 11 - 1 - -
\]
25. **Dyck paths** of length $2n$, i.e., lattice paths from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$, never falling below the $x$-axis
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Walther von Dyck (1856–1934)
Bijection with ballot sequences

For each upstep, record 1.
For each downstep, record \(-1\).
312-avoiding permutations

116. Permutations $a_1a_2\cdots a_n$ of $1, 2, \ldots, n$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-avoiding permutations)

$$
123 \quad 132 \quad 213 \quad 231 \quad 321
$$
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34251768
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\begin{align*}
123 & \quad 132 & \quad 213 & \quad 231 & \quad 321 \\
3425 & \quad 768
\end{align*}
\]
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\[123 \quad 132 \quad 213 \quad 231 \quad 321\]

\[3425 \ 768 \quad \text{(note red < blue)}\]
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$$123 \quad 132 \quad 213 \quad 231 \quad 321$$

$$3425 \quad 768$$ (note red $<$ blue)

part of the subject of pattern avoidance
Another example of pattern avoidance:

**115.** Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called **321-avoiding** permutations:

123  213  132  312  231
Another example of pattern avoidance:

115. Permutations \(a_1 a_2 \cdots a_n\) of 1, 2, \ldots, \(n\) with longest decreasing subsequence of length at most two (i.e., there does not exist \(i < j < k, a_i > a_j > a_k\)), called \textbf{321-avoiding} permutations

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123 \quad 213 \quad 132 \quad 312 \quad 231
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more subtle: no obvious decomposition into two pieces
Bijection with ballot sequences

\[ w = 412573968 \]
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\[
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & - & - & - & 1 & - 1 1 & - - 1 1 & - -
\end{array}
\]
An unexpected interpretation

92. $n$-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2 \cdots a_n1$, each $a_i$ divides the sum of its two neighbors

$14321$  $13521$  $13231$  $12531$  $12341$
Bijection with ballot sequences

remove largest; insert bar before the element to its left; continue until only 1’s remain; then replace bar with 1 and an original number with $-1$, except last two

1 2 5 3 4 1
Bijection with ballot sequences

remove largest; insert bar before the element to its left; continue until only 1’s remain; then replace bar with 1 and an original number with $-1$, except last two

$$1 \mid 2 \; 5 \; 3 \; 4 \; 1$$
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$1 \mid 2 \ 5 \mid 3 \ 4 \ 1$
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1|2 5 3 4 1
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\[ |1| 2 \ 5 \ 3 \ 4 \ 1 \]
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<table>
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```
|1|2 5|3 4 1
```

