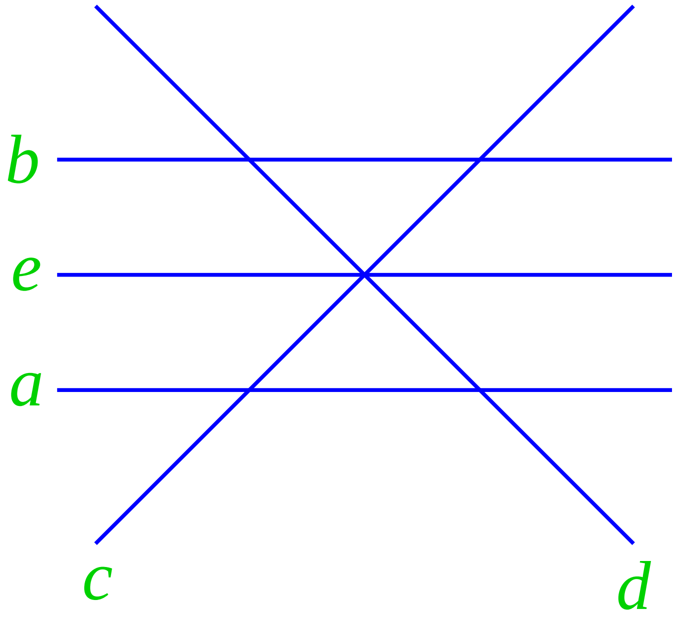
arrangement: a finite set $\mathcal{A}$ of affine hyperplanes in K^n, where K is a field

