Mastery of

Convex

Mathematics

Unerringly

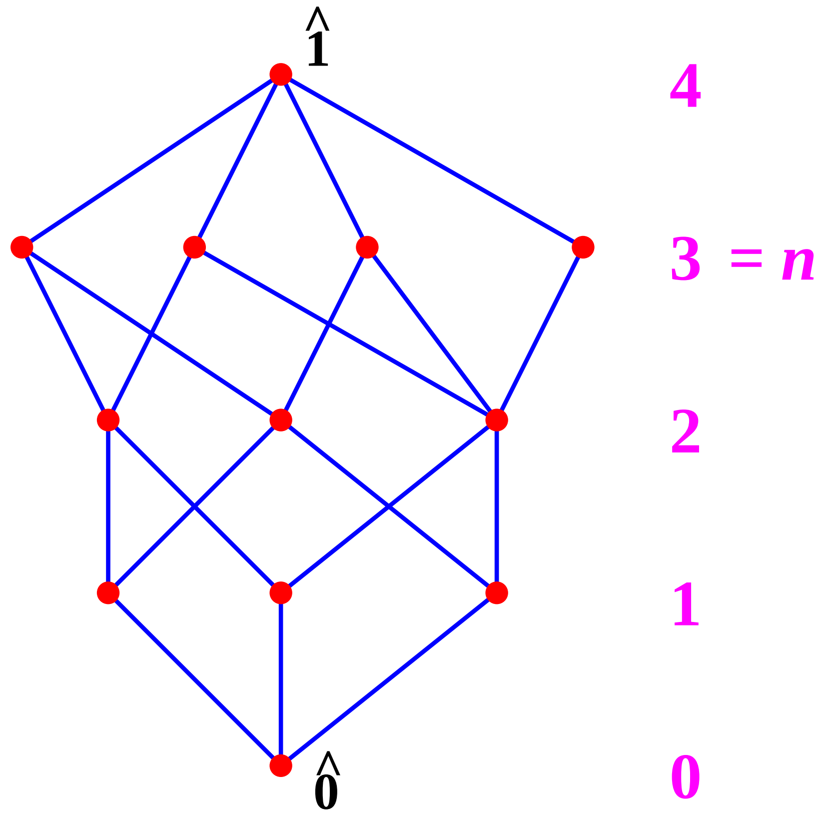
Led to

Lovely &

Enlightening

Novelties

Let P be a finite graded poset of rank $n + 1$ with $\hat{0}$ and $\hat{1}$, and with rank function ρ. Thus $\rho(\hat{0}) = 0$ and $\rho(\hat{1}) = n + 1$.



Let $S \subseteq [n] = \{1, 2, \ldots, n\}$, say

$$S = \{a_1 < a_2 < \cdots < a_k\}.$$

Define the **flag f-vector**

$$\tilde{f}(P) : 2^{[n]} \to \mathbb{N} = \{0, 1, \ldots\}$$

of P by

$$\boldsymbol{\tilde{f}_S(P)} = \#\{\hat{0} < t_1 < \cdots < t_k < \hat{1} \ : \ \rho(t_i) = a_i\}.$$

Define the **flag h-vector** $\tilde{h}(P) : 2^{[n]} \to \mathbb{N}$ of P by

$$\boldsymbol{\tilde{h}_S(P)} = \sum_{T \subseteq S} (-1)^{\#(S-T)} \tilde{f}_T(P).$$

Equivalently,

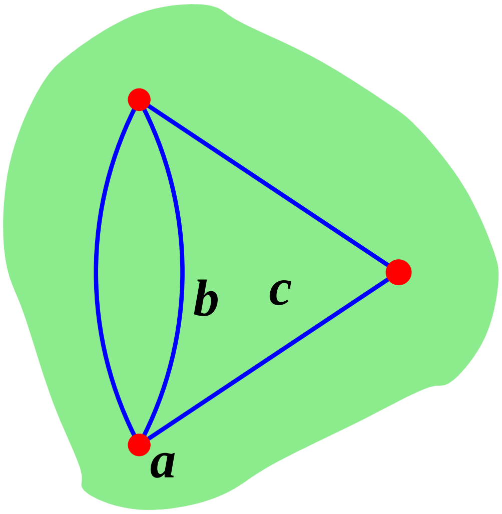
$$\tilde{f}_S(P) = \sum_{T \subseteq S} \tilde{h}_T(P).$$

EXAMPLE: P = face-lattice of 3-cube.

S	$\tilde{f}_S(P)$	$\tilde{h}_S(P)$
$\emptyset$	1	1
1	8	7
2	12	11
3	6	5
1, 2	24	5
1, 3	24	11
2, 3	24	7
1, 2, 3	48	1

Define the **order complex** $\Delta(P)$ to be the abstract simplicial complex whose faces are the chains of $P - \{\hat{0}, \hat{1}\}$. If P is the face poset of a regular CW-complex Γ (e.g., a polyhedral complex) with $\hat{1}$ adjoined, then $\Delta(P) = \mathrm{sd}(\Gamma)$, the first barycentric subdivision of Γ. Note:

$$n := \mathrm{rank}(P) - 1 = \dim(\Delta(P)) + 1.$$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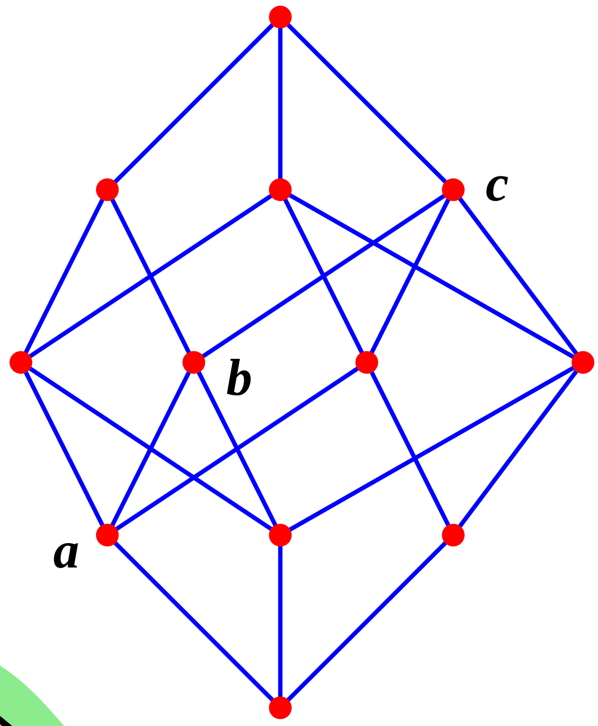
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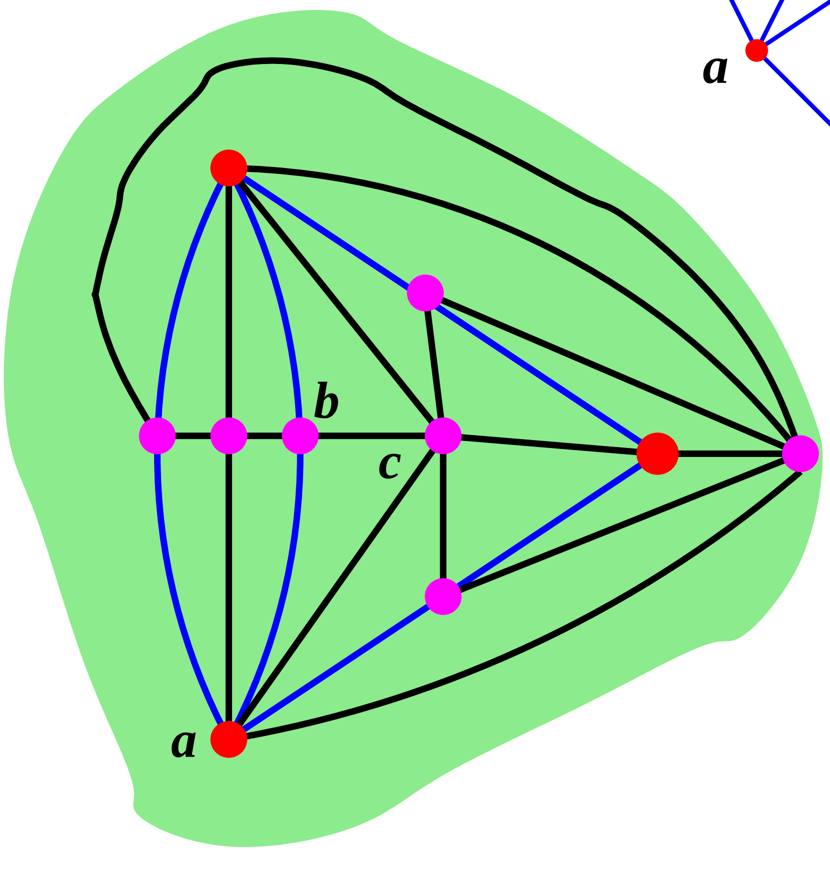
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If $\Delta \neq \emptyset$ is any $(n-1)$-dimensional simplicial complex, define the **f-vector** $(f_0, \ldots, f_{n-1})$ (with $f_{-1} = 1$) and **h-vector** $(h_0, h_1, \ldots, h_n)$ of Δ by

$$\boldsymbol{f_i} = \#\{F \in \Delta \,:\, \dim(F) = i\}$$

$$\sum_{i=0}^{n} f_{i-1}(x-1)^{n-i} = \sum_{i=0}^{n} \boldsymbol{h_i} x^{n-i}.$$

Then

$$f_i(\Delta(P)) = \sum_{\#S=i+1} \tilde{f}_S(P)$$

$$h_i(\Delta(P)) = \sum_{\#S=i} \tilde{h}_S(P).$$

Rank-selection and homology. Given $S \subseteq [n]$, define the **rank-selected subposet** $P_S \subseteq P$ by

$$\boldsymbol{P_S} = \{t \in P \,:\, t = \hat{0}, \hat{1} \text{ or } \mathrm{rank}(t) \in S\}.$$

Then

$$\begin{aligned} \tilde{f}_S(P) &= \#\text{ maximal chains of } P_S \\ \tilde{h}_S(P) &= \tilde{\chi}(\Delta(P_S)), \end{aligned}$$

where $\tilde{\chi}$ denotes reduced Euler characteritic.

Thus $\tilde{h}_S(P)$ can be investigated purely topologically, unlike h_i.

Let Δ be a simplicial complex. If $F \in \Delta$, define the **link**

$$\mathbf{lk}(\boldsymbol{F}) = \{G \in \Delta \,:\, F \cap G = \emptyset,\ F \cup G \in \Delta\},$$

so $\mathrm{lk}(\emptyset) = \Delta$.

Definition. Δ is **Cohen-Macaulay (C-M)** over the field K if

$$\tilde{H}_i(\mathrm{lk}(F); K) = 0, \quad i < \dim(\mathrm{lk}(F)),$$

for all $F \in \Delta$. Equivalently, the **face ring** $K[\Delta]$ is a Cohen-Macaulay ring.

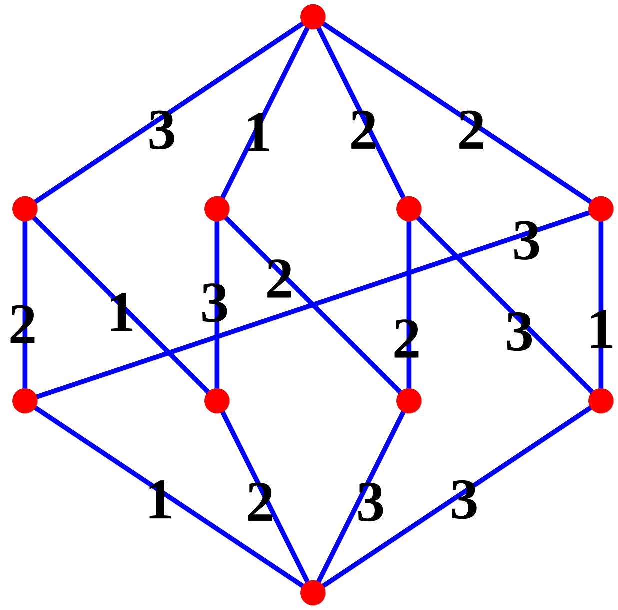
Theorem (rank-selection). *If P is C-M and $S \subseteq [n]$, then P_S is C-M.*

Corollary. *If P is C-M and $S \subseteq [n]$, then $\tilde{h}_S(P) \geq 0$.*

Examples of C-M P:

- semimodular lattices (e.g., distributive, modular, and geometric lattices)
- face lattices of convex polytopes (or of regular CW-spheres and balls)

Edge labelings and shellability: the fundamental combinatorial tool for proving C-M.



Maximal chains: 123, 132, 213, 231, 321, 322, 332, 312

***E*-labeling**: unique weakly increasing chain between any $s < t$ in P.

***L*-labeling**: in addition, this chain is lexicographically least among all chain from s to t.

Theorem. (a) *If λ is an E-labeling of P, then $\tilde{h}_S(P)$ is the number of maximal chains in P whose label $(a_1, a_2, \ldots, a_{n+1})$ satisfies $a_i > a_{i+1}$ if and only if $i \in S$.*

(b) *If λ is an EL-labeling of P, then ordering all maximal chains of P lexicographically by their labels gives a shelling order. Hence P is C-M.*

Example. P = face-lattice of a square.

label	descent set
123	$\emptyset$
132	2
213	1
231	2
321	1,2
322	1
332	2
312	1

$$\Rightarrow \tilde{h}_{\emptyset} = 1, \quad \tilde{h}_1 = 3$$
$$\tilde{h}_2 = 3, \quad \tilde{h}_{1,2} = 1.$$

Recall: If Δ is C-M simplicial complex, then $\exists$ a **multicomplex** Γ with $f(\Gamma) = h(\Delta)$. I.e., $\Gamma \subset \mathbb{N}^k$,

$$(a_1, \ldots, a_k) \in \Gamma,\ (b_1, \ldots, b_k) \leq (a_1, \ldots, a_k)$$
$$\Rightarrow (b_1, \ldots, b_k) \in \Gamma,$$

$$h_i(\Delta) = \#\left\{(b_1, \ldots, b_k) \in \Gamma \,:\, \sum b_j = i\right\}.$$

Example. $\Delta = \partial$(simplicial 3-polytope with 5 vertices).

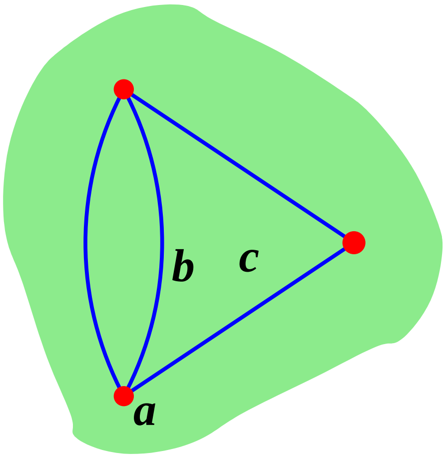
$$\begin{aligned} f(\Delta) &= (5, 9, 6) \\ h(\Delta) &= (1, 2, 2, 1) \\ \Gamma &= \{00, 10, 01, 11, 20, 30\}. \end{aligned}$$

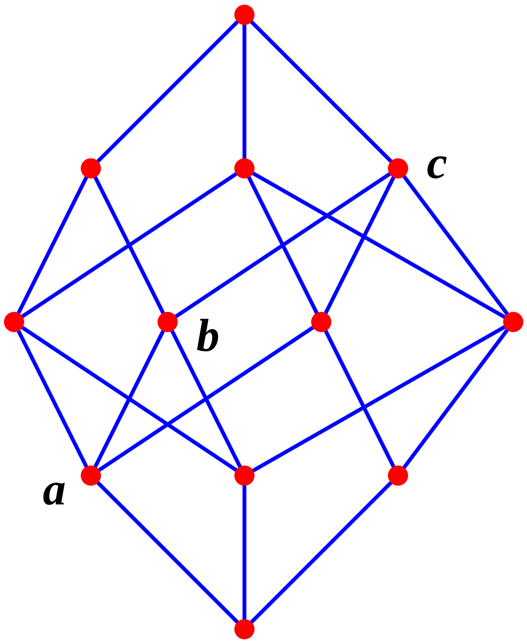
Proved using $K[\Delta]$.

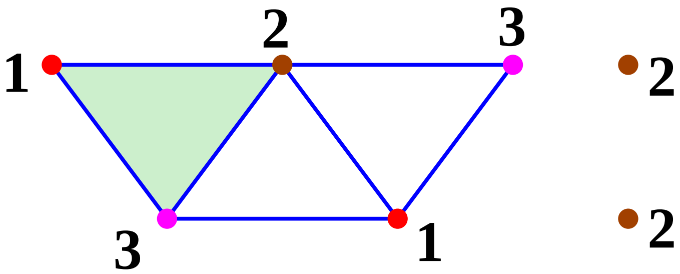
What about $\tilde{h}(P)$ for C-M P?

Theorem. Let P be C-M. Then $\exists$ a **colored simplicial complex** Γ, i.e., each vertex v has a "color" $c(v) \in \mathbb{P}$ such that no face uses a color more than once, and

$$\tilde{h}_S(P) = \#\{F \in \Gamma \,:\, \{c(v) \,:\, v \in F\} = S\}.$$







S	$\tilde{f}_S(P)$	$\tilde{h}_S(P)$
$\emptyset$	1	1
1	3	3
2	4	3
3	3	2
$1,2$	8	2
$1,3$	8	8
$2,3$	8	2
$1,2,3$	16	1

Definition. A pure simplicial complex of dimension $n-1$ is **Eulerian** if

$$\tilde{\chi}(\mathrm{lk}(F)) = (-1)^{\dim F}$$

for all $F \in \Delta$ (e.g., triangulations of spheres). Δ is **Gorenstein*** if C-M and Eulerian, i.e.,

$$\tilde{H}_i(\mathrm{lk}(F); K) = \begin{cases} K, & i = \dim(\mathrm{lk}(F)) \\ 0, & \text{otherwise.} \end{cases}$$

Dehn-Sommerville equations: If Δ is Eulerian then $h_i = h_{n-i}$.

GLBC: If Δ is Gorenstein* then in addition

$$1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor n/2 \rfloor}.$$

(True for ∂(simplicial polytope).)

"Naive" analogue of Dehn-Sommerville: if P is Eulerian, then

$$\tilde{h}_S(P) = \tilde{h}_{[n]-S}(P).$$

These give 2^{n-1} linear relations. But there are more!

Theorem (Bayer-Billera). *Let $\boldsymbol{\mathcal{B}_n}$ be the subspace of the 2^n dimensional space of functions $f : [n] \to \mathbb{R}$ spanned by $\{\tilde{f}(P) : P$ is Eulerian of rank $n+1\}$. Then*

$$\dim \mathcal{B}_n = F_{n+1},$$

where $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$.

The cd-index (a "seedy" area of mathematics). Alternative formulation of Bayer-Billera relations conjectured by J. Fine, proved by Bayer-Klapper.

Given $S \subseteq n$ define $\boldsymbol{u_S} = u_1 \cdots u_n$ by

$$u_i = \begin{cases} a, \text{ if } i \notin S \\ b, \text{ if } i \in S. \end{cases}$$

where a, b are **noncommutative** indeterminates.

For any graded poset P of rank $n+1$, define

$$\begin{aligned} \boldsymbol{\Upsilon_P(a,b)} &= \sum_{S \subseteq [n]} \tilde{f}_S(P) u_S \\ \boldsymbol{\Psi_P(a,b)} &= \sum_{S \subseteq [n]} \tilde{h}_S(P) u_S. \end{aligned}$$

Thus

$$\begin{aligned} \Psi_P(a,b) &= \Upsilon_P(a, b-a) \\ \Upsilon_P(a,b) &= \Psi_P(a, a+b). \end{aligned}$$

Example. P = face-lattice of 3-cube:

$$\begin{aligned}\Upsilon_P(a,b) &= aaa + 8baa + 12aba + 6aab \\ &\quad +24bba + 24bab + 24abb + 48bbb\end{aligned}$$

$$\begin{aligned}\Psi_P(a,b) &= aaa + 7baa + 11aba + 5aab \\ &= +5bba + 11bab + 7abb + bbb \\ &= (a+b)^3 + 6(ab+ba)(a+b) \\ &\quad +4(a+b)(ab+ba).\end{aligned}$$

Theorem. *If P is Eulerian, then $\exists$ a polynomial* $\boldsymbol{\Phi_P(c,d)}$, *called the* ***cd-index*** *of P, in the noncommutative variables c, d such that*

$$\Psi_P(a,b) = \Phi_P(a+b, ab+ba).$$

Even for $P = B_{n+1}$ (the face lattice of an n-simplex), $\Phi_P(c,d)$ is interesting. For instance, if

$$\boldsymbol{E_{n+1}} = \Phi_{B_{n+1}}(1,1),$$

then (Purtill)

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

In general:

Proposition. *We have*

$$\begin{aligned}\Phi_P(1,1) &= \tilde{h}_{\{1,3,5,\ldots\}}(P)\\ &= \tilde{h}_{\{2,4,6,\ldots\}}(P).\end{aligned}$$

Main open problem on cd-index (analogue of GLBC for Gorenstein* simplicial complexes):

Conjecture. Suppose P is Gorenstein* (i.e, C-M and Eulerian). Then every coefficient of $\Phi_P(c,d)$ is nonnegative.

Is there a sensible conjecture for a **complete characterization** of flag f-vectors of Gorenstein* posets (flag analogue of McMullen's g-conjecture)?

Theorem. *The above conjecture, if true, gives* ***all*** *linear inequalities satisfied by the coefficients of* $\Phi_P(c, d)$ *for all Gorenstein* P of rank* $n+1$*. Equivalently, the above conjecture determines the smallest polyhedral cone containing the flag* f*-vectors of all Gorenstein* posets of rank* $n+1$*.*

Theorem. *If* P *is the face poset (with* $\hat{1}$ *adjoined) of a "shellable" regular CW-sphere (e.g., the face lattice of a convex polytope), then every coefficient of* $\Phi_P(c, d)$ *is nonnegative.*

The Charney-Davis conjecture. A **flag complex** is a simplicial complex Δ for which every "missing face" (minimal set of vertices not forming a face) has two elements. E.g., $\Delta(P)$ for any poset P.

Let Δ be an $(n-1)$-dimensional Gorenstein* flag complex (e.g., $\Delta(P)$ for a Gorenstein* poset P) with

$$h(\Delta) = (h_0, h_1, \ldots, h_n).$$

If $n = 2m+1$, then

$$h_0 - h_1 + h_2 - \cdots - h_n = 0,$$

since $h_i = h_{n-i}$.

Conjecture. If $n = 2m$ then

CD(Δ) $:= (-1)^m(h_0 - h_1 + h_2 - \cdots + h_n) \geq 0.$

If $\Delta = \Delta(P)$ then

$$\mathrm{CD}(\Delta) = [d^m]\Phi_P(c,d),$$

the coefficient of d^m in $\Phi_P(c,d)$. Hence:

Proposition. *If $\Phi_P x(c,d)$ has nonnegative coefficients for Gorenstein* P, then the Charney-Davis conjecture is true for (Gorenstein*) order complexes.*