

OK

Increasing and decreasing subsequences

3 1 8 4 9 6 7 2 5 (i.s.)

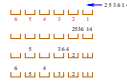
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is(w) = |longest i.s.| = 4

ds(w) = |longest d.s.| = 3

Application: airplane boarding

Naive model: passengers board in order $w = a_1 a_2 \dots a_n$ for seats $1, 2, \dots, n$. Each passenger takes one time unit to be seated after arriving at his seat.



Easy: Total waiting time = is(w).

Bachmat, et al.: more sophisticated model.

Two conclusions:

- Usual system (back-to-front) not much better than random.
- Better: first board window seats, then center, then aisle.

partition $\lambda \vdash n$: $\lambda = (\lambda_1, \lambda_2, \dots)$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq 0$$

$$\sum \lambda_i = n$$

(Young) diagram of $\lambda = (4, 4, 3, 1)$:



Young diagram of the **conjugate** partition $\lambda' = (4, 3, 3, 2)$:



standard Young tableau (SYT) of shape $\lambda \vdash n$, e.g., $\lambda = (4, 4, 3, 1)$:



$$f^\lambda = \# \text{ of SYT of shape } \lambda$$

E.g., $f^{(3,2)} = 5$:

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5
4 5	3 5	3 4	2 5	2 4

\exists simple formula for f^λ (Frame-Robinson-Thrall **hook-length formula**)

Note. $f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n)$, where \mathfrak{S}_n is the **symmetric group** of all permutations of $1, 2, \dots, n$.

RSK algorithm: a bijection

$$w \xrightarrow{\text{RSK}} (P, Q),$$

where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

Write $\lambda = \mathbf{sh}(w)$, the **shape** of w .

R = Gilbert de Beauregard Robinson

S = Craige Schensted (= Ea Ea)

K = Donald Ervin Knuth

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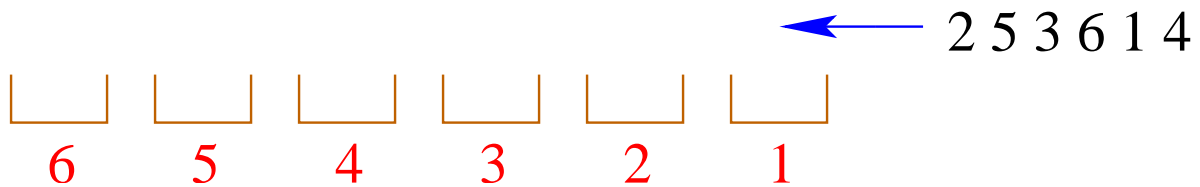
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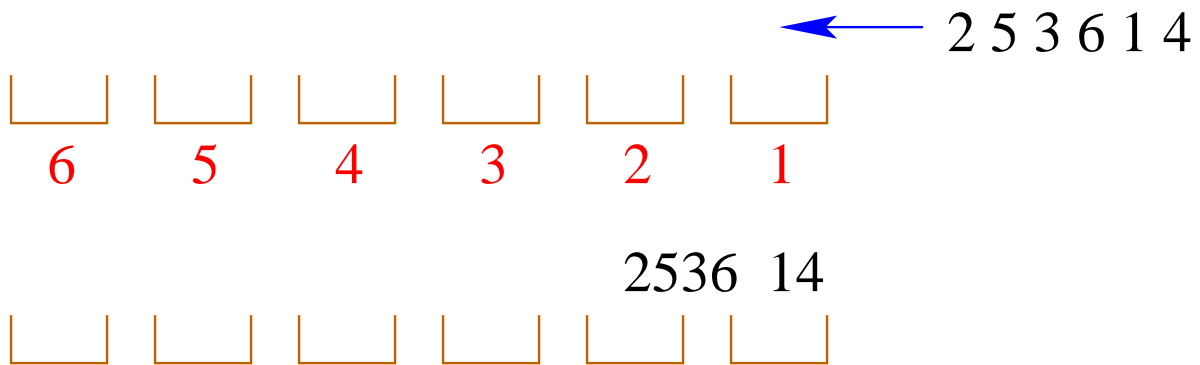
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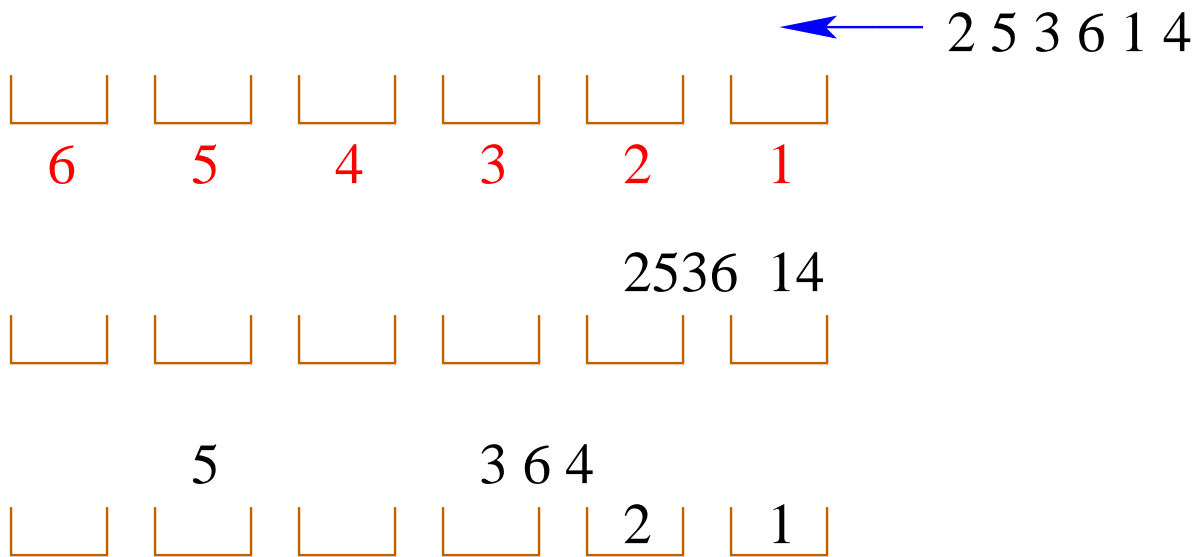
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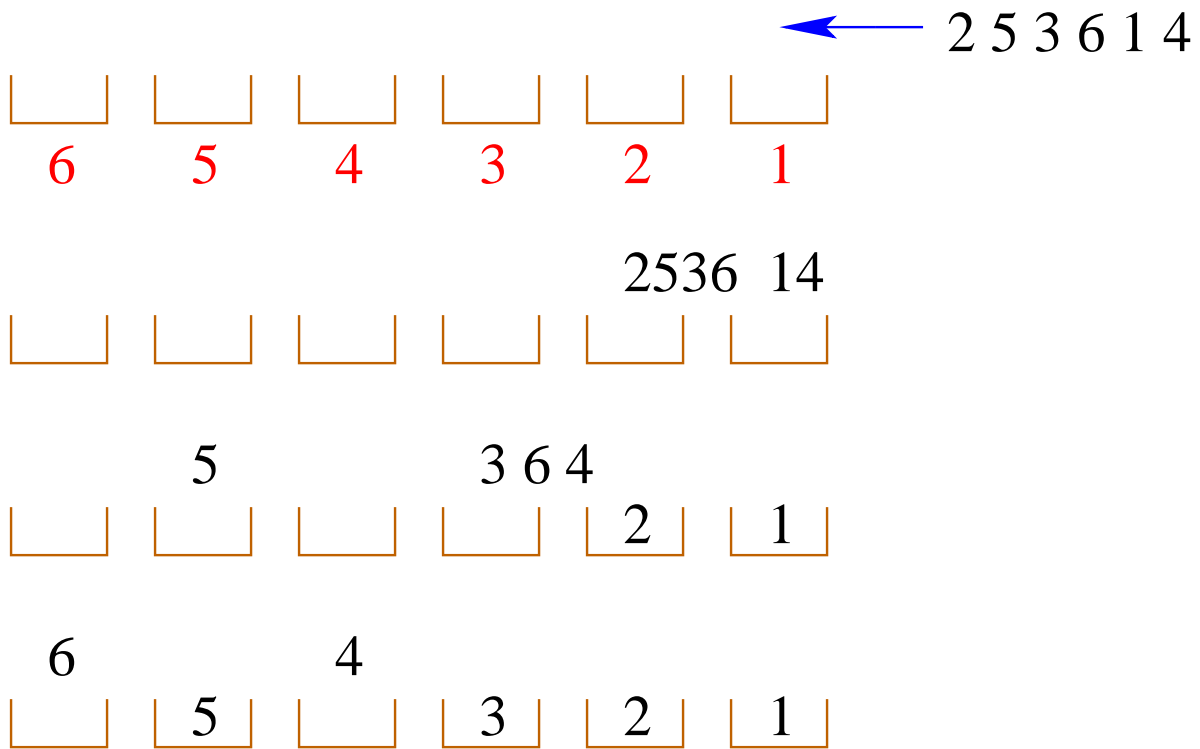
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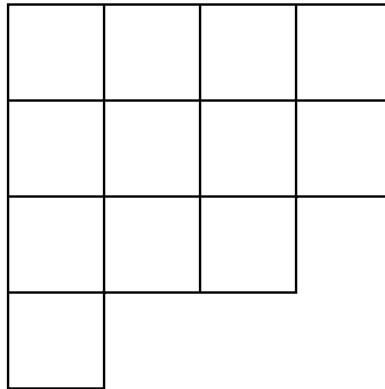
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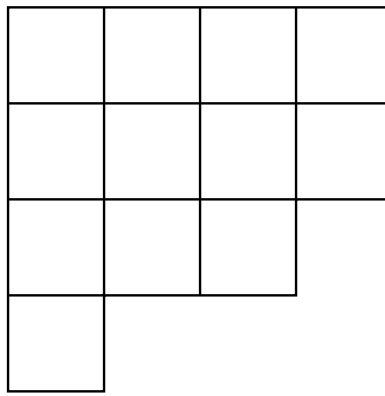


