Increasing and Decreasing Subsequences

Richard P. Stanley

M.I.T.
**Definitions**

- **is**$(w) = |\text{longest i.s.}| = 4$
- **ds**$(w) = |\text{longest d.s.}| = 3$
Naive model: passengers board in order $w = a_1 a_2 \cdots a_n$ for seats 1, 2, \ldots, n. Each passenger takes one time unit to be seated after arriving at his seat.
Boarding process

Increasing and Decreasing Subsequences – p. 4
Boarding process

6 5 4 3 2 1

2 5 3 6 1 4

2536 14
Boarding process

6 5 4 3 2 1

2536 14

5 3 6 4

2 1
Boarding process

6 5 4 3 2 1

2536 14

5 3 6 4

6 5 4 3 2 1

2536 14

5 3 6 4

6 5 4 3 2 1

Increasing and Decreasing Subsequences – p. 4
Results

**Easy:** Total waiting time $= iS(w)$.

**Bachmat, et al.** more sophisticated model.
Results

**Easy:** Total waiting time $= i_s(w)$.

**Bachmat, et al.:** more sophisticated model.

**Two conclusions:**

- Usual system (back-to-front) not much better than random.
- Better: first board window seats, then center, then aisle.
**partition** $\lambda \vdash n$: $\lambda = (\lambda_1, \lambda_2, \ldots)$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$

$$\sum \lambda_i = n$$
(Young) diagram of $\lambda = (4, 4, 3, 1)$:
\( \lambda' = (4, 3, 3, 2) \), the **conjugate** partition to 
\( \lambda = (4, 4, 3, 2) \)
standard Young tableau (SYT) of shape $\lambda \vdash n$, e.g., $\lambda = (4, 4, 3, 1)$:
\[ f^\lambda = \# \text{ of SYT of shape } \lambda \]

E.g., \( f^{(3,2)} = 5: \)

\[
\begin{array}{cccccc}
1 & 2 & 3 & 1 & 2 & 4 \\
4 & 5 & 3 & 5 & 3 & 4 \\
1 & 2 & 5 & 1 & 3 & 4 \\
1 & 2 & 5 & 1 & 3 & 5 \\
1 & 3 & 4 & 2 & 5 & 2 \\
1 & 3 & 5 & 2 & 4 & \\
\end{array}
\]
\( f^\lambda = \# \text{ of SYT of shape } \lambda \)

E.g., \( f^{(3,2)} = 5 \):

\[
\begin{array}{cccccc}
1 & 2 & 3 & 1 & 2 & 4 \\
4 & 5 & 3 & 5 & 3 & 4 \\
\end{array}
\begin{array}{cccccc}
1 & 2 & 5 & 1 & 3 & 4 \\
3 & 4 & 2 & 5 & 2 & 4 \\
\end{array}
\begin{array}{cccccc}
1 & 3 & 5 \\
2 & 4 \\
\end{array}
\]

∃ simple formula for \( f^\lambda \) (Frame-Robinson-Thrall hook-length formula)
$f^\lambda = \# \text{ of SYT of shape } \lambda$

E.g., $f^{(3,2)} = 5$:

$$\begin{array}{cccccc}
1 & 2 & 3 & 1 & 2 & 4 \\
1 & 2 & 5 & 1 & 3 & 4 \\
4 & 5 & 3 & 5 & 3 & 4 \\
4 & 5 & 3 & 5 & 3 & 4 \\
\end{array}$$

∃ simple formula for $f^\lambda$ (Frame-Robinson-Thrall hook-length formula)

**Note.** $f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n)$, where $\mathfrak{S}_n$ is the symmetric group of all permutations of $1, 2 \ldots, n$. 
RSK algorithm: a bijection

\[ w \xrightarrow{\text{rsk}} (P, Q), \]

where \( w \in \mathfrak{S}_n \) and \( P, Q \) are SYT of the same shape \( \lambda \vdash n \).