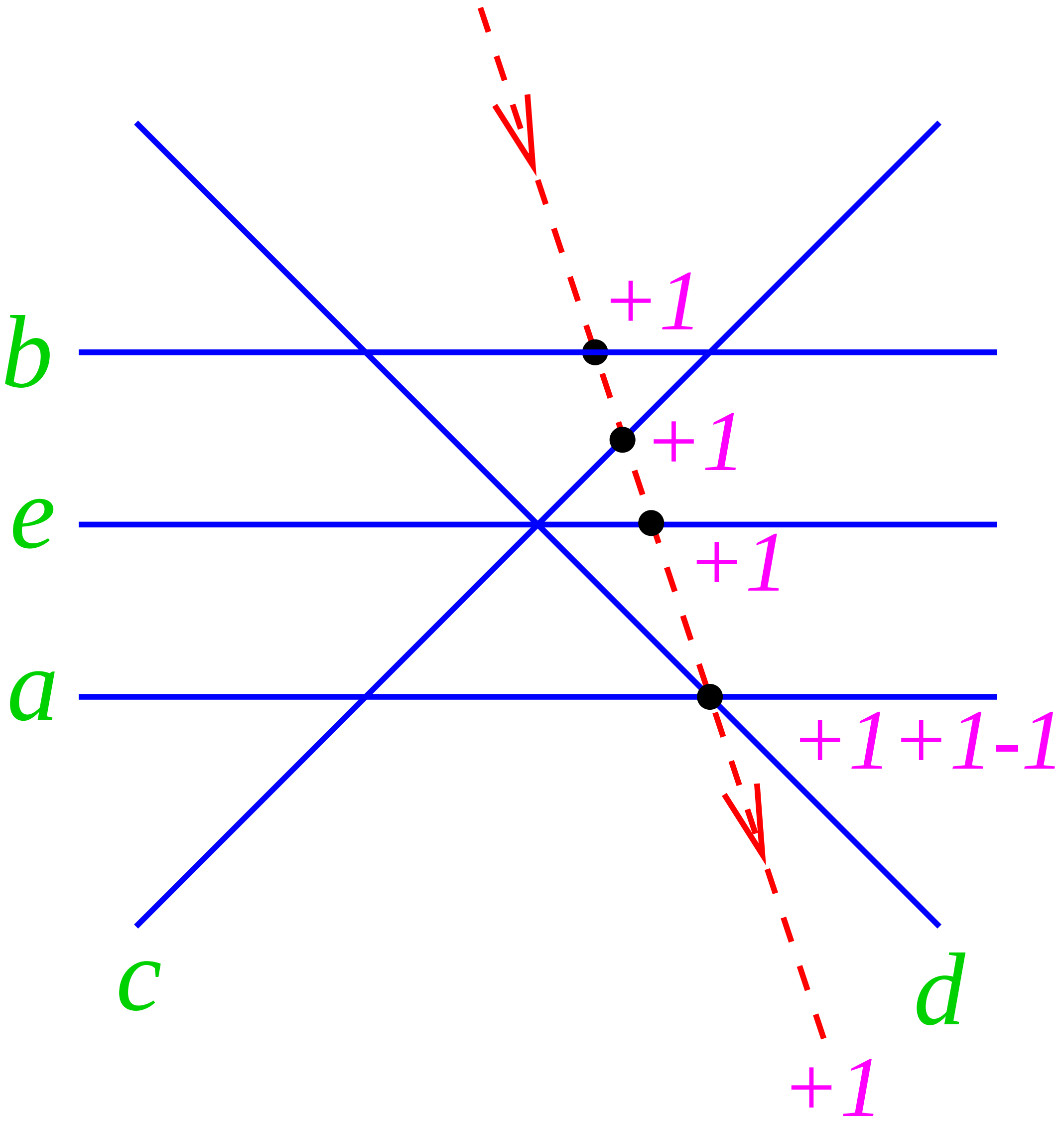


region: a connected component of $\mathbb{R}^n - \bigcup_{H\in\mathcal{A}} H$

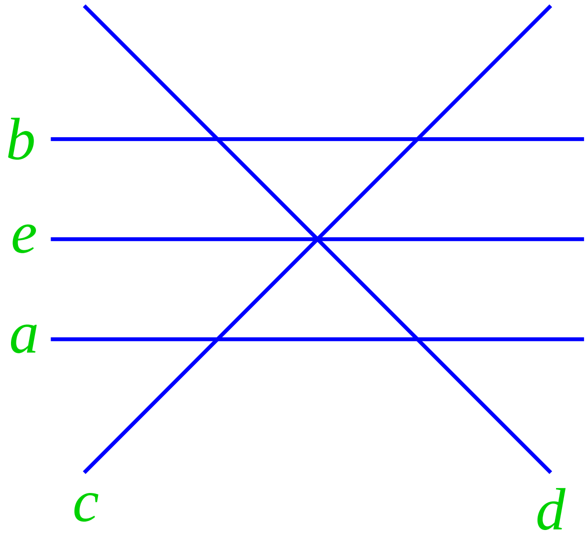
$r(\mathcal{A})$ = number of regions

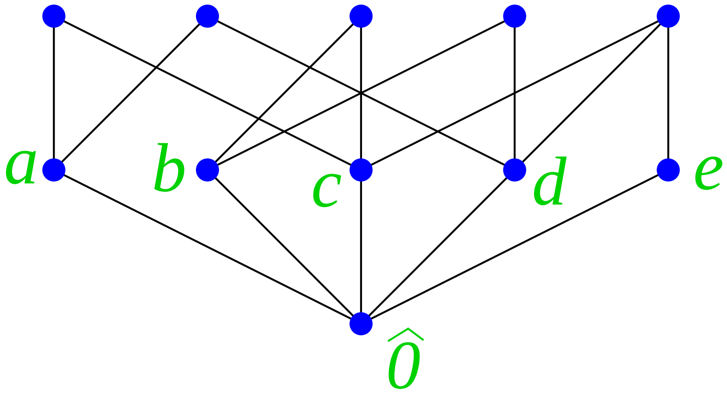
E.g, for above arrangement,

$$r(\mathcal{A}) = 12.$$


+1
b
+1
e
+1
a
+1+1-1
c
d
+1

intersection poset $P_{\mathcal{A}}$: set of **nonempty** intersections of hyperplanes in $\mathcal{A}$, ordered by reverse inclusion (including $\mathbb{R}^n$, denoted $\hat{0}$)