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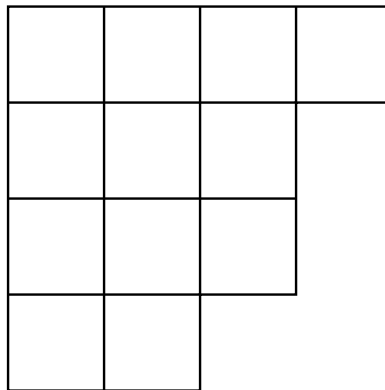
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$f^\lambda = \#$ of SYT of shape λ

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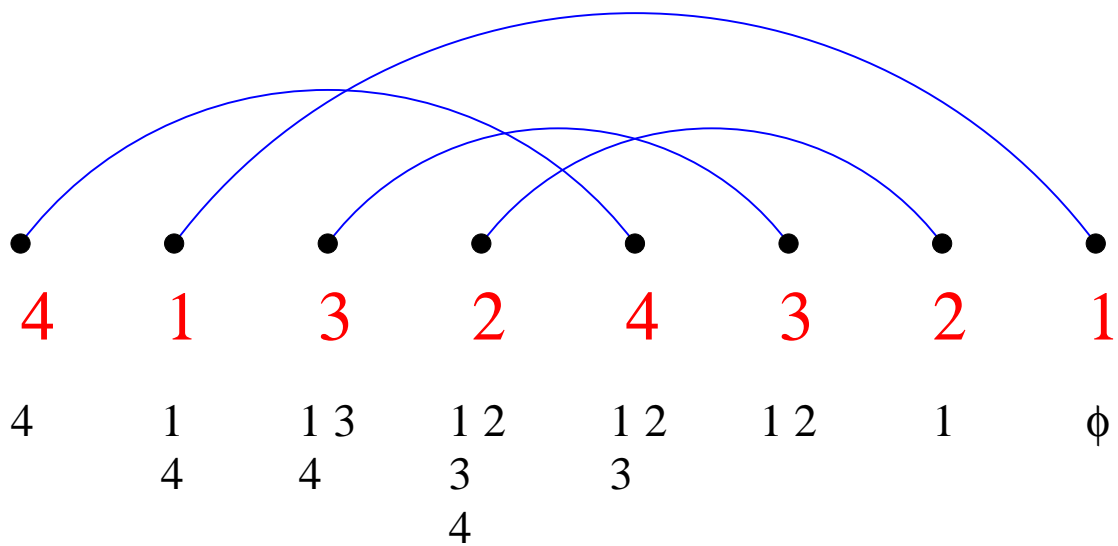
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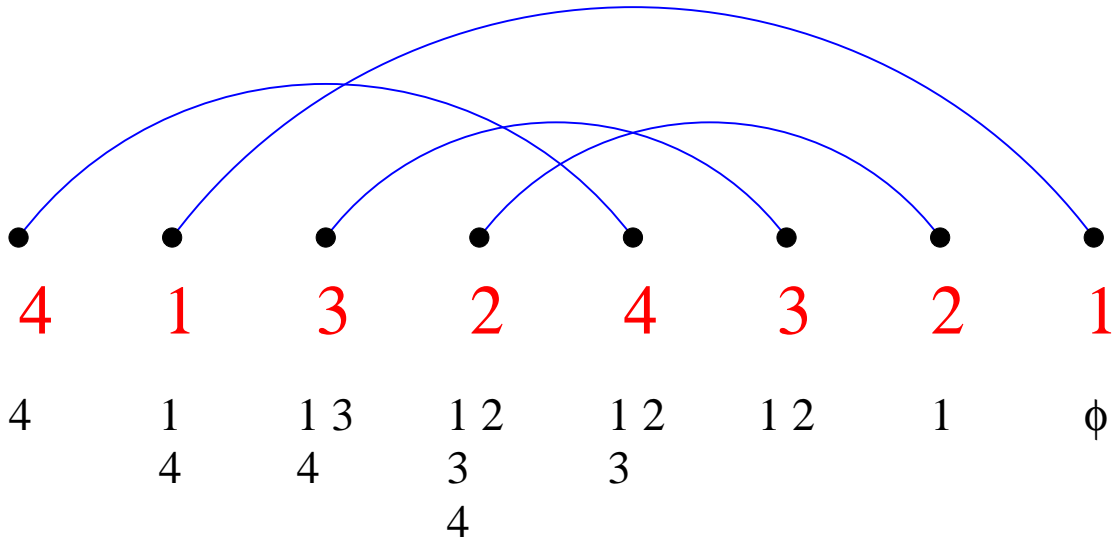
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$$(P, Q) = \begin{pmatrix} 12 & 13 \\ 3 & 2 \\ 4 & 4 \end{pmatrix}$$