```
| 1 | | 2 5 | 3 4 1
1 1 1 1
```

tricky to prove
(a) Number of two-sided ideals of the algebra of all \((n - 1) \times (n - 1)\) upper triangular matrices over a field
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A symmetric group representation

Dimension of the irreducible representation of \( S_{2n-1} \) indexed by the partition \((n, n - 1)\), and of \( S_{2n} \) indexed by \((n, n)\).
A symmetric group representation

Dimension of the irreducible representation of $\mathfrak{S}_{2n-1}$ indexed by the partition $(n, n - 1)$, and of $\mathfrak{S}_{2n}$ indexed by $(n, n)$.

Is there a “natural” action of $\mathfrak{S}_{2n-1}$ and/or $\mathfrak{S}_{2n}$ on the space $QX$, where $X$ is some family of Catalan objects indexed by $2n - 1$ and/or $2n$?
(i) Let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $A = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by
\[ w \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{w(1)}, \ldots, x_{w(n)}, y_{w(1)}, \ldots, y_{w(n)}) \]
for all $w \in \mathfrak{S}_n$. Let $I$ be the ideal generated by all invariants of positive degree, i.e.,
\[ I = \langle f \in A : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n, \text{ and } f(0) = 0 \rangle. \]
Then $C_n$ is the dimension of the subspace of $A/I$ affording the sign representation, i.e.,

$$C_n = \dim\{f \in A/I : w \cdot f = (\text{sgn } w)f \text{ for all } w \in S_n\}.$$
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$$C_n = \dim\{ f \in A/I : w \cdot f = (\text{sgn } w)f \text{ for all } w \in \mathfrak{S}_n \}.$$ 

A12. *k*-triangulation of *n*-gon: maximal collections of diagonals such that no *k* + 1 of them pairwise intersect in their interiors

*k* = 1: an ordinary triangulation

**superfluous edge**: an edge between vertices at most *k* steps apart (along the boundary of the *n*-gon). They appear in all *k*-triangulations and are irrelevant.
Example. 2-triangulations of a hexagon (superfluous edges omitted):
Some theorems

**Theorem** (Nakamigawa, Dress-Koolen-Moulton). All \( k \)-triangulations of an \( n \)-gon have \( k(n - 2k - 1) \) nonsuperfluous edges.
Some theorems

**Theorem** (Nakamigawa, Dress-Koolen-Moulton). All $k$-triangulations of an $n$-gon have $k(n - 2k - 1)$ nonsuperfluous edges.

**Theorem** (Jonsson, Serrano-Stump). The number $T_k(n)$ of $k$-triangulations of an $n$-gon is given by

$$T_k(n) = \det \left[ C_{n-i-j}^k \right]_{i,j=1}$$

$$= \prod_{1 \leq i < j \leq n-2k} \frac{2k + i + j - 1}{i + j - 1}.$$
Note. The number $T_k(n)$ is the dimension of an irreducible representation of the symplectic group $\text{Sp}(2n - 4)$. 
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Is there a direct connection?
A61. Let $b(n)$ denote the number of 1’s in the binary expansion of $n$. Using Kummer’s theorem on binomial coefficients modulo a prime power, show that the exponent of the largest power of 2 dividing $C_n$ is equal to $b(n + 1) - 1$. 
Let $f(n)$ denote the number of integers $1 \leq k \leq n$ such that $k$ is the sum of three squares (of nonnegative integers). Well-known:

$$\lim_{n \to \infty} \frac{f(n)}{n} = \frac{5}{6}.$$
Sums of three squares

Let $f(n)$ denote the number of integers $1 \leq k \leq n$ such that $k$ is the sum of three squares (of nonnegative integers). Well-known:

$$\lim_{n \to \infty} \frac{f(n)}{n} = \frac{5}{6}.$$ 

A63. Let $g(n)$ denote the number of integers $1 \leq k \leq n$ such that $C_k$ is the sum of three squares. Then

$$\lim_{n \to \infty} \frac{g(n)}{n} = ??.$$
Let \( f(n) \) denote the number of integers \( 1 \leq k \leq n \) such that \( k \) is the sum of three squares (of nonnegative integers). Well-known:

\[
\lim_{n \to \infty} \frac{f(n)}{n} = \frac{5}{6}.
\]

**A63.** Let \( g(n) \) denote the number of integers \( 1 \leq k \leq n \) such that \( C_k \) is the sum of three squares. Then

\[
\lim_{n \to \infty} \frac{g(n)}{n} = \frac{7}{8}.
\]
A65.(b)

\[ \sum_{n \geq 0} \frac{1}{C_n} = ?? \]
Analysis

A65.(b)

$$\sum_{n \geq 0} \frac{1}{C_n} = ??$$

$$1 + 1 + \frac{1}{2} + \frac{1}{5} = 2.7$$
Analysis

A65.(b)

\[ \sum_{n \geq 0} \frac{1}{C_n} = 2 + \frac{4\sqrt{3}\pi}{27} \]

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A65.(b)

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\[ 2 + \frac{4\sqrt{3}\pi}{27} = 2.806133 \cdots \]
Why?

A65.(a)

\[ \sum_{n \geq 0} \frac{x^n}{C_n} = \frac{2(x + 8)}{(4 - x)^2} + \frac{24\sqrt{x} \sin^{-1}\left(\frac{1}{2}\sqrt{x}\right)}{(4 - x)^{5/2}}. \]
Based on a (difficult) calculus exercise: let

\[ y = 2 \left( \sin^{-1} \frac{1}{2} \sqrt{x} \right)^2. \]

Then \( y = \sum_{n \geq 1} \frac{x^n}{n^2 \binom{2n}{n}}. \) Use \( \sin^{-1} x = \sum_{n \geq 0} 4^{-n} \binom{2n}{n} \frac{x^{2n+1}}{2n+1}. \)
Recall $y = \sum_{n \geq 1} \frac{x^n}{n^2 \binom{2n}{n}}$. Note that:
Recall $y = \sum_{n \geq 1} \frac{x^n}{n^2 \binom{2n}{n}}$. Note that:

$$\frac{d}{dx} y = \sum_{n \geq 1} \frac{x^{n-1}}{n \binom{2n}{n}}$$
Recall \( y = \sum_{n \geq 1} \frac{x^n}{n^2 \binom{2n}{n}} \). Note that:

\[
x \frac{d}{dx} y = \sum_{n \geq 1} \frac{x^n}{n \binom{2n}{n}}
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Recall $y = \sum_{n \geq 1} \frac{x^n}{n^2 \binom{2n}{n}}$. Note that:

$$x^2 \frac{d}{dx} \frac{dx}{x} y = \sum_{n \geq 1} \frac{x^{n+1}}{\binom{2n}{n}}$$
Recall $y = \sum_{n \geq 1} \frac{x^n}{n^2 \binom{2n}{n}}$. Note that:

$$\frac{d}{dx} x^2 \frac{d}{dx} x \frac{dx}{x} y = \sum_{n \geq 1} \frac{(n + 1)x^n}{\binom{2n}{n}}$$
Recall $y = \sum_{n \geq 1} \frac{x^n}{n^2 \binom{2n}{n}}$. Note that:

$$\frac{d}{dx} x^2 \frac{d}{dx} x \frac{dy}{dx} = \sum_{n \geq 1} \frac{(n + 1)x^n}{\binom{2n}{n}}$$

$$= -1 + \sum_{n \geq 0} \frac{x^n}{C_n},$$

etc.
What’s next?

Next topic: Euler numbers