Write \( \lambda = \text{sh}(w) \), the shape of \( w \).
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\textbf{R} = Gilbert de Beauregard Robinson
\textbf{S} = Craige Schensted (= Ea Ea)
\textbf{K} = Donald Ervin Knuth
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[ea.ea.home.mindspring.com](http://ea.ea.home.mindspring.com)
Example of RSK: \( w = 4132 \)

- insert 4, record 1: \( \begin{array}{cc} 4 & 1 \\ \end{array} \)
- insert 1, record 2: \( \begin{array}{cc} 1 & 1 \\ 4 & 2 \\ \end{array} \)
- insert 3, record 3: \( \begin{array}{cc} 1 & 1 \\ 3 & 2 \\ 4 & 4 \\ \end{array} \)
- insert 2, record 4: \( \begin{array}{cc} 1 & 2 \\ 2 & 3 \\ 4 & 4 \\ \end{array} \)
Example of RSK: \( w = 4132 \)

insert 4, record 1: 
\[
\begin{array}{cc}
4 & 1 \\
\end{array}
\]

insert 1, record 2: 
\[
\begin{array}{cc}
1 & 1 \\
4 & 2 \\
\end{array}
\]

insert 3, record 3: 
\[
\begin{array}{cc}
1 & 3 \\
4 & 2 \\
\end{array}
\]

insert 2, record 4: 
\[
\begin{array}{cc}
1 & 2 \\
2 & 1 \\
3 & 2 \\
4 & 4 \\
\end{array}
\]

\[
(P, Q) = \begin{pmatrix}
1 & 2 \\
3 & 1 \\
4 & 3 \\
\end{pmatrix}
\]

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**Schensted’s theorem**

**Theorem.** Let \( w \overset{\text{rsk}}{\rightarrow} (P, Q) \), where \( \text{sh}(P) = \text{sh}(Q) = \lambda \). Then

\[
\begin{align*}
\text{is}(w) & = \text{longest row length} = \lambda_1 \\
\text{ds}(w) & = \text{longest column length} = \lambda'_1.
\end{align*}
\]
Schensted’s theorem

Theorem. Let $w \overset{\text{rsk}}{\rightarrow} (P, Q)$, where $sh(P) = sh(Q) = \lambda$. Then

$$\begin{align*}
is(w) &= \text{longest row length} = \lambda_1 \\
ds(w) &= \text{longest column length} = \lambda'_1.
\end{align*}$$

Example. $4132 \overset{\text{rsk}}{\rightarrow} \begin{pmatrix} 1 & 2 & & \\
3 & & & 1 & 3 \\
4 & 1 & & 2 \\
& & & & 4 \\
& & & & 4
\end{pmatrix}$

$$\begin{align*}
is(w) &= 2, \\
ds(w) &= 3.
\end{align*}$$
Corollary (Erdős-Szekeres, Seidenberg). Let \( w \in \mathcal{S}_{pq+1} \). Then either \( \text{is}(w) > p \) or \( \text{ds}(w) > q \).
Corollary (Erdős-Szekeres, Seidenberg). Let $w \in \mathcal{S}_{pq+1}$. Then either $\text{is}(w) > p$ or $\text{ds}(w) > q$.

Proof. Let $\lambda = \text{sh}(w)$. If $\text{is}(w) \leq p$ and $\text{ds}(w) \leq q$ then $\lambda_1 \leq p$ and $\lambda'_1 \leq q$, so $\sum \lambda_i \leq pq$. $\square$
An extremal case

**Corollary.** Say $p \leq q$. Then

$$\# \{ w \in \mathfrak{S}_{pq} : \text{is}(w) = p, \text{ds}(w) = q \}$$

$$= \left( f(p^q) \right)^2$$
Corollary. Say $p \leq q$. Then

$$\# \{ w \in \mathcal{S}_{pq} : \text{is}(w) = p, \text{ds}(w) = q \}$$

$$= \left( f(p^q) \right)^2$$

By hook-length formula, this is

$$\left( \frac{(pq)!}{1!^2 \cdots p^p(p + 1)^{p-1} \cdots q^q(q + 1)^{q-1} \cdots (p + q - 1)^1} \right)^2.$$
Romik’s theorem