Schensted's theorem: Let $w \xrightarrow{\text{rsk}} (P, Q)$, where $sh(P) = sh(Q) = \lambda$.
Then

$$is(w) = \text{longest row length} = \lambda_1$$

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$$4132 \xrightarrow{\text{rsk}} \begin{pmatrix} 1\ 2 & 1\ 3 \\ 3 & 2 \\ 4 & 4 \end{pmatrix}$$

$$\text{is}(w) = 2, \quad \text{ds}(w) = 3$$

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Corollary (Erdős-Szekeres, Seidenberg). Let $w \in \mathfrak{S}_{pq+1}$. Then either $\text{is}(w) > p$ or $\text{ds}(w) > q$.

Proof. Let $\lambda = \text{sh}(w)$. If $\text{is}(w) \leq p$ and $\text{ds}(w) \leq q$ then $\lambda_1 \leq p$ and $\lambda'_1 \leq q$, so $\sum \lambda_i \leq pq$. \square

Corollary. *Say $p \leq q$. Then*

$$\begin{aligned} \#\{w \in \mathfrak{S}_{pq} : \text{is}(w) = p, \text{ds}(w) = q\} \\ = \left(f(p^q)\right)^2 \end{aligned}$$

By hook-length formula, this is

$$\left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1} \right)^2.$$

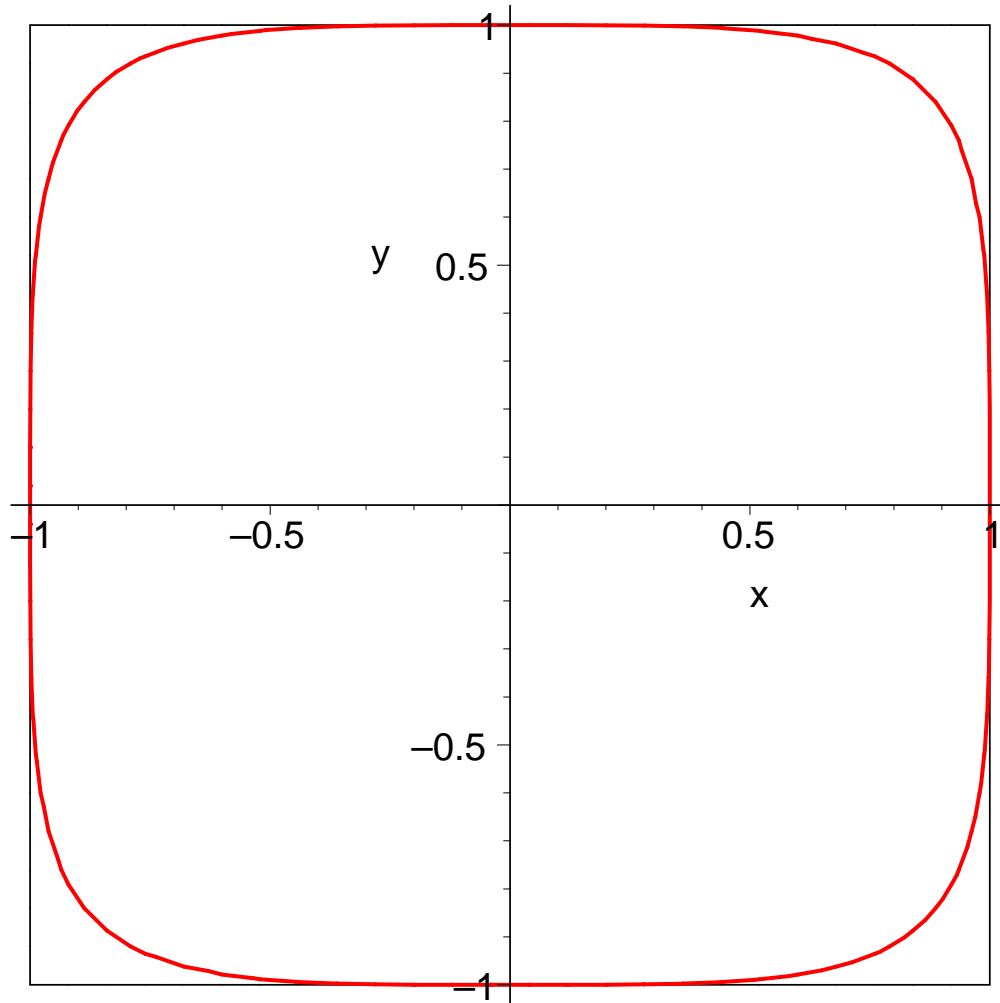
Romik: let

$$w \in \mathfrak{S}_{p^2}, \quad \text{is}(w) = \text{ds}(w) = p.$$

Let P_w be the permutation matrix of w with corners $(\pm 1, \pm 1)$. Then (informally) as $p \rightarrow \infty$ almost surely the 1's in P_w will become dense in the region bounded by the curve

$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3,$$

and will remain isolated outside this region.



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Distribution of $\text{is}(w)$

$$\begin{aligned} \mathbf{E}(n) &= \text{expectation of } \text{is}(w), \quad w \in \mathfrak{S}_n \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda \right)^2 \end{aligned}$$

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Hammersley (1972):

$$\exists c = \lim_{n \rightarrow \infty} \frac{E(n)}{\sqrt{n}},$$

and

$$\frac{\pi}{2} \leq c \leq e.$$

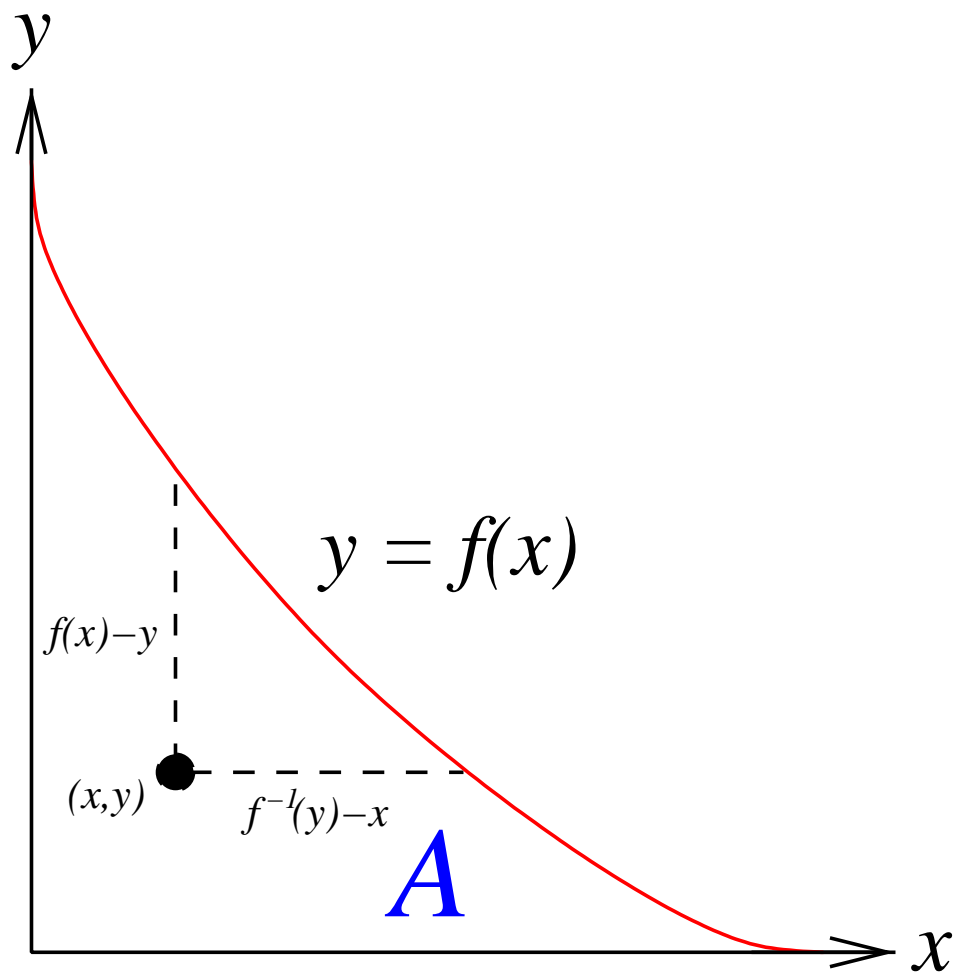
Conjectured $c = 2$.

Logan-Shepp, Vershik-Kerov (1977):
 $c = 2$

Idea of proof.

$$\begin{aligned} E(n) &= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda \right)^2 \\ &\approx \frac{1}{n!} \max_{\lambda \vdash n} \lambda_1 \left(f^\lambda \right)^2. \end{aligned}$$

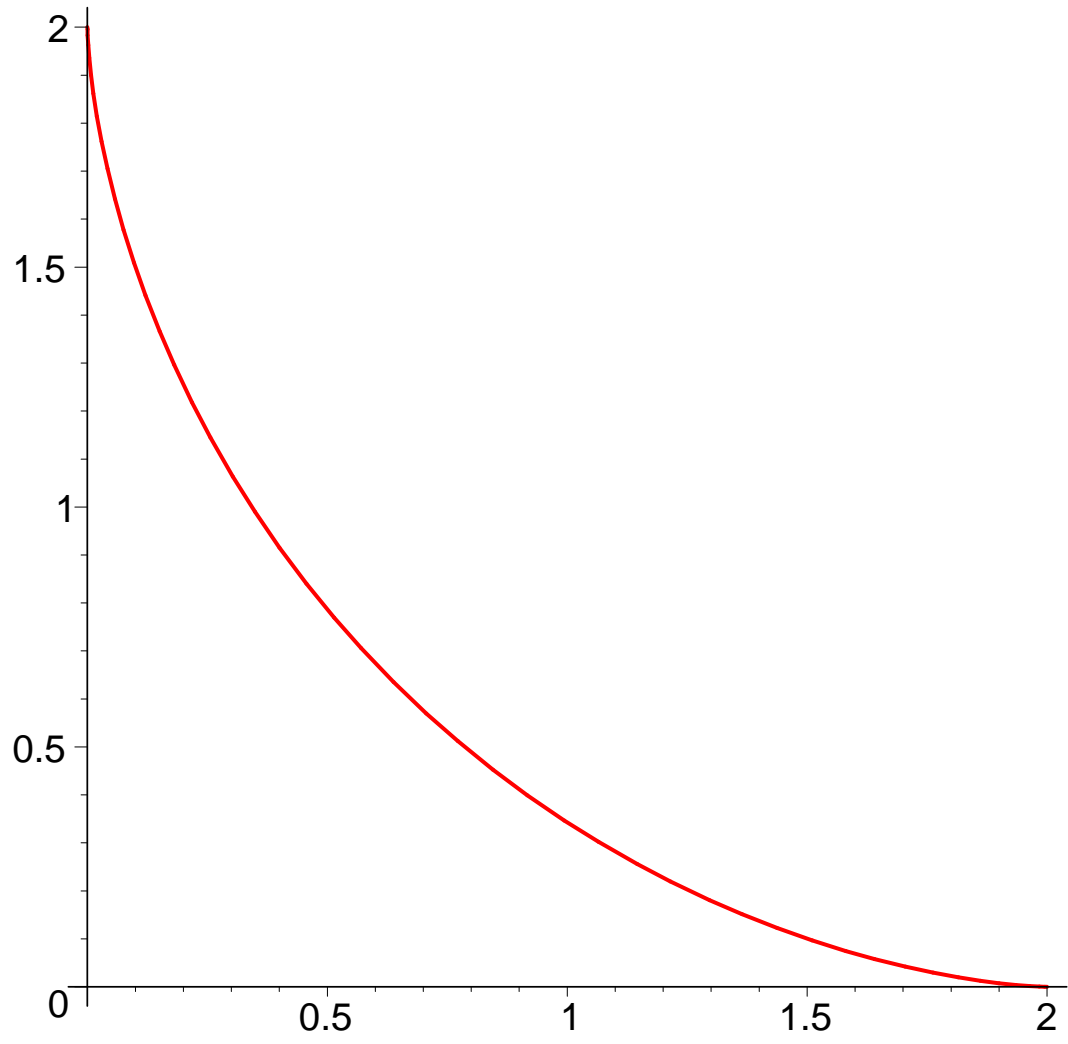
Find “limiting shape” of $\lambda \vdash n$ maximizing λ as $n \rightarrow \infty$ using hook-length formula.

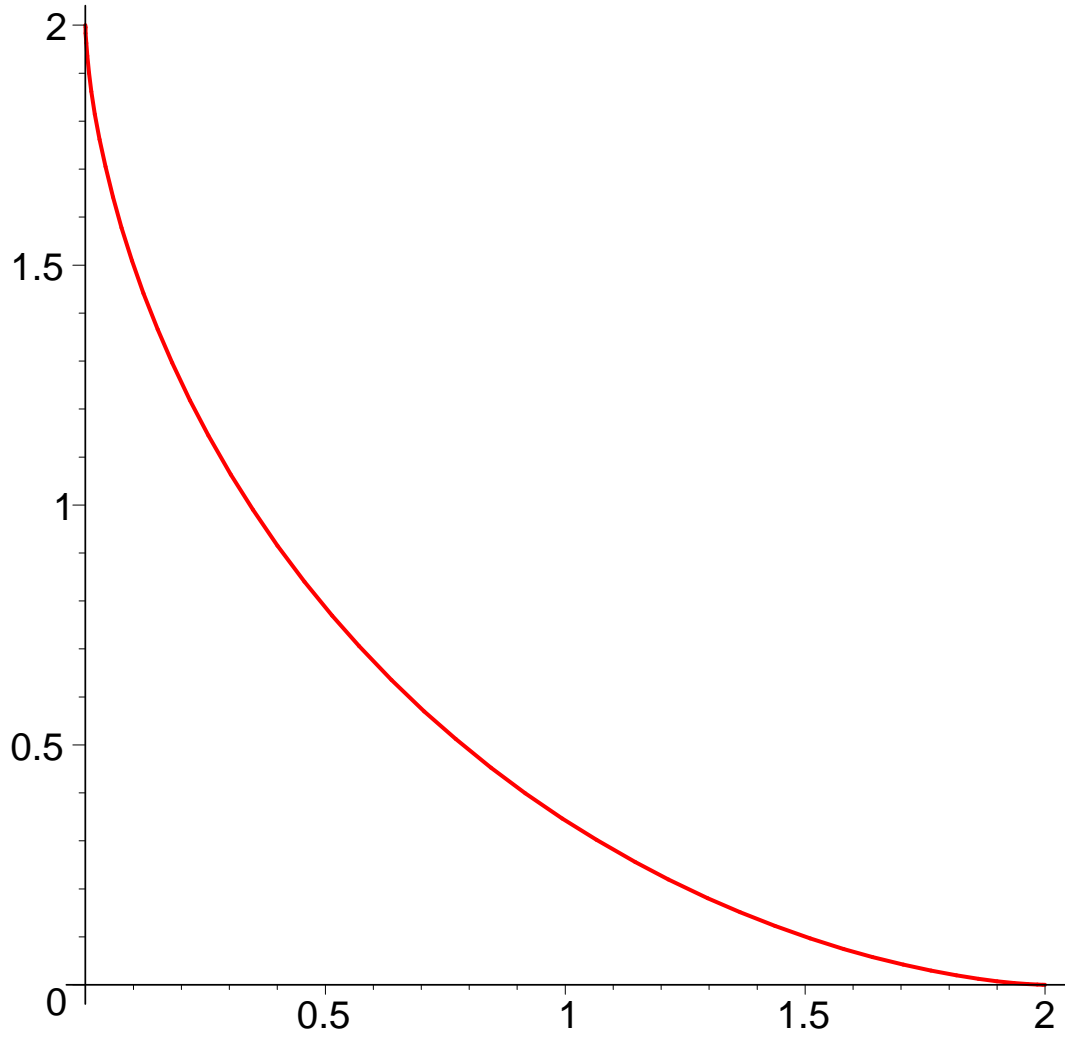


$$\min \iint_A \log(f(x) + f^{-1}(y) - x - y) dx dy,$$

subject to

$$\iint_A dx dy = 1.$$





$$x = y + 2 \cos \theta$$

$$y = \frac{2}{\pi}(\sin \theta - \theta \cos \theta)$$

$$0 \leq \theta \leq \pi$$

$$u_k(n) := \#\{w \in \mathfrak{S}_n : \text{is}_n(w) \leq k\}.$$

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$$u_2(n) = C_n = \frac{1}{n+1} \binom{2n}{n},$$

a **Catalan number**.

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For ≥ 130 combinatorial interpretations of C_n , see

`www-math.mit.edu/~rstan/ec`

I. Gessel (1990):

$$\sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det \left[I_{|i-j|}(2x) \right]_{i,j=1}^k,$$

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$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!},$$

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E.g.,

$$\begin{aligned} \sum_{n \geq 0} u_2(n) \frac{x^{2n}}{n!^2} &= U_0(2x)^2 - U_1(2x)^2 \\ &= \sum_{n \geq 0} C_n \frac{x^{2n}}{n!^2}. \end{aligned}$$

Corollary. For fixed k , $u_k(n)$ is **P-recursive**, e.g.,

$$\begin{aligned} & (n+4)(n+3)^2 u_4(n) \\ = & (20n^3 + 62n^2 + 22n - 24)u_4(n-1) \\ & - 64n(n-1)^2 u_4(n-2) \end{aligned}$$

$$\begin{aligned} & (n+6)^2(n+4)^2 u_5(n) \\ = & (375 - 400n - 843n^2 - 322n^3 - 35n^4)u_5(n-1) \\ & + (259n^2 + 622n + 45)(n-1)^2 u_5(n-2) \\ & - 225(n-1)^2(n-2)^2 u_5(n-3). \end{aligned}$$

Conjectures on form of recurrence due to Bergeron, Favreau, and Krob.

Baik-Deift-Johansson:

Define $u(x)$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

with certain initial conditions.

$(*)$ is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

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Tracy-Widom distribution:

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Theorem (Baik-Deift-Johansson) *For random (uniform) $w \in \mathfrak{S}_n$ and all $t \in \mathbb{R}$ we have*

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t).$$

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Corollary.

$$\begin{aligned} \text{is}_n(w) &= 2\sqrt{n} + \left(\int t dF(t) \right) n^{1/6} + o(n^{1/6}) \\ &= 2\sqrt{n} - (1.7711 \dots) n^{1/6} + o(n^{1/6}) \end{aligned}$$

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Where did the Tracy-Widom distribution $F(t)$ come from?

$$F(t) = \exp \left(- \int_t^\infty (x - t) u(x)^2 dx \right)$$

$$\frac{d^2}{dx^2} u(x) = 2u(x)^3 + xu(x) \quad (*),$$

Gaussian Unitary Ensemble (GUE):

Consider an $n \times n$ hermitian matrix $\mathbf{M} = (M_{ij})$ with probability density

$$Z_n^{-1} e^{-\text{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii} \cdot \prod_{i < j} d(\text{Re}(M_{ij})) d(\text{Im}(M_{ij})),$$

where Z_n is a normalization constant.

Tracy-Widom (1994): let α_1 denote the largest eigenvalue of M . Then

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\left(\alpha_1 - \sqrt{2n} \right) \sqrt{2n}^{1/6} \leq t \right) = F(t).$$

Is the connection between $\text{is}(w)$ and GUE a coincidence?

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Okounkov provides a connection, via the theory of **random topologies on surfaces**. Very briefly, a surface can be described in two ways:

- Gluing polygons along their edges, connected to random matrices via quantum gravity.
- Ramified covering of a sphere, which can be formulated in terms of permutations.

Joint with:

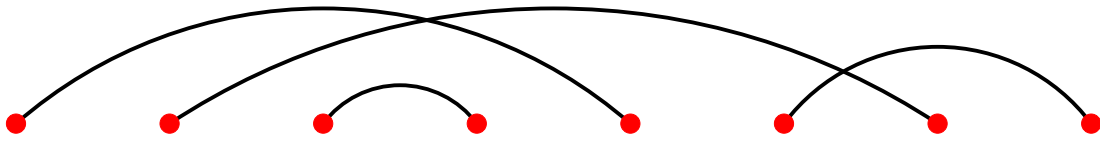
Bill Chen 陈永川

Eva Deng 邓玉平

Rosena Du 杜若霞

Catherine Yan 颜华菲

(complete) matching:



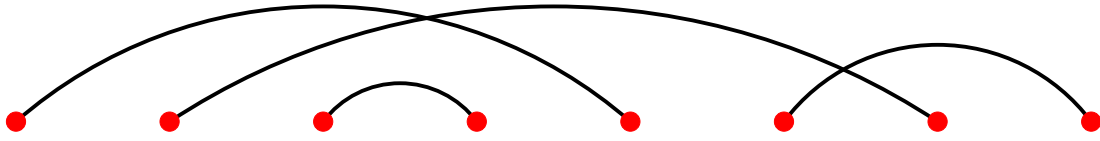
crossing:



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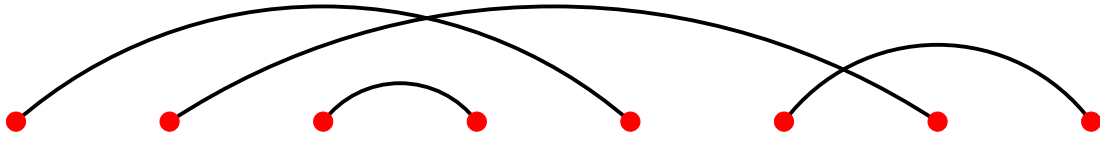
nesting:



total number of matchings on $[2n] := \{1, 2, \dots, 2n\}$ is

$$(2n - 1)!! := 1 \cdot 3 \cdot 5 \cdots (2n - 1).$$

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Theorem. *The number of matchings on $[2n]$ with no crossings (or with no nestings) is*

$$C_n := \frac{1}{n+1} \binom{2n}{n}.$$

Well-known:

$$C_n = \#\{a_1 \cdots a_{2n} : a_i = \pm 1,$$

$$a_1 + \cdots + a_i \geq 0, \sum a_i = 0\}$$

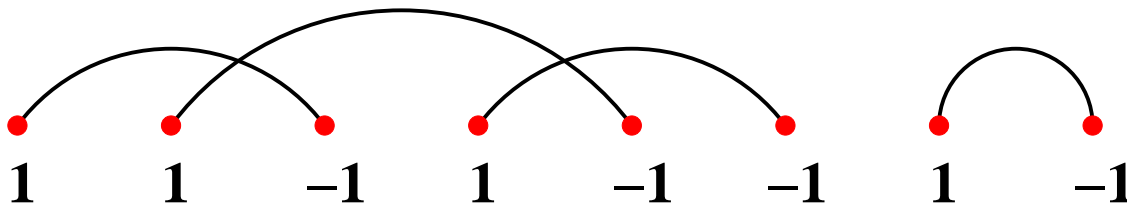
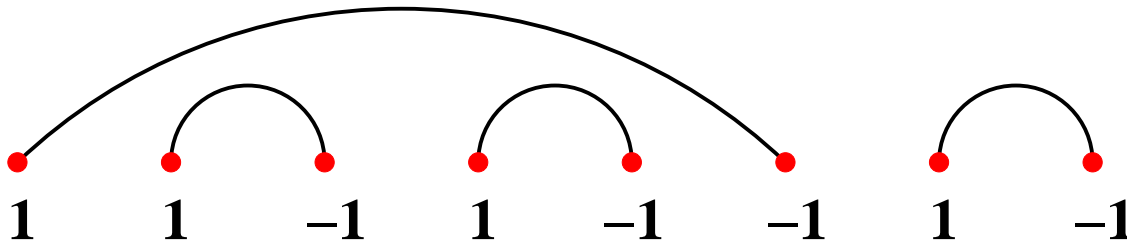
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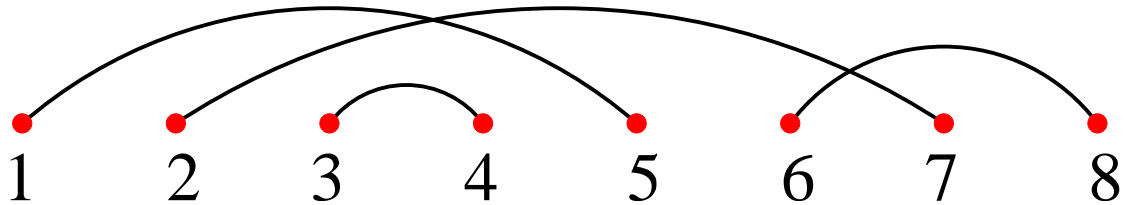
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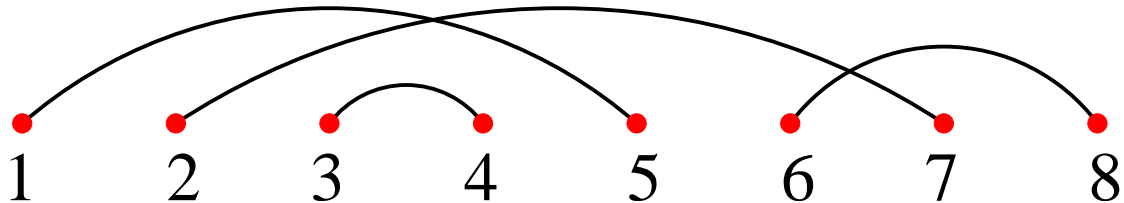
Associate with a matching M on the vertices $1, 2, \dots, 2n$ a fixed-point free involution $w_M \in \mathfrak{S}_{2n}$:



$$w_M = (1, 5)(2, 7)(3, 4)(6, 8)$$

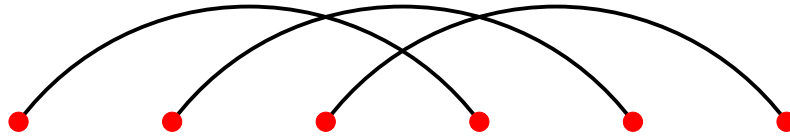
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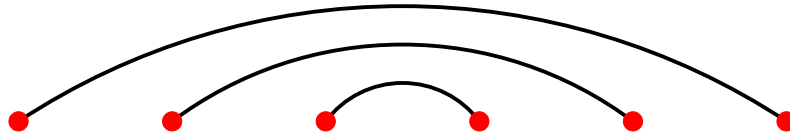


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Flaw: no symmetry between is and ds (different distributions on fixed-point free involutions).



3-crossing

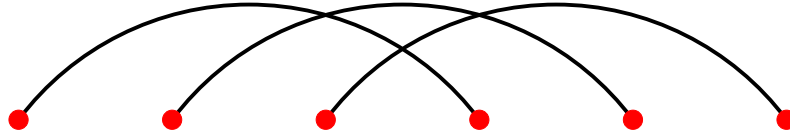


3-nesting

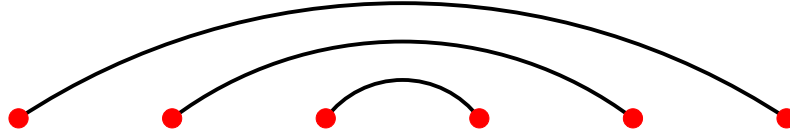
M = matching

$$\mathbf{cr}(M) = \max\{k : \exists k\text{-crossing}\}$$

$$\mathbf{ne}(M) = \max\{k : \exists k\text{-nesting}\} = \frac{1}{2}\mathbf{ds}(w_M)$$



3-crossing



3-nesting

M = matching

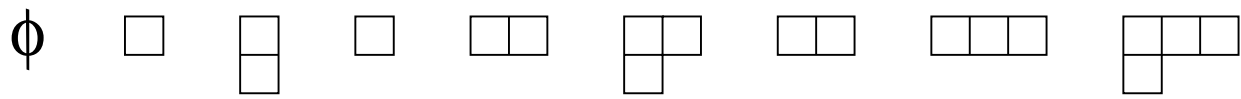
$\mathbf{cr}(M) = \max\{k : \exists k\text{-crossing}\}$

$\mathbf{ne}(M) = \max\{k : \exists k\text{-nesting}\} = \frac{1}{2}\mathbf{ds}(w_M)$

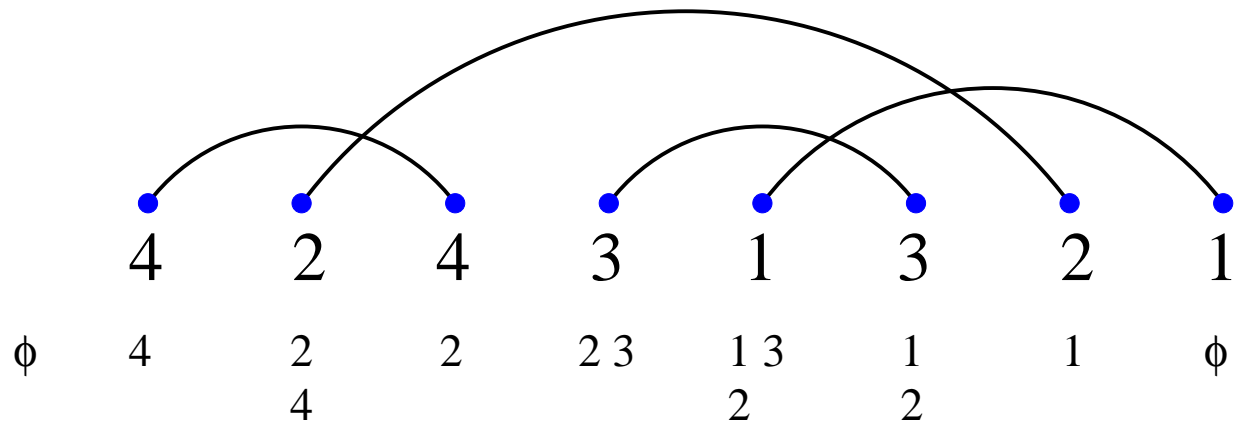
Theorem. Let $f_n(i, j) = \#$ matchings M on $[2n]$ with $\mathbf{cr}(M) = i$ and $\mathbf{ne}(M) = j$. Then $\mathbf{f}_n(\mathbf{i}, \mathbf{j}) = \mathbf{f}_n(\mathbf{j}, \mathbf{i})$.

Corollary. $\#$ matchings M on $[2n]$ with $\mathbf{cr}(M) = k$ equals $\#$ matchings M on $[2n]$ with $\mathbf{ne}(M) = k$.

Main tool: oscillating tableaux.



shape $(3, 1)$, length 8



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Schensted's theorem for matchings. *Let*

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset).$$

Then

$$\begin{aligned} \text{cr}(M) &= \max\{(\lambda^i)'_1 : 0 \leq i \leq n\} \\ \text{ne}(M) &= \max\{\lambda^i_1 : 0 \leq i \leq n\}. \end{aligned}$$

Proof. Reduce to ordinary RSK.

Now let $\text{cr}(M) = i$, $\text{ne}(M) = j$, and
 $\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset)$.

Define M' by

$$\Phi(M') = (\emptyset = (\lambda^0)', (\lambda^1)', \dots, (\lambda^{2n})' = \emptyset).$$

By Schensted's theorem for matchings,

$$\text{cr}(M') = j, \quad \text{ne}(M') = i.$$

Thus $M \mapsto M'$ is an involution on matchings of $[2n]$ interchanging cr and ne .

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Open: simple description of $M \mapsto M'$, the analogue of

$$a_1 a_2 \cdots a_n \mapsto a_n \cdots a_2 a_1,$$

which interchanges is and ds .

Enumeration of k -noncrossing matchings (or nestings).

Recall: The number of matchings M on $[2n]$ with no crossings, i.e., $\text{cr}(M) = 1$, (or with no nestings) is $C_n = \frac{1}{n+1} \binom{2n}{n}$.

What about the number with $\text{cr}(M) \leq k$?

Assume $\text{cr}(M) \leq k$. Let

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset).$$

Regard each $\lambda^i = (\lambda_1^i, \dots, \lambda_k^i) \in \mathbb{N}^k$.

Corollary. *The number $f_k(n)$ of matchings M on $[2n]$ with $\text{cr}(M) \leq k$ is the number of lattice paths of length $2n$ from $\mathbf{0}$ to $\mathbf{0}$ in the region*

$$\mathcal{C}_n := \{(a_1, \dots, a_k) \in \mathbb{N}^k : a_1 \leq \dots \leq a_k\}$$

with steps $\pm e_i$ ($e_i = i$ th unit coordinate vector).

$\mathcal{C}_n \otimes \mathbb{R}_{\geq 0}$ is a fundamental chamber for the Weyl group of type B_k .

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Theorem. *Define*

$$\mathbf{H}_k(\mathbf{x}) = \sum_n f_k(n) \frac{x^{2n}}{(2n)!}.$$

Then

$$H_k(x) = \det \left[I_{|i-j|}(2x) - I_{i+j}(2x) \right]_{i,j=1}^k$$

where

$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$$

as before.

Example. $k = 1$ (noncrossing matchings):

$$\begin{aligned} H_1(x) &= I_0(2x) - I_2(2x) \\ &= \sum_{j \geq 0} C_j \frac{x^{2j}}{(2j)!}. \end{aligned}$$

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Compare:

$u_k(n) := \#\{w \in \mathfrak{S}_n : \text{longest increasing subsequence of length } \leq k\}.$

$$\sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n! 2^n} = \det [I_{i-j}(2x)]_{i,j=1}^k.$$

Baik-Rains (implicitly):

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\frac{\text{cr}_n(M) - \sqrt{2n}}{(2n)^{1/6}} \leq \frac{t}{2} \right) = F_1(t),$$

where

$$F_1(t) = \sqrt{F(t)} \exp \left(\frac{1}{2} \int_t^\infty u(x) dx \right),$$

where $F(t)$ is the Tracy-Widom distribution and $u(x)$ the Painlevé II function.

$$F(t) = \exp \left(- \int_t^\infty (x - t) u(x)^2 dx \right)$$

$$\frac{d^2}{dx^2} u(x) = 2u(x)^3 + xu(x)$$

$$g_{j,k}(n) := \#\{\text{matchings } M \text{ on } [2n], \\ \text{cr}(M) \leq j, \text{ne}(M) \leq k\}$$

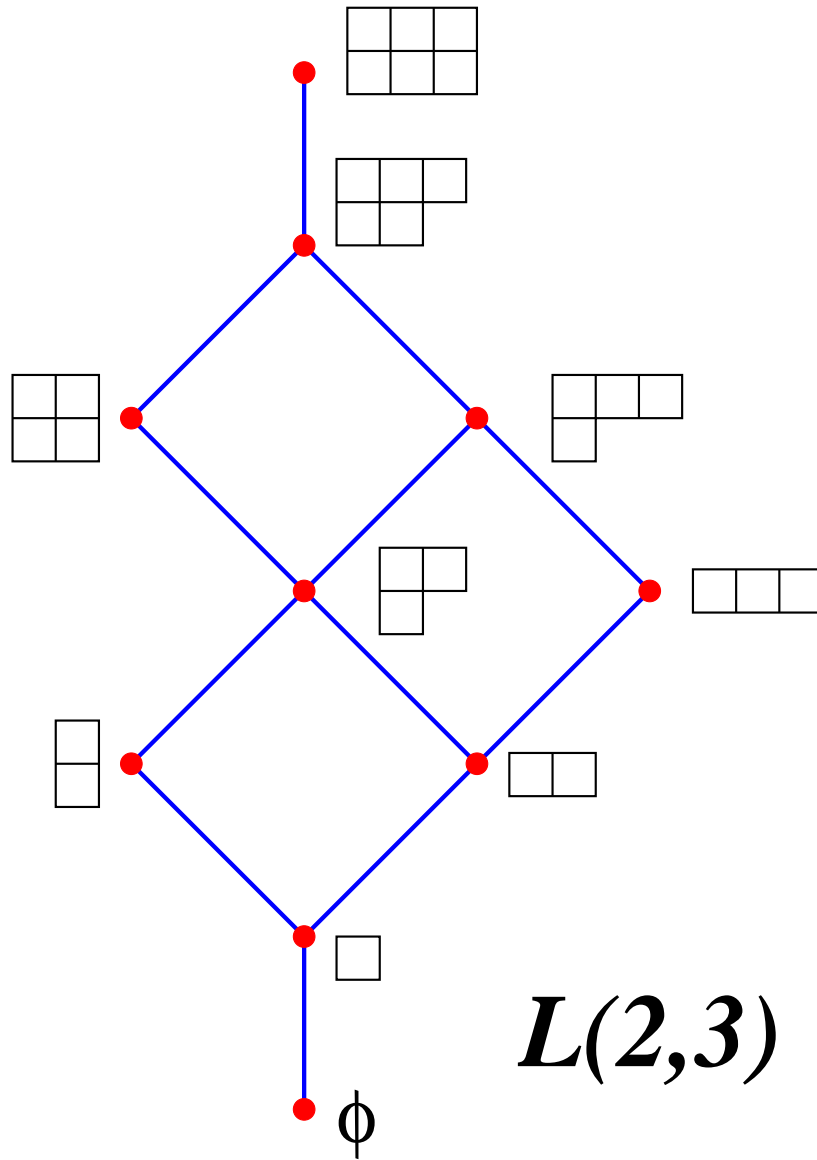
Now

$$g_{j,k}(n) = \#\{(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^{2n} = \emptyset) : \\ \lambda^{i+1} = \lambda^i \pm \square, \lambda^i \subseteq j \times k \text{ rectangle}\},$$

a walk on the Hasse diagram $\mathcal{H}(j, k)$ of

$$\mathbf{L}(j, k) := \{\lambda \subseteq j \times k \text{ rectangle}\},$$

ordered by inclusion.



$L(2,3)$

\mathbf{A} = adjacency matrix of $\mathcal{H}(j, k)$

\mathbf{A}_0 = adjacency matrix of $\mathcal{H}(j, k) - \{\emptyset\}$.

Transfer-matrix method \Rightarrow

$$\sum_{n \geq 0} g_{j,k}(n) x^{2n} = \frac{\det(I - xA_0)}{\det(I - xA)}.$$

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where each $r_i \in \mathbb{Z}$ and $m = j + k + 1$.

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where each $r_i \in \mathbb{Z}$ and $m = j + k + 1$.

Corollary. *Every factor of $\det(I - xA)$ over \mathbb{Q} has degree dividing*

$$\frac{1}{2}\phi(2(j + k + 1)),$$

where ϕ is the Euler phi-function.

Example.

$$j = 2, k = 5, \frac{1}{2}\phi(16) = 4:$$

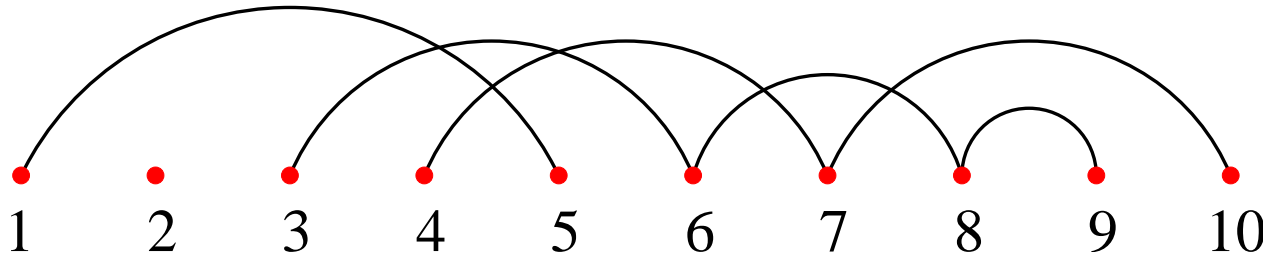
$$\begin{aligned} \det(I - xA) &= (1 - 2x^2)(1 - 4x^2 + 2x^4) \\ &\quad (1 - 8x^2 + 8x^4)(1 - 8x^2 + 8x^3 - 2x^4) \\ &\quad (1 - 8x^2 - 8x^3 - 2x^4) \end{aligned}$$

$$j = k = 3, \frac{1}{2}\phi(14) = 3:$$

$$\begin{aligned} \det(I - xA) &= (1 - x)(1 + x)(1 + x - 9x^2 - x^3) \\ &\quad (1 - x - 9x^2 + x^3)(1 - x - 2x^2 + x^3)^2 \\ &\quad (1 + x - 2x^2 - x^3)^2 \end{aligned}$$

Partition of the set $[n]$:

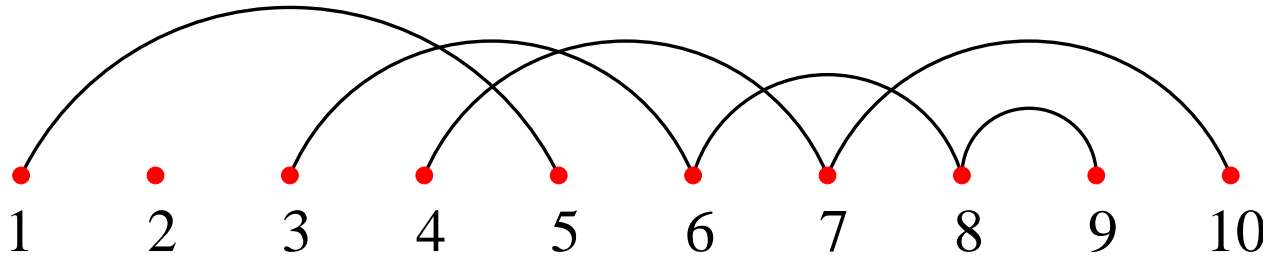
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Many other variations: see paper!