OK
Increasing and decreasing subsequences

\[3 \ 1 \ 8 \ 4 \ 9 \ 6 \ 7 \ 2 \ 5\] (i.s.)

\[3 \ 1 \ 8 \ 4 \ 9 \ 6 \ 7 \ 2 \ 5\] (d.s.)

is(\(w\)) = \(|\text{longest i.s.}| = 4\)

ds(\(w\)) = \(|\text{longest d.s.}| = 3\)

**Application:** airplane boarding

Naive model: passengers board in order \(w = a_1 a_2 \ldots a_n\) for seats 1, 2, \ldots, \(n\). Each passenger takes one time unit to be seated after arriving at his seat.

Easy: Total waiting time = is(\(w\)).

Bachmat, et al.: more sophisticated model.

Two conclusions:
- Usual system (back-to-front) not much better than random.
- Better: first board window seats, then center, then aisle.

**partition** \(\lambda \vdash n\): \(\lambda = (\lambda_1, \lambda_2, \ldots)\)

\[\lambda_1 \geq \lambda_2 \geq \cdots \geq 0\]

\[\sum \lambda_i = n\]

(Young) diagram of \(\lambda = (4, 4, 3, 1)\):

\(<\)

Young diagram of the **conjugate** partition \(\lambda' = (4, 3, 3, 2)\):

\(<\)

**standard Young tableau** (SYT) of shape \(\lambda \vdash n\), e.g., \(\lambda = (4, 4, 3, 1)\):

\(<\)

\[f^\lambda = \# \text{ of SYT of shape } \lambda\]

E.g., \(f^{(3,3)} = 5\):

\begin{array}{cccccc}
123 & 124 & 125 & 134 & 135 \\
45 & 35 & 34 & 25 & 24 \\
\end{array}

\[\exists \text{ simple formula for } f^\lambda \text{ (Frame-Robinson-Thrall hook-length formula)}\]

**Note.** \(f^\lambda = \dim(\text{irrep. of } S_n)\), where \(S_n\) is the **symmetric group** of all permutations of 1, 2, \ldots, \(n\).

**RSK algorithm:** a bijection \(w \mapsto (P, Q)\),

where \(w \in S_n\) and \(P, Q\) are SYT of the same shape \(\lambda \vdash n\).

Write \(\lambda = \text{sh}(w)\), the **shape** of \(w\).

\(R = \) Gilbert de Beauregard Robinson
\(S = \) Craige Schensted (= Ea Ea)
\(K = \) Donald Ervin Knuth
Increasing and decreasing subsequences

\[318496725 \quad \text{(i.s.)} \]
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\[ \begin{array}{ccccccc}
\, & \, & \, & \, & \, & \, & \, \\
6 & 5 & 4 & 3 & 2 & 1 & \, \\
\end{array} \]
**Application:** airplane boarding

**Naive model:** passengers board in order $w = a_1a_2 \cdots a_n$ for seats $1, 2, \ldots, n$. Each passenger takes one time unit to be seated after arriving at his seat.

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6 & 5 & 4 & 3 & 2 & 1 \\
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Easy: Total waiting time $= \text{is}(w)$. 

\[
\begin{array}{ccccccc}
6 & 5 & 4 & 3 & 2 & 1 & \textbf{2 5 3 6 1 4} \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\textbf{5} & \textbf{3 6 4} \\
\end{array}
\]

\[
\begin{array}{ccccccc}
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![Young diagram of $\lambda = (4, 4, 3, 1)$](image)

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\[
\begin{array}{cccc}
1 & 2 & 7 & 10 \\
3 & 5 & 8 & 12 \\
4 & 6 & 11 \\
9 \\
\end{array}
\]

$f^\lambda = \# \text{ of SYT of shape } \lambda$

E.g., $f^{(3,2)} = 5$:

\begin{align*}
1 & 2 & 3 & 1 & 2 & 4 & 1 & 2 & 5 & 1 & 3 & 4 & 1 & 3 & 5 \\
4 & 5 & 3 & 5 & 3 & 4 & 2 & 5 & 2 & 4 \\
\end{align*}

$\exists$ simple formula for $f^\lambda$ (Frame-Robinson-Thrall hook-length formula)
Note. \( f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n) \), where \( \mathfrak{S}_n \) is the \textit{symmetric group} of all permutations of \( 1, 2\ldots, n \).
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\begin{itemize}
  \item \textbf{R} = Gilbert de Beauregard Robinson
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\end{itemize}
$w = 4132$: 
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\[
(P, Q) = \begin{pmatrix}
1 & 2 & 1 & 13 \\
3 & 2 & 1 & 12 \\
4 & 3 & 1 & 12 \\
4 & 4 & 1 & \phi
\end{pmatrix}
\]
Schensted’s theorem: Let \( w \overset{\text{rsk}}{\rightarrow} (P,Q) \), where \( sh(P) = sh(Q) = \lambda \). Then
\[
\begin{align*}
is(w) &= \text{longest row length} = \lambda_1 \\
ds(w) &= \text{longest column length} = \lambda'_1.
\end{align*}
\]
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\]

\[
4132 \overset{\text{rsk}}{\rightarrow} \begin{pmatrix}
1 & 2 & 13 \\
3 & 3 & 2 \\
4 & 4 & 4
\end{pmatrix}
\]

\[
\text{is}(w) = 2, \quad \text{ds}(w) = 3
\]
Schensted’s theorem: Let $w \stackrel{\text{rsk}}{\to} (P, Q)$, where $sh(P) = sh(Q) = \lambda$. Then

$\text{is}(w) = \text{longest row length} = \lambda_1$
$\text{ds}(w) = \text{longest column length} = \lambda_1'$. 

Corollary (Erdős-Szekeres, Seidenberg). Let $w \in S_{pq+1}$. Then either

$\text{is}(w) > p$ or $\text{ds}(w) > q$. 

Proof. Let $\lambda = sh(w)$. If $\text{is}(w) \leq p$ and $\text{ds}(w) \leq q$ then $\lambda_1 \leq p$ and $\lambda_1' \leq q$, so $\sum \lambda_i \leq pq$. $\Box$
Corollary. Say \( p \leq q \). Then

\[
\#\{w \in \mathfrak{S}_{pq} : \text{is}(w) = p, \, \text{ds}(w) = q\}
\]

\[
= \left(f(p^q)\right)^2
\]

By hook-length formula, this is

\[
\left(\frac{(pq)!}{1^12^2 \cdots p^p(p + 1)^p \cdots q^q(q + 1)^{p-1} \cdots (p + q - 1)^1}\right)^2.
\]
Romik: let

\[ w \in \mathcal{S}_p^2, \quad \text{is}(w) = \text{ds}(w) = p. \]

Let \( P_w \) be the permutation matrix of \( w \) with corners \((\pm 1, \pm 1)\). Then (informally) as \( p \to \infty \) almost surely the 1’s in \( P_w \) will become dense in the region bounded by the curve

\[ (x^2 - y^2)^2 + 2(x^2 + y^2) = 3, \]

and will remain isolated outside this region.
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\]

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\[ w = 9, 11, 6, 14, 2, 10, 1, 5, 13, 3, 16, 8, 15, 4, 12, 7 \]
\[(x^2 - y^2)^2 + 2(x^2 + y^2) = 3\]
Distribution of $\text{is}(w)$

\[ E(n) = \text{expectation of } \text{is}(w), \ w \in \mathcal{S}_n \]

\[ = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left( f^\lambda \right)^2 \]
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**Ulam**: what is distribution of $\text{is}(w)$? rate of growth of $E(n)$?
Distribution of is($w$)

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**Ulam:** what is distribution of is($w$)?
rate of growth of $E(n)$?

**Hammersley** (1972):

$$\exists \ c = \lim_{n \to \infty} \frac{E(n)}{\sqrt{n}},$$

and

$$\frac{\pi}{2} \leq c \leq e.$$  

Conjectured $c = 2$. 
Logan-Shepp, Vershik-Kerov (1977): $c = 2$

Idea of proof.

$$E(n) = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left( f^\lambda \right)^2$$

$$\approx \frac{1}{n!} \max_{\lambda \vdash n} \lambda_1 \left( f^\lambda \right)^2.$$

Find “limiting shape” of $\lambda \vdash n$ maximizing $\lambda$ as $n \to \infty$ using hook-length formula.
\[
\min \iint_A \log(f(x)+f^{-1}(y)-x-y) \, dx \, dy, \\
\text{subject to} \\
\iint_A dx \, dy = 1.
\]
\[ x = y + 2 \cos \theta \]
\[ y = \frac{2}{\pi} (\sin \theta - \theta \cos \theta) \]
\[ 0 \leq \theta \leq \pi \]
\( u_k(n) := \#\{w \in S_n : is_n(w) \leq k \} \).
\[ u_k(n) := \#\{w \in \mathcal{S}_n : \text{is}_n(w) \leq k \}. \]

**J. M. Hammersley (1972):**

\[ u_2(n) = C_n = \frac{1}{n+1} \binom{2n}{n}, \]

a **Catalan number**.
\[ u_k(n) := \#\{w \in S_n : is_n(w) \leq k\}. \]

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For \( \geq 130 \) combinatorial interpretations of \( C_n \), see

www-math.mit.edu/~rstan/ec
I. Gessel (1990):

\[ \sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det \left[ I_{|i-j|}(2x) \right]_{i,j=1}^k, \]

where

\[ I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}, \]

a hyperbolic Bessel function of the first kind of order \( m \).
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a hyperbolic Bessel function of the first kind of order \( m \).

E.g.,

\[ \sum_{n \geq 0} u_2(n) \frac{x^{2n}}{n!^2} = U_0(2x)^2 - U_1(2x)^2 \]

\[ = \sum_{n \geq 0} C_n \frac{x^{2n}}{n!^2}. \]
Corollary. For fixed \( k \), \( u_k(n) \) is \( P \)-recursive, e.g.,

\[
(n + 4)(n + 3)^2 u_4(n) \\
= (20n^3 + 62n^2 + 22n - 24)u_4(n - 1) \\
\quad - 64n(n - 1)^2 u_4(n - 2)
\]

\[
(n + 6)^2(n + 4)^2 u_5(n) \\
= (375 - 400n - 843n^2 - 322n^3 - 35n^4)u_5(n - 1) \\
\quad + (259n^2 + 622n + 45)(n - 1)^2 u_5(n - 2) \\
\quad - 225(n - 1)^2(n - 2)^2 u_5(n - 3).
\]

Conjectures on form of recurrence due to Bergeron, Favreau, and Krob.
Baik-Deift-Johansson:

Define $u(x)$ by

$$\frac{d^2}{dx^2} u(x) = 2u(x)^3 + xu(x) \quad (\star),$$

with certain initial conditions.

$(\star)$ is the Painlevé II equation (roughly, the branch points and essential singularities are independent of the initial conditions).
Paul Painlevé

Paul Painlevé


1890: Grand Prix des Sciences Mathématiques
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1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.
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1933: died in Paris.
Tracy-Widom distribution:

\[ F(t) = \exp \left( - \int_{t}^{\infty} (x - t)u(x)^2 \, dx \right) \]
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**Theorem (Baik-Deift-Johansson)** For random (uniform) \( w \in \mathcal{S}_n \) and all \( t \in \mathbb{R} \) we have

\[
\lim_{n \to \infty} \operatorname{Prob}\left( \frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t).
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**Theorem** (Baik-Deift-Johansson) For random (uniform) \( w \in \mathcal{S}_n \) and all \( t \in \mathbb{R} \) we have

\[ \lim_{n \to \infty} \operatorname{Prob} \left( \frac{\text{i} s_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t). \]

**Corollary.**

\[ \text{i} s_n(w) = 2\sqrt{n} + \left( \int t \, dF(t) \right) n^{1/6} + o(n^{1/6}) \]

\[ = 2\sqrt{n} - (1.7711 \cdots) n^{1/6} + o(n^{1/6}) \]
Gessel’s theorem reduces the problem to “just” analysis, viz., the **Riemann-Hilbert problem** in the theory of integrable systems, and the **method of steepest descent** to analyze the asymptotic behavior of integrable systems.
Gessel’s theorem reduces the problem to “just” analysis, viz., the **Riemann-Hilbert problem** in the theory of integrable systems, and the **method of steepest descent** to analyze the asymptotic behavior of integrable systems.

Where did the Tracy-Widom distribution $F(t)$ come from?

\[
F(t) = \exp \left( - \int_{t}^{\infty} (x - t)u(x)^2 \, dx \right)
\]

\[
\frac{d^2}{dx^2} u(x) = 2u(x)^3 + xu(x) \quad (*),
\]
Gaussian Unitary Ensemble (GUE):

Consider an $n \times n$ hermitian matrix $M = (M_{ij})$ with probability density

$$Z_n^{-1} e^{-\text{tr}(M^2)} dM,$$

where $Z_n$ is a normalization constant.

$$dM = \prod_i dM_{ii}$$

$$\cdot \prod_{i<j} d(\text{Re}(M_{ij}))d(\text{Im}(M_{ij})),$$

where $Z_n$ is a normalization constant.
Tracy-Widom (1994): let $\alpha_1$ denote the largest eigenvalue of $M$. Then

$$\lim_{n \to \infty} \text{Prob}\left(\left(\alpha_1 - \sqrt{2n}\right) \sqrt{2n^{1/6}} \leq t\right) = F(t).$$
Is the connection between \( \text{is}(w) \) and GUE a coincidence?
Is the connection between $\text{is}(w)$ and GUE a coincidence?

Okounkov provides a connection, via the theory of random topologies on surfaces. Very briefly, a surface can be described in two ways:

- Gluing polygons along their edges, connected to random matrices via quantum gravity.
- Ramified covering of a sphere, which can be formulated in terms of permutations.
Joint with:

Bill Chen  陈永川
Eva Deng  邓玉平
Rosena Du  杜若霞
Catherine Yan  颜华菲
(complete) matching:

crossing:

nesting:
(complete) matching:

\begin{center}
\begin{tikzpicture}
\foreach \i in {1,2,3,4} {
    \node[circle,fill,inner sep=2pt] at (2*\i,0) {};
}\end{tikzpicture}
\end{center}

crossing:

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total number of matchings on \([2n] := \{1, 2, \ldots, 2n\}\) is

\((2n - 1)!! := 1 \cdot 3 \cdot 5 \cdots (2n - 1)\).
(complete) matching:

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nesting:

total number of matchings on \([2n]\) := \{1, 2, \ldots, 2n\} is

\[(2n - 1)!! := 1 \cdot 3 \cdot 5 \cdots (2n - 1).\]

**Theorem.** *The number of matchings on \([2n]\) with no crossings (or with no nestings) is*

\[C_n := \frac{1}{n + 1} \binom{2n}{n}.\]
Well-known:

\[ C_n = \# \{ a_1 \cdots a_{2n} : a_i = \pm 1, \]
\[ a_1 + \cdots + a_i \geq 0, \quad \sum a_i = 0 \} \]

(ballot sequence).
Well-known:

\[ C_n = \#\{a_1 \cdots a_{2n} : a_i = \pm 1, \ a_1 + \cdots + a_i \geq 0, \ \sum a_i = 0 \} \]

(ballot sequence).
What is the analogue of increasing and decreasing subsequences for matchings $M$?
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Associate with a matching $M$ on the vertices $1, 2, \ldots, 2n$ a fixed-point free involution $w_M \in \mathfrak{S}_{2n}$:

$$w_M = (1, 5)(2, 7)(3, 4)(6, 8)$$
What is the analogue of increasing and decreasing subsequences for matchings $M$?

Associate with a matching $M$ on the vertices $1, 2, \ldots, 2n$ a fixed-point free involution $w_M \in S_{2n}$:

$$w_M = (1, 5)(2, 7)(3, 4)(6, 8)$$

**Flaw:** no symmetry between is and ds (different distributions on fixed-point free involutions).
\[
\begin{align*}
M &= \text{matching} \\
\text{cr}(M) &= \max\{k : \exists k\text{-crossing}\} \\
\text{ne}(M) &= \max\{k : \exists k\text{-nesting}\} = \frac{1}{2} \text{ds}(w_M)
\end{align*}
\]
\(M\) = matching
\(\text{cr}(M) = \max\{k : \exists k\text{-crossing}\}\)
\(\text{ne}(M) = \max\{k : \exists k\text{-nesting}\} = \frac{1}{2}\text{ds}(w_M)\)

**Theorem.** Let \(f_n(i, j) = \#\) matchings \(M\) on \([2n]\) with \(\text{cr}(M) = i\) and \(\text{ne}(M) = j\). Then \(f_n(i, j) = f_n(j, i)\).

**Corollary.** \# matchings \(M\) on \([2n]\) with \(\text{cr}(M) = k\) equals \# matchings \(M\) on \([2n]\) with \(\text{ne}(M) = k\).
Main tool: oscillating tableaux.

shape (3, 1), length 8
Main tool: oscillating tableaux.

\[ \phi \quad \square \quad \square \quad \square \quad \square \quad \square \quad \square \quad \square \quad \square \]

shape \((3, 1)\), length 8

\[ \Phi(M) = ( \phi \quad \square \quad \square \quad \square \quad \square \quad \square \quad \square \quad \square \quad \phi ) \]
\[ \Phi \text{ is a bijection from matchings on } 1, 2, \ldots, 2n \text{ to oscillating tableaux of length } 2n, \text{ shape } \emptyset. \]
Φ is a bijection from matchings on 1, 2, . . . , 2n to oscillating tableaux of length 2n, shape $\emptyset$.

**Corollary.** Number of oscillating tableaux of length 2n, shape $\emptyset$, is $(2n-1)!!$ (related to Brauer algebra of dimension $(2n - 1)!!$).
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**Corollary.** Number of oscillating tableaux of length 2n, shape \(\emptyset\), is 
\((2n−1)!!\) (related to **Brauer algebra** of dimension \((2n − 1)!!\)).

**Schensted’s theorem for matchings.** Let

\[ \Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset) . \]

Then

\[ \text{cr}(M) = \max\{ (\lambda^i_1)' : 0 \leq i \leq n \} \]

\[ \text{ne}(M) = \max\{ \lambda^i_1 : 0 \leq i \leq n \} . \]

**Proof.** Reduce to ordinary RSK.
Now let \( \text{cr}(M) = i, \) \( \text{ne}(M) = j, \) and 
\[
\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).
\]
Define \( M' \) by
\[
\Phi(M') = (\emptyset = (\lambda^0)', (\lambda^1)', \ldots, (\lambda^{2n})' = \emptyset).
\]
By Schensted’s theorem for matchings, 
\[
\text{cr}(M') = j, \quad \text{ne}(M') = i.
\]

Thus \( M \mapsto M' \) is an involution on matchings of \([2n]\) interchanging \text{cr} and \text{ne}.  

70
Now let \( \text{cr}(M) = i \), \( \text{ne}(M) = j \), and
\[
\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).
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Define \( M' \) by
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\]

Thus \( M \mapsto M' \) is an involution on matchings of \([2n]\) interchanging \( \text{cr} \) and \( \text{ne} \).

\[\Rightarrow \text{Theorem.} \quad \text{Let } f_n(i, j) = \# \text{ matchings } M \text{ on } [2n] \text{ with } \text{cr}(M) = i \text{ and } \text{ne}(M) = j. \text{ Then } f_n(i, j) = f_n(j, i).\]
Now let \( \text{cr}(M) = i \), \( \text{ne}(M) = j \), and
\[
\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).
\]
Define \( M' \) by
\[
\Phi(M') = (\emptyset = (\lambda^0)', (\lambda^1)', \ldots, (\lambda^{2n})' = \emptyset).
\]
By Schensted’s theorem for matchings,
\[
\text{cr}(M') = j, \quad \text{ne}(M') = i.
\]

Thus \( M \mapsto M' \) is an involution on matchings of \([2n]\) interchanging \( \text{cr} \) and \( \text{ne} \).

\( \Rightarrow \) Theorem. Let \( f_n(i, j) = \# \text{ matchings } M \text{ on } [2n] \text{ with } \text{cr}(M) = i \text{ and } \text{ne}(M) = j \). Then \( f_n(i, j) = f_n(j, i) \).

Open: simple description of \( M \mapsto M' \), the analogue of
\[a_1a_2\cdots a_n \mapsto a_n\cdots a_2a_1,\]
which interchanges is and ds.
Enumeration of $k$-noncrossing matchings (or nestings).

**Recall:** The number of matchings $M$ on $[2n]$ with no crossings, i.e., $\text{cr}(M) = 1$, (or with no nestings) is $C_n = \frac{1}{n+1} \binom{2n}{n}$.

What about the number with $\text{cr}(M) \leq k$?

Assume $\text{cr}(M) \leq k$. Let

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Regard each $\lambda^i = (\lambda^i_1, \ldots, \lambda^i_k) \in \mathbb{N}^k$. 

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Corollary. The number $f_k(n) \leq k$ of matchings $M$ on $[2n]$ with $\text{cr}(M) \leq k$ is the number of lattice paths of length $2n$ from $0$ to $0$ in the region

$$C_n := \{(a_1, \ldots, a_k) \in \mathbb{N}^k : a_1 \leq \cdots \leq a_k\}$$

with steps $\pm e_i$ ($e_i = \text{i\text{-th unit coordinate vector)}$).

$C_n \otimes \mathbb{R}_{\geq 0}$ is a fundamental chamber for the Weyl group of type $B_k$. 
Grabiner-Magyar: applied **Gessel-Zeilberger reflection principle** to solve this lattice path problem (not knowing connection with matchings).
Grabiner-Magyar: applied Gessel-Zeilberger reflection principle to solve this lattice path problem (not knowing connection with matchings).

**Theorem.** Define

\[ H_k(x) = \sum_{n} f_k(n) \frac{x^{2n}}{(2n)!}. \]

Then

\[ H_k(x) = \det \left[ I_{|i-j|}(2x) - I_{i+j}(2x) \right]_{i,j=1}^k \]

where

\[ I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!} \]

as before.
Example. $k = 1$ (noncrossing matchings):

$$H_1(x) = I_0(2x) - I_2(2x)$$

$$= \sum_{j \geq 0} C_j \frac{x^{2j}}{(2j)!}.$$
Example. \( k = 1 \) (noncrossing matchings):

\[
H_1(x) = I_0(2x) - I_2(2x)
\]

\[
= \sum_{j\geq 0} C_j \frac{x^{2j}}{(2j)!}.
\]

Compare:

\( u_k(n) := \#\{w \in \mathcal{S}_n : \text{longest increasing subsequence of length } \leq k}\}.\)

\[
\sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det \left[ I_{i-j}(2x) \right]_{i,j=1}^{k}.
\]
Baik-Rains (implicitly):

$$\lim_{n \to \infty} \text{Prob} \left( \frac{cr_n(M) - \sqrt{2n}}{(2n)^{1/6}} \leq \frac{t}{2} \right) = F_1(t),$$

where

$$F_1(t) = \sqrt{F(t)} \exp \left( \frac{1}{2} \int_t^\infty u(x) dx \right),$$

where $F(t)$ is the Tracy-Widom distribution and $u(x)$ the Painlevé II function.

$$F(t) = \exp \left( - \int_t^\infty (x - t) u(x)^2 \, dx \right)$$

$$\frac{d^2}{dx^2} u(x) = 2u(x)^3 + xu(x)$$
\[ g_{j,k}(n) := \#\{ \text{matchings } M \text{ on } [2n], \]
\[ \text{cr}(M) \leq j, \text{ ne}(M) \leq k \} \]

Now
\[ g_{j,k}(n) = \#\{ (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset) : \lambda^{i+1} = \lambda^i \pm \square, \lambda^i \subseteq j \times k \text{ rectangle} \}, \]

a walk on the Hasse diagram \( \mathcal{H}(j, k) \) of

\[ L(j, k) := \{ \lambda \subseteq j \times k \text{ rectangle} \}, \]

ordered by inclusion.
$L(2,3)$
\( A = \text{adjacency matrix of } \mathcal{H}(j, k) \)
\( A_0 = \text{adjacency matrix of } \mathcal{H}(j, k) - \{\emptyset\} .\)

Transfer-matrix method \(\Rightarrow\)

\[ \sum_{n \geq 0} g_{j,k}(n)x^{2n} = \frac{\det(I - xA_0)}{\det(I - xA)}. \]
\[ A = \text{adjacency matrix of } \mathcal{H}(j, k) \]
\[ A_0 = \text{adjacency matrix of } \mathcal{H}(j, k) - \{\emptyset\}. \]

Transfer-matrix method ⇒
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\sum_{n \geq 0} g_{j,k}(n)x^{2n} = \frac{\det(I - xA_0)}{\det(I - xA)}.
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**Theorem** (Grabiner, implicitly) Every zero of \( \det(I - xA) \) has the form
\[
2(\cos(\pi r_1/m) + \cdots + \cos(\pi r_j/m)),
\]
where each \( r_i \in \mathbb{Z} \) and \( m = j + k + 1 \).
\( A \) = adjacency matrix of \( \mathcal{H}(j, k) \)
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Transfer-matrix method \( \Rightarrow \)

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**Theorem** (Grabiner, implicitly) Every zero of \( \det(I - xA) \) has the form

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where each \( r_i \in \mathbb{Z} \) and \( m = j + k + 1 \).

**Corollary.** Every factor of \( \det(I - xA) \) over \( \mathbb{Q} \) has degree dividing

\[
\frac{1}{2}\phi(2(j + k + 1)),
\]

where \( \phi \) is the Euler phi-function.
Example.

\( j = 2, \; k = 5, \; \frac{1}{2} \phi(16) = 4: \)

\[
\det(I - xA) = (1 - 2x^2)(1 - 4x^2 + 2x^4) \\
(1 - 8x^2 + 8x^4)(1 - 8x^2 + 8x^3 - 2x^4) \\
(1 - 8x^2 - 8x^3 - 2x^4)
\]

\( j = k = 3, \; \frac{1}{2} \phi(14) = 3: \)

\[
\det(I - xA) = (1 - x)(1 + x)(1 + x - 9x^2 - x^3) \\
(1 - x - 9x^2 + x^3)(1 - x - 2x^2 + x^3)^2 \\
(1 + x - 2x^2 - x^3)^2
\]
**Partition** of the set $[n]$:

$\{1, 5\}, \{2\}, \{3, 6, 8, 9\}, \{4, 7, 10\}$

Generalize oscillating tableaux to **vacillating tableaux** (related to the partition algebra).
**Partition** of the set \([n]\):

\{1, 5\}, \{2\}, \{3, 6, 8, 9\}, \{4, 7, 10\}

Generalize oscillating tableaux to **vacillating tableaux** (related to the **partition algebra**).

Many other variations: see paper!