Romik: let

\[ w \in \mathfrak{S}_{n^2}, \text{ is}(w) = \text{ds}(w) = n. \]

Let \( P_w \) be the permutation matrix of \( w \) with corners \((\pm 1, \pm 1)\). Then (informally) as \( n \to \infty \) almost surely the 1’s in \( P_w \) will become dense in the region bounded by the curve

\[
(x^2 - y^2)^2 + 2(x^2 + y^2) = 3,
\]

and will remain isolated outside this region.
An example

\[
w = 9, 11, 6, 14, 2, 10, 1, 5, 13, 3, 16, 8, 15, 4, 12, 17
\]
\[(x^2 - y^2)^2 + 2(x^2 + y^2) = 3\]
Area enclosed by curve

\[
\alpha = 8 \int_0^1 \frac{1}{\sqrt{(1 - t^2)(1 - (t/3)^2)}} dt \\
-6 \int_0^1 \sqrt{\frac{1 - (t/3)^2}{1 - t^2}} dt \\
= 4(0.94545962 \cdots)
\]
\[ E(n) = \text{expectation of } is(w), \ w \in S_n \]
\[ = \frac{1}{n!} \sum_{w \in S_n} \text{is}(w) \]
\[ = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 (f^\lambda)^2 \]
**Expectation of is(\(w\))**

\[
E(n) = \text{expectation of is}(w), \ w \in \mathfrak{S}_n
\]

\[
= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{is}(w)
\]

\[
= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left( f^\lambda \right)^2
\]

**Ulam:** what is distribution of is(\(w\))? rate of growth of \(E(n)\)?
Hammersley (1972):

\[ \exists c = \lim_{n \to \infty} n^{-1/2} E(n), \]

and

\[ \frac{\pi}{2} \leq c \leq e. \]
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\[ \exists c = \lim_{n \to \infty} n^{-1/2} E(n), \]

and

\[ \frac{\pi}{2} \leq c \leq e. \]

Conjectured \( c = 2. \)
Logan-Shepp, Vershik-Kerov (1977): $c = 2$
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Idea of proof.

$$E(n) = \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 (f^{\lambda})^2 \approx \frac{1}{n!} \max_{\lambda \vdash n} \lambda_1 (f^{\lambda})^2.$$ 

Find “limiting shape” of $\lambda \vdash n$ maximizing $\lambda$ as $n \to \infty$ using hook-length formula.
The limiting curve
Equation of limiting curve

\[ x = y + 2 \cos \theta \]
\[ y = \frac{2}{\pi} (\sin \theta - \theta \cos \theta) \]
\[ 0 \leq \theta \leq \pi \]
is(w) ≤ 2

\[ u_k(n) := \# \{ w \in S_n : is_n(w) \leq k \} . \]
\( \text{is}(w) \leq 2 \)

\[ u_k(n) := \# \{ w \in \mathcal{S}_n : \text{is}_n(w) \leq k \}. \]

**J. M. Hammersley (1972):**

\[ u_2(n) = C_n = \frac{1}{n + 1} \binom{2n}{n}, \]

a **Catalan number**.
\[u_k(n) := \# \{ w \in S_n : \text{is}_n(w) \leq k \} .\]

**J. M. Hammersley (1972):**

\[u_2(n) = C_n = \frac{1}{n+1} \binom{2n}{n},\]

a **Catalan number**.

For \(\geq 160\) combinatorial interpretations of \(C_n\), see

[www-math.mit.edu/~rstan/ec](http://www-math.mit.edu/~rstan/ec)
Gessel’s theorem

I. Gessel (1990):

\[ \sum_{n \geq 0} u_k(n) \frac{x^{2n}}{n!^2} = \det [I_{\mid i-j \mid}(2x)]_{i,j=1}^{k}, \]

where

\[ I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}, \]

a hyperbolic Bessel function of the first kind of order \( m \).
The case $k = 2$

Example. \[
\sum_{n \geq 0} u_2(n) \frac{x^{2n}}{n!^2} \]

\[= I_0(2x)^2 - I_1(2x)^2 \]

\[= \sum_{n \geq 0} C_n \frac{x^{2n}}{n!^2}.\]
Baik-Deift-Johansson:
Define \( u(x) \) by
\[
\frac{d^2}{dx^2} u(x) = 2u(x)^3 + xu(x) \quad (*),
\]
with certain initial conditions.
Baik-Deift-Johansson:
Define \( u(x) \) by

\[
\frac{d^2}{dx^2} u(x) = 2u(x)^3 + xu(x) \quad (*),
\]

with certain initial conditions.

\( (*) \) is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

1890: Grand Prix des Sciences Mathématiques
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1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.
Paul Painlevé


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1917, 1925: Prime Minister of France.
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1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.
1917, 1925: Prime Minister of France.
1933: died in Paris.
The Tracy-Widom distribution

\[ F(t) = \exp \left( -\int_t^\infty (x - t)u(x)^2 \, dx \right) \]

where \( u(x) \) is the Painlevé II function.
The Baik-Deift-Johansson theorem

Let \( \chi \) be a random variable with distribution \( F \), and let \( \chi_n \) be the random variable on \( \mathcal{S}_n \):

\[
\chi_n(w) = \frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}}.
\]
The Baik-Deift-Johansson theorem

Let \( \chi \) be a random variable with distribution \( F \), and let \( \chi_n \) be the random variable on \( \mathcal{S}_n \):

\[
\chi_n(w) = \frac{is_n(w) - 2\sqrt{n}}{n^{1/6}}.
\]

**Theorem.** As \( n \to \infty \),

\[
\chi_n \to \chi \quad \text{in distribution,}
\]

i.e.,

\[
\lim_{n \to \infty} \text{Prob}(\chi_n \leq t) = F(t).
\]
Recall $E(n) \sim 2\sqrt{n}$. 
Recall $E(n) \sim 2\sqrt{n}$.

Corollary to BDJ theorem.

\[
E(n) = 2\sqrt{n} + \left(\int t \, dF(t)\right) n^{1/6} + o(n^{1/6})
\]

\[
= 2\sqrt{n} - (1.7711 \cdots) n^{1/6} + o(n^{1/6})
\]
Gessel’s theorem reduces the problem to “just” analysis, viz., the **Riemann-Hilbert problem** in the theory of integrable systems, and the **method of steepest descent** to analyze the asymptotic behavior of integrable systems.
Where did the Tracy-Widom distribution $F(t)$ come from?

$$F(t) = \exp \left( - \int_t^\infty (x - t)u(x)^2 \, dx \right)$$

$$\frac{d^2}{dx^2} u(x) = 2u(x)^3 + xu(x)$$
Analogue of normal distribution for $n \times n$ hermitian matrices $M = (M_{ij})$: 

$$Z_n$$ is a normalization constant.
Gaussian Unitary Ensemble (GUE)

Analogue of normal distribution for $n \times n$ hermitian matrices $M = (M_{ij})$:

$$Z_n^{-1} e^{-\text{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii} \cdot \prod_{i<j} d(\Re M_{ij}) d(\Im M_{ij}),$$

where $Z_n$ is a normalization constant.
Tracy-Widom theorem (1994): let $\alpha_1$ denote the largest eigenvalue of $M$. Then

$$\lim_{n \to \infty} \text{Prob} \left( \left( \alpha_1 - \sqrt{2n} \right) \sqrt{2n^{1/6}} \leq t \right) = F(t).$$
Random topologies

Is the connection between \( w \) and GUE a coincidence?
Is the connection between $\pi_1(w)$ and GUE a coincidence?

Okounkov provides a connection, via the theory of **random topologies on surfaces**. Very briefly, a surface can be described in two ways:

- **Gluing polygons along their edges**, connected to random matrices via quantum gravity.

- **Ramified covering of a sphere**, which can be formulated in terms of permutations.
A variation

**Alternating sequence** of length $k$:

\[ b_1 > b_2 < b_3 > b_4 < \cdots < b_k \]

$E_n$: number of alternating $w \in \mathcal{S}_n$ (**Euler number**)

$E_4 = 5$: 2134, 3142, 3241, 4132, 4231
Alternating sequence of length $k$:

$$b_1 > b_2 < b_3 > b_4 < \cdots < b_k$$

$E_n$: number of alternating $w \in \mathcal{S}_n$ (Euler number)

$E_4 = 5$: 2134, 3142, 3241, 4132, 4231

Désiré André (1840–1917): showed in 1879 that

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$
Alternating subsequences?

\[ \text{as}(w) = \text{length of longest alternating subseq. of } w \]
Alternating subsequences?

\[ as(w) = \text{length of longest alternating subseq. of } w \]

\[ w = 56218347 \Rightarrow as(w) = 5 \]
\textbf{MAIN LEMMA.} \( \forall \ w \in \mathcal{S}_n \ \exists \text{ alternating subsequence of maximal length that contains } n \).
The main lemma

**MAIN LEMMA.** \( \forall w \in S_n \exists \text{ alternating subsequence of maximal length that contains } n. \)

\[
\begin{align*}
a_k(n) &= \# \{ w \in S_n : \text{as}(w) = k \} \\
b_k(n) &= a_1(n) + a_2(n) + \cdots + a_k(n) \\
&= \# \{ w \in S_n : \text{as}(w) \leq k \}.
\end{align*}
\]
Recurrence for $a_k(n)$

$$a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} (a_{2r}(j-1) + a_{2r+1}(j-1)) a_s(n-j)$$
Define

\[ B(x, t) = \sum_{k,n \geq 0} b_k(n) t^k \frac{x^n}{n!} \]

\[ A(x, t) = \sum_{k,n \geq 0} a_k(n) t^k \frac{x^n}{n!} \]
The main generating function

**Theorem.**

\[
B(x, t) = \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho}
\]

\[
A(x, t) = (1 - t) B(x, t),
\]

where \( \rho = \sqrt{1 - t^2} \).
Formulas for $b_k(n)$

Corollary.

\[ 1 \Rightarrow b_1(n) = 1 \]
\[ b_2(n) = n \]
\[ b_3(n) = \frac{1}{4}(3^n - 2n + 3) \]
\[ b_4(n) = \frac{1}{8}(4^n - (2n - 4)2^n) \]
\[ \vdots \]
Formulas for $b_k(n)$

Corollary.

\[ \Rightarrow b_1(n) = 1 \]
\[ b_2(n) = n \]
\[ b_3(n) = \frac{1}{4}(3^n - 2n + 3) \]
\[ b_4(n) = \frac{1}{8}(4^n - (2n - 4)2^n) \]
\[ \vdots \]

no such formulas for longest increasing subsequences
Mean (expectation) of $as(w)$

$$D(n) = \frac{1}{n!} \sum_{w \in S_n} as(w),$$

the expectation of $as(w)$ for $w \in S_n$
A formula for $D(n)$

$$\sum_{n \geq 1} D(n)x^n = \frac{\partial}{\partial t} A(x, 1)$$

$$= 6x - 3x^2 + x^3$$

$$= \frac{6(1 - x)^2}{6(1 - x)^2}$$

$$= x + \sum_{n \geq 2} \frac{4n + 1}{6} x^n.$$
A formula for $D(n)$

$$\sum_{n \geq 1} D(n) x^n = \frac{\partial}{\partial t} A(x, 1)$$

$$= \frac{6x - 3x^2 + x^3}{6(1 - x)^2}$$

$$= x + \sum_{n \geq 2} \frac{4n + 1}{6} x^n.$$ 

$$\Rightarrow D(n) = \frac{4n + 1}{6}, \quad n \geq 2$$
Comparison of $E(n)$ and $D(n)$

\[ D(n) = \frac{4n + 1}{6}, \quad n \geq 2 \]

\[ E(n) \sim 2\sqrt{n} \]
Variance of \( as(w) \)

\[
V(n) = \frac{1}{n!} \sum_{w \in S_n} \left( as(w) - \frac{4n + 1}{6} \right)^2, \quad n \geq 2
\]

the variance of \( as(n) \) for \( w \in S_n \)
Variance of $a_s(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} \left( a_s(w) - \frac{4n + 1}{6} \right)^2, \ n \geq 2$$

the **variance** of $a_s(n)$ for $w \in \mathcal{S}_n$

**Corollary.**

$$V(n) = \frac{8}{45}n - \frac{13}{180}, \ n \geq 4$$
Variance of $\text{as}(w)$

\[
V(n) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} \left( \text{as}(w) - \frac{4n + 1}{6} \right)^2, \quad n \geq 2
\]

the variance of $\text{as}(n)$ for $w \in \mathcal{S}_n$

Corollary.

\[
V(n) = \frac{8}{45} n - \frac{13}{180}, \quad n \geq 4
\]

similar results for higher moments
A new distribution?

\[ P(t) = \lim_{n \to \infty} \text{Prob}_{w \in \mathcal{S}_n} \left( \frac{a_{sn}(w) - 2n/3}{\sqrt{n}} \leq t \right) \]
A new distribution?

$$P(t) = \lim_{n \to \infty} \text{Prob}_{w \in S_n} \left( \frac{a_{s_n}(w) - 2n/3}{\sqrt{n}} \leq t \right)$$

Stanley distribution?
Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

\[
\lim_{n \to \infty} \text{Prob}_{w \in \mathfrak{S}_n} \left( \frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right)
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45}/4} e^{-s^2} \, ds
\]

(Gaussian distribution)
Theorem (Pemantle, Widom, (Wilf)).

$$\lim_{n \to \infty} \operatorname{Prob}_{w \in S_n} \left( \frac{as(w) - 2n/3}{\sqrt{n}} \leq t \right)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45/4}} e^{-s^2} \, ds$$

(Gaussian distribution)
Given $k \geq 1$, define a sequence $a_1a_2\cdots a_n$ of integers to be $k$-alternating if

$$a_i > a_{i+1} \iff i \equiv 1 \pmod{k}.$$
**k-alternating sequences**

Given $k \geq 1$, define a sequence $a_1 a_2 \cdots a_n$ of integers to be **$k$-alternating** if

$$a_i > a_{i+1} \iff i \equiv 1 \pmod{k}.$$ 

**Example.** 61482572 is 3-alternating
$a_k(w)$ and $E_k(n)$

$a_k(w)$ : length of longest $k$–alt. subsequence of $w$
$a_k(w)$ and $E_k(n)$

$a_k(w)$: length of longest $k$-alt. subsequence of $w$

$$a_{n-1}(w) = \text{is}(w)$$

$$a_2(w) = \text{as}(w)$$
$a_k(w)$ and $E_k(n)$

$a_k(w)$: length of longest $k$-alt. subsequence of $w$

\[
a_{n-1}(w) = \text{is}(w) \\
a_2(w) = \text{as}(w)
\]

$E_k(n) = \text{expectation of } a_k(w)$

\[
= \frac{1}{n!} \sum_{w \in \mathcal{S}_n} a_k(w)
\]
A problem

$E_k(n)$ interpolates between $E(n) \sim 2\sqrt{n}$ and $D(n) \sim 2n/3$. Is there a sharp cutoff between $c\sqrt{n}$ and $cn$ behavior, or do we get intermediate values like $cn^\alpha$, $\frac{1}{2} < \alpha < 1$, say for $k = \sqrt{n}$?
A problem

$E_k(n)$ interpolates between $E(n) \sim 2\sqrt{n}$ and $D(n) \sim 2n/3$. Is there a sharp cutoff between $c\sqrt{n}$ and $cn$ behavior, or do we get intermediate values like $cn^\alpha$, $\frac{1}{2} < \alpha < 1$, say for $k = \sqrt{n}$?

Similar questions for the limiting distribution: do we interpolate between Tracy-Widom and Gaussian?
A variant

Same questions if we replace $k$-alternating with:

$$a_i > a_{i+1} \iff \left\lfloor i/k \right\rfloor \text{ is even.}$$

E.g., $k = 3$:

$$a_1 > a_2 > a_3 < a_4 < a_5 > a_6 > a_7 < \cdots$$