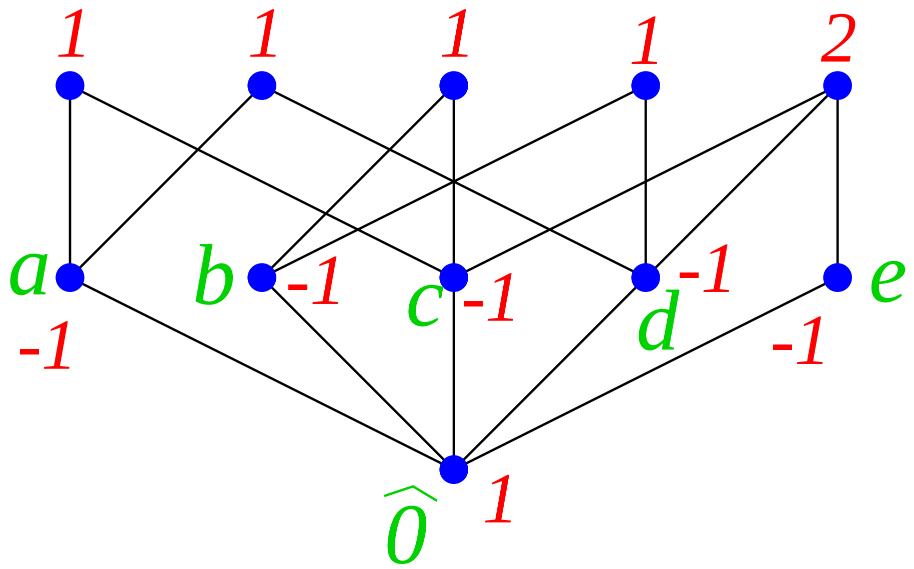


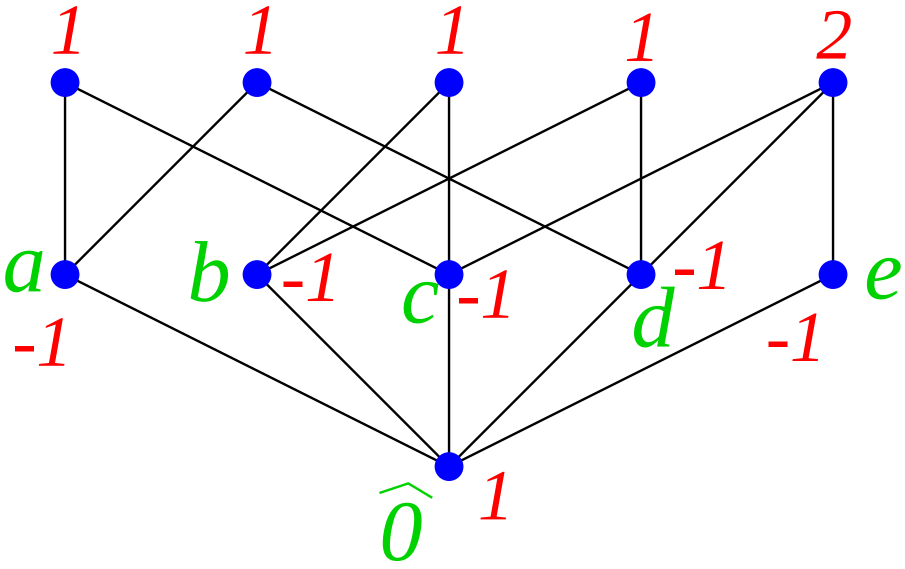


Möbius function $\mu : P_{\mathcal{A}} \to \mathbb{Z}$:

$$\mu(\hat{0}) = 1$$

$$t > 0 \Rightarrow \sum_{s \leq t} \mu(s) = 0$$





characteristic polynomial:

$$\chi_{\mathcal{A}}(q) = \sum_{t \in P_{\mathcal{A}}} \mu(t) q^{\dim t}$$

For example above,

$$\chi_{\mathcal{A}}(q) = q^2 - 5q + 6.$$

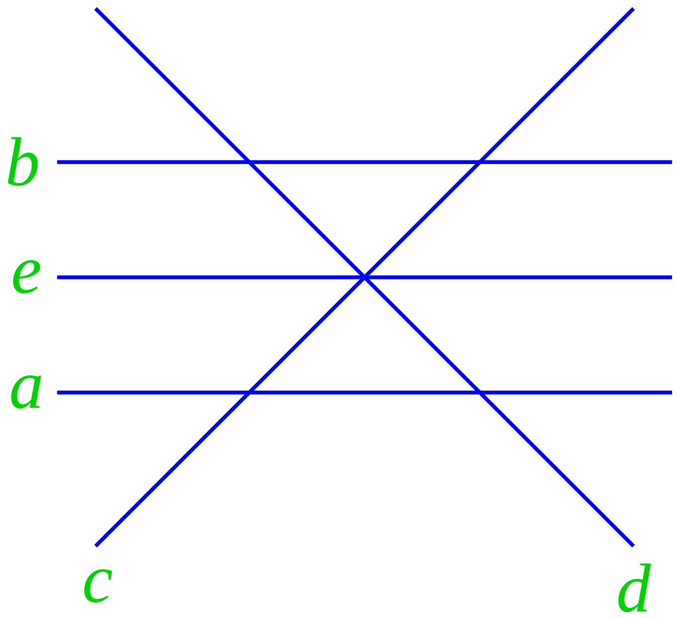
Note: $(-1)^{\operatorname{codim} t}\mu(t) > 0$, so coefficients of $\chi_{\mathcal{A}}(q)$ **alternate in sign**.

Theorem (Zaslavsky, 1975)

(a) $r(\mathcal{A}) = \sum_{t \in P_{\mathcal{A}}} |\mu(t)| = (-1)^n \chi_{\mathcal{A}}(-1)$

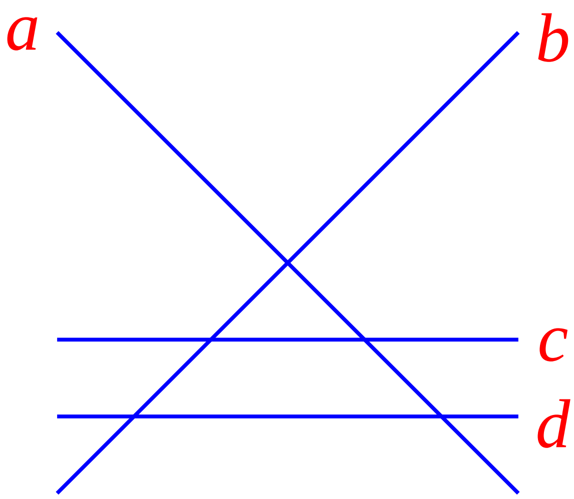
(b) *The number* $b(\mathcal{A})$ *of (relatively)* **bounded** *regions of* $\mathcal{A}$ *is equal to*

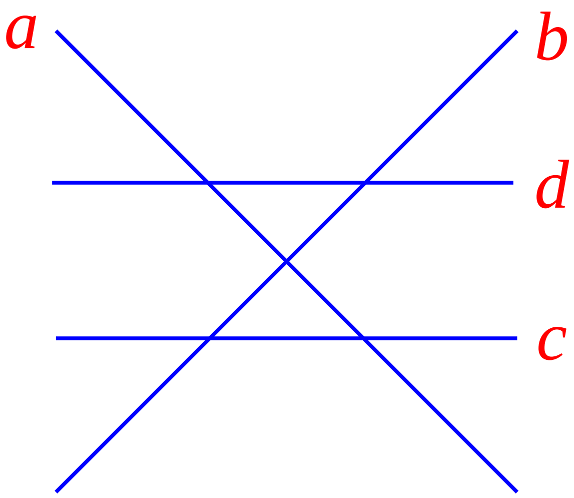
$$b(\mathcal{A}) = \left| \sum_{t \in P_{\mathcal{A}}} \mu(t) \right| = \pm \chi_{\mathcal{A}}(1).$$



$$\chi_{\mathcal{A}}(q) = q^2 - 5q + 6$$

Corollary. $r(\mathcal{A})$ and $b(\mathcal{A})$ depend only on $P_{\mathcal{A}}$.





OTHER APPEARANCES OF $\chi_{\mathcal{A}}(q)$

(1) Let $\mathcal{A}_{\mathbb{C}}$ denote the complexification of $\mathcal{A}$. Let

$$X = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}_{\mathbb{C}}} H$$

$H_i(X;\mathbb{Z}) = i$th homology group of X.

Theorem (Orlik-Solomon, 1980)

$$\sum_i (\operatorname{rank} H_i(X;\mathbb{Z}))\, q^i = (-q)^n \chi_{\mathcal{A}}(-1/q).$$

(2) Suppose that $\mathcal{A}$ is **central**, i.e., $0 \in H$ for all $H \in \mathcal{A}$. Let

$$x = (x_1, \ldots, x_n).$$

Define the **Terao module**

$$T(\mathcal{A}) =$$

$$\{p(x) = (p_1(x), \ldots, p_n(x)) \in \mathbb{R}[x]^n : \\ p(\alpha) \in H \text{ for all } \alpha \in H,\ H \in \mathcal{A}\}.$$

Theorem (Terao, 1980) *If $T(\mathcal{A})$ is a* **free** *$\mathbb{R}[x]$-module with homogeneous generators of degrees $d_1, \ldots, d_n$, then*

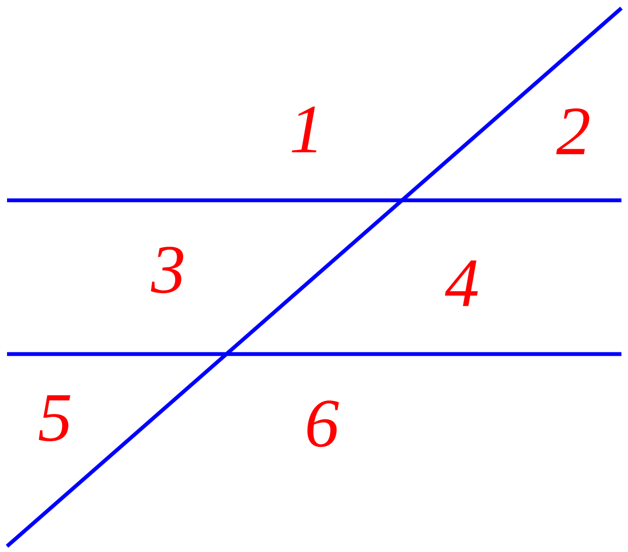
$$\chi_{\mathcal{A}}(q) = \prod_{i=1}^{n} (q - d_i).$$

Open: Does the freeness of $T(\mathcal{A})$ depend only on $P_{\mathcal{A}}$? (probably not)

(3) Given two regions R, R' of $\mathcal{A}$, let

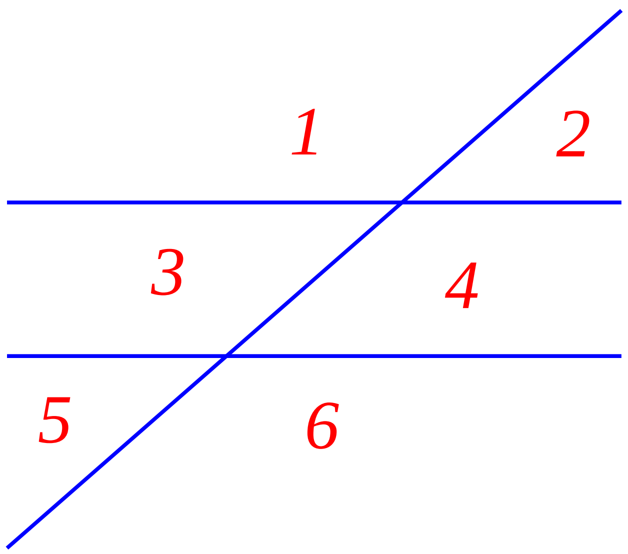
$d(R, R') = \# H \in \mathcal{A}$ separating R and R'.

$$D(\mathcal{A}) = \left[q^{d(R,R')} \right]$$



Write $\mathbf{i} = q^i$.

$$D(\mathcal{A}) = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$



Smith normal form of $D(\mathcal{A})$:

$$\operatorname{diag}\Big(1, q^2-1, q^2-1, q^2-1,$$
$$(q^2-1)^2, (q^2-1)^2\Big).$$

Let a_i = # entries of SNF exactly divisible by $(q-1)^i$.

$$a_0 = 1, \quad a_1 = 3, \quad a_2 = 2$$

Theorem (Varchenko, 1993). *Entries of SNF are products of cyclotomic polynomials.*

Theorem (Denham-Hanlon, 1997).

$$\chi_{\mathcal{A}}(q) = \sum_i (-1)^{n-i} a_{n-i} q^i.$$

OPEN: SNF of $D(\mathcal{A})$

(4) A **face** of $\mathcal{A}$ is a (nonempty) face of some region of $\mathcal{A}$.

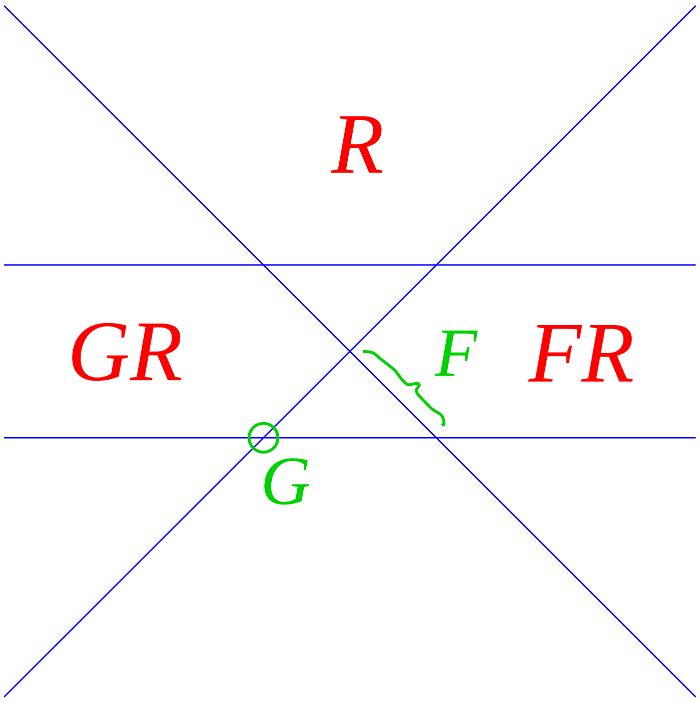
$$\begin{aligned}\mathcal{F} &= \text{set of faces of } \mathcal{A}\\ \mathcal{R} &= \text{set of regions of } \mathcal{A}\end{aligned}$$

If $F \in \mathcal{F}$ and $R \in \mathcal{R}$, define

$$\mathbf{FR} = \text{nearest region to } R$$
$$\text{with } F \text{ as a face.}$$

$$p = \text{probability measure on } \mathcal{F}$$

Define a random walk on $\mathcal{R}$: from $R \in \mathcal{R}$, choose F from p and move to FR.



Transition matrix:

$$K(R,R') = \sum_{FR=R'} p(F)$$

Theorem (Bridigare-Hanlon-Rockmore, 1997) *For each $x \in P_{\mathcal{A}}$ there is an eigenvalue*

$$\lambda_x = \sum_{\substack{F \in \mathcal{F} \\ F \subseteq x}} p(F)$$

of K with multiplicity $|\mu(x)|$.

(5) Theorem (Crapo-Rota 1971, Orlik-Terao 1992, Athanasiadis 1996) *Let* $\mathcal{A}$ *be defined over* $\mathbb{Z}$. *For* $q > 0$, *let*

$$\mathcal{A}_{\mathbf{q}} = \mathcal{A} \text{ reduced modulo } q.$$

Then for q *prime,* $q >> 0$,

$$\chi_{\mathcal{A}}(q) = \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H\right).$$

$*$ Second method for computing $\chi_{\mathcal{A}}(q)$.

Example. G = graph with vertices $1, 2, \ldots, n$, edge set E.

$$\mathcal{A}_{\mathbf{G}}: \ x_i = x_j, \quad ij \in E$$

(graphical arrangement)

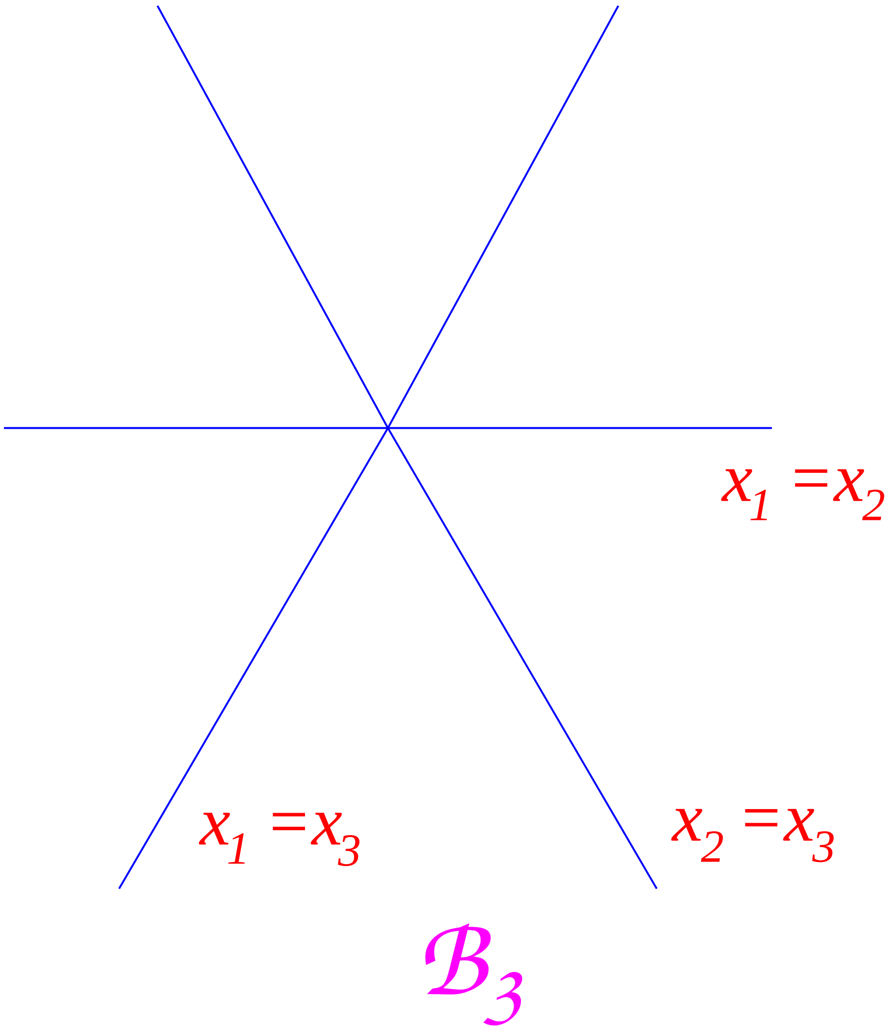
$$\begin{aligned}\chi_G(q) &= \mathbb{F}_q^n - \#\{(x_1, \ldots, x_n) : \\ &\qquad x_i = x_j \text{ for some } ij \in E\} \\ &= \#\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : \\ &\qquad x_i \neq x_j \text{ if } ij \in E\} \\ &= \# \text{ proper } q\text{-colorings of } G\end{aligned}$$

(chromatic polynomial)

braid arrangement $\mathcal{B}_n$:

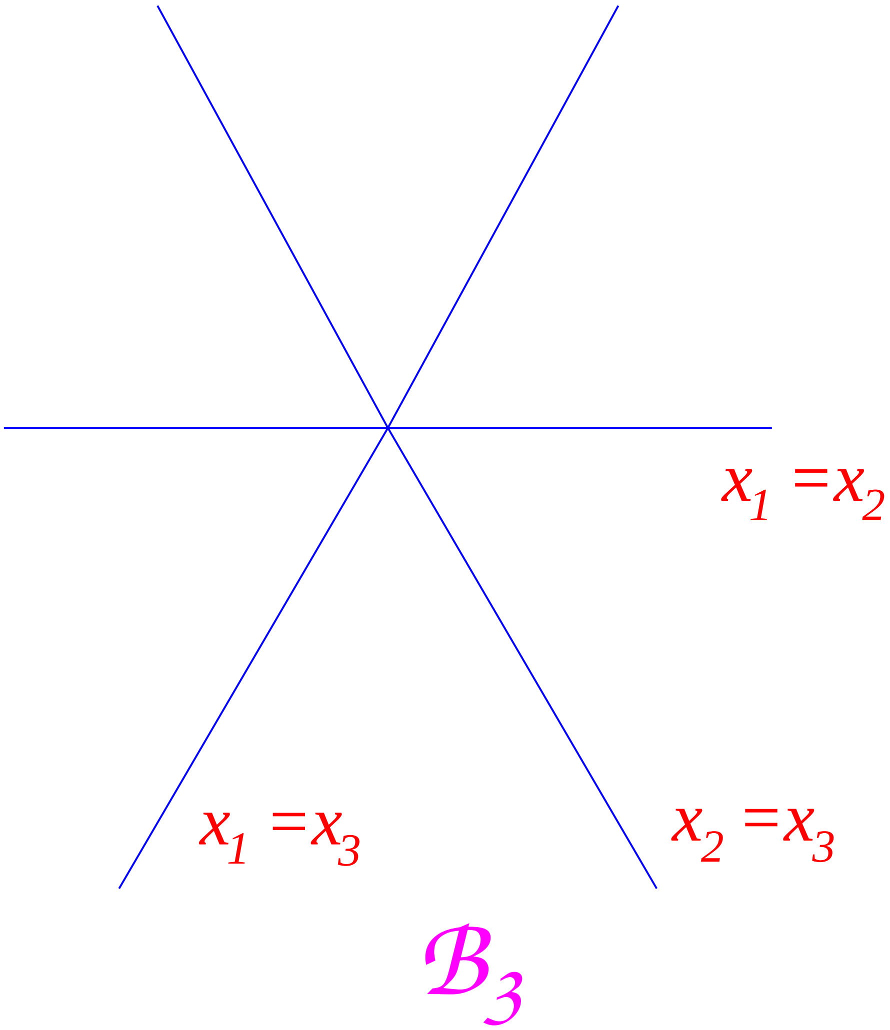
$$x_i - x_j = 0, \quad 1 \le i < j \le n$$

$$\mathcal{B}_n = \mathcal{A}_{K_n}$$



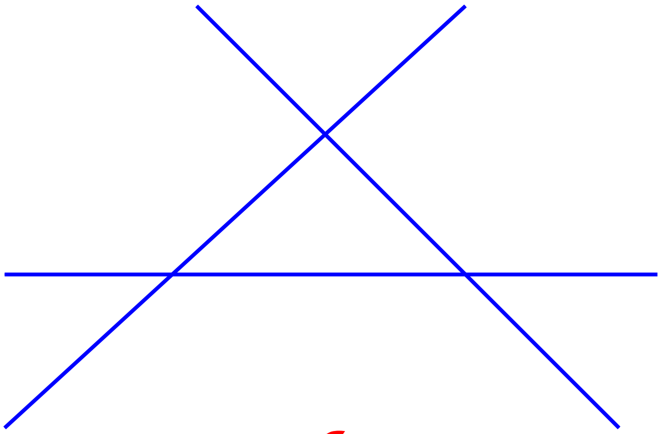
$$\chi_{\mathcal{B}_n} = q(q-1)\cdots(q-n+1)$$

$$r(\mathcal{B}_n) = n!$$



THE LINIAL ARRANGEMENT

$$\mathcal{L}_{\mathbf{n}}: \ x_i = x_j + 1, \ 1 \le i < j \le n$$



$$\mathcal{L}_3$$

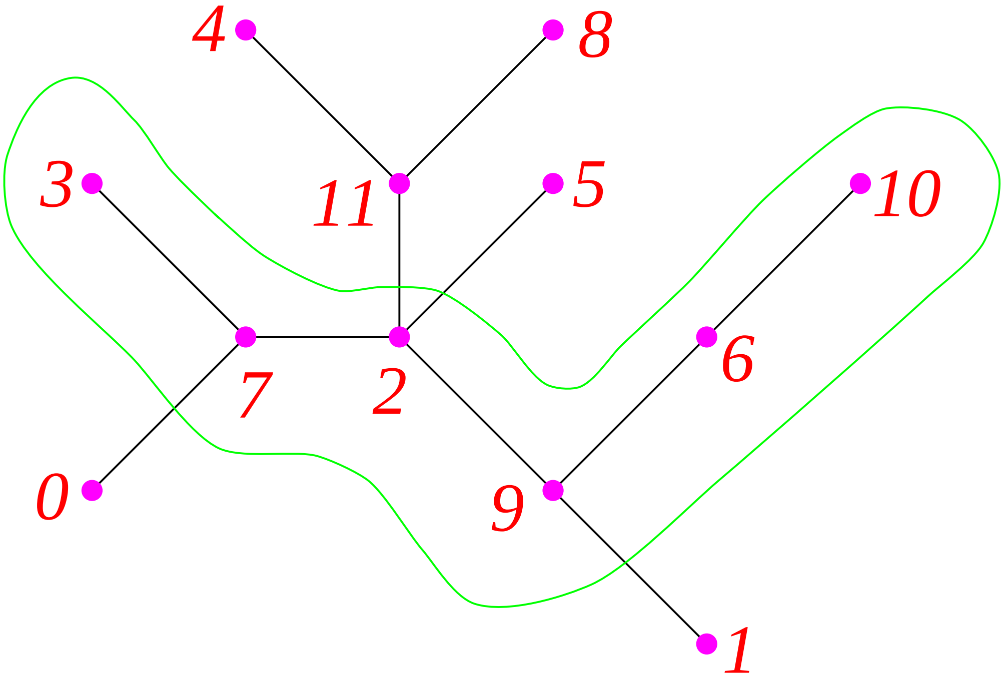
$$\chi_{\mathcal{L}_3}(q) = q^3 - 3q^2 + 3q$$

An **alternating** (or **intransitive**) **tree** on $0, 1, \ldots, n$ is a tree with vertices $0, 1, \ldots, n$ such that every path has the form

$$a_1 < a_2 > a_3 < a_4 > \cdots \text{ or}$$

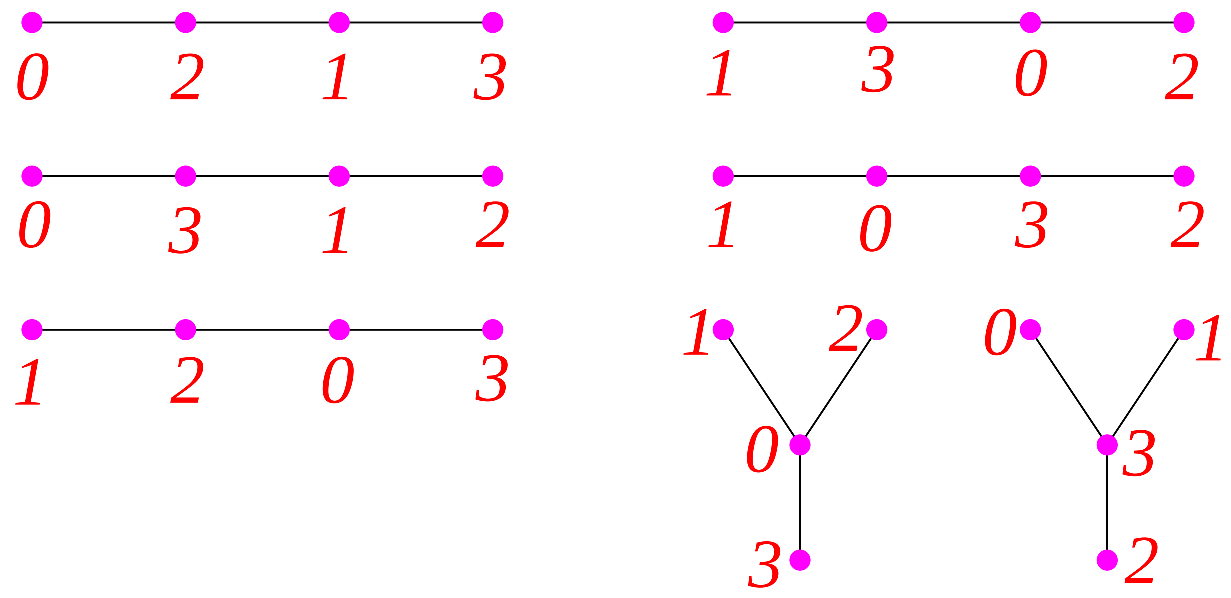
$$a_1 > a_2 < a_3 > a_4 < \cdots.$$

Equivalently, every vertex is either less than all its neighbors or greater than all its neighbors.



$$\mathbf{f_n} = \#\text{ alt. trees on } 0, 1, \ldots, n$$

n	1	2	3	4	5	6
f(n)	1	2	7	36	246	2104



$$y = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

$$\Rightarrow y = e^{\frac{x}{2}(1+y)}$$

Lagrange inversion $\Rightarrow$

$$\begin{aligned} f_n &= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1} \\ &= \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-1}{k-1} k^{n-2}. \end{aligned}$$

No bijective proof known.

Note. Let e_n be the number of alternating permutations of $1, 2, \ldots, n$, i.e.,

$$a_1 > a_2 < a_3 > a_4 < \cdots a_n.$$

$e_4 = 5$: 2143 3142 3241 4132 4231

Theorem (D. André, 1879)

$$\sum_{n \geq 0} e_n \frac{x^n}{n!} = \sec x + \tan x.$$

$\longrightarrow$ **Combinatorial Trigonometry**

Exercise. Prove **combinatorially** that

$$1 + \tan^2 x = \sec^2 x.$$

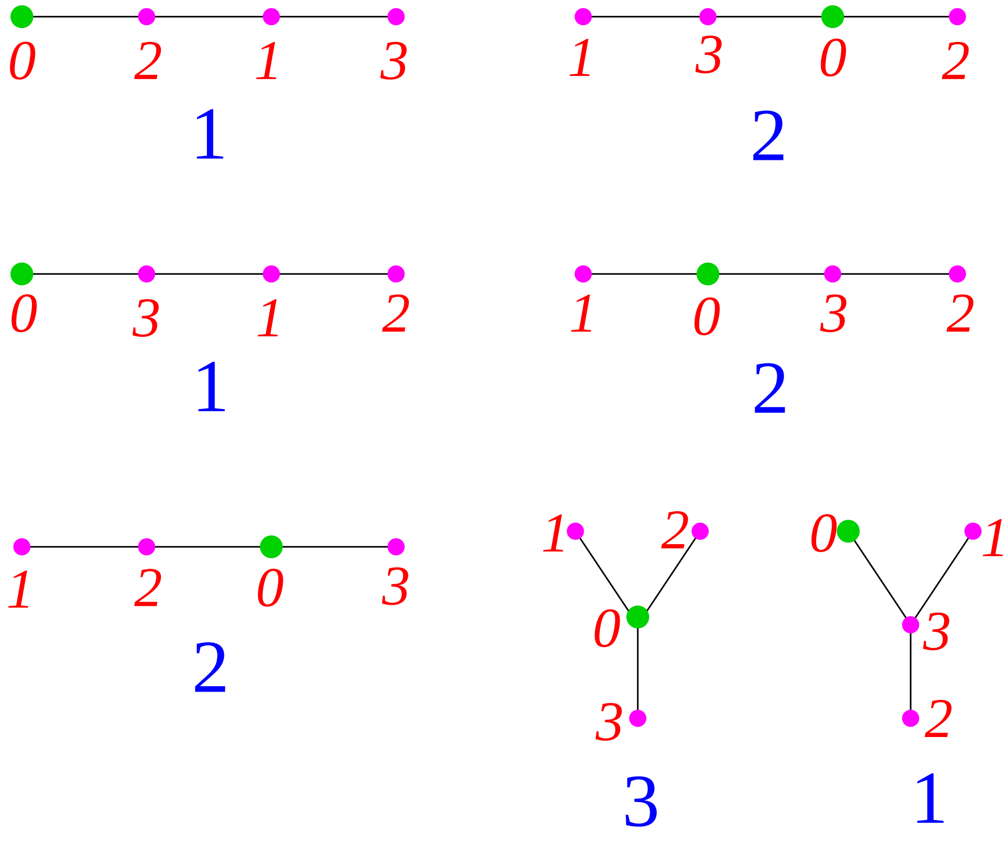
Theorem (Athanasiadis 1996, Postnikov 1996)

$$r(\mathcal{L}_n) = f_n.$$

(No bijective proof known.)

What about $\chi_{\mathcal{L}_n}(q)$?

Let $\mathbf{f_n(q)} = \sum_T q^{\deg(0)}$, summed over all alternating trees on $0, 1, \ldots, n$.



$$f_3(q) = q^3 + 3q^2 + 3q$$

$$\sum_{n\geq 0} f_n(q)\frac{x^n}{n!} = \left(\sum_{n\geq 0} f_n\frac{x^n}{n!}\right)^q$$

$$f_n(q) = \frac{q}{2^n}\sum_{k=0}^{n}\binom{n}{k}(q+k)^{n-1}$$

Theorem (Athanasiadis, Postnikov 1996)

$$\chi_{\mathcal{L}_n}(q) = (-1)^n f_n(-q)$$

Theorem (Riemann hypothesis for $\mathcal{L}_n$) *Every zero of* $\chi_{\mathcal{L}_n}(q)$*, except* $q = 0$*, has real part* $n/2$*.*

Corollary (functional equation for $\mathcal{L}_n$)

$$\frac{\chi_{\mathcal{L}_n}(q)}{q} = \frac{(-1)^n \chi_{\mathcal{L}_n}(-q+n)}{-q+n}$$

Proof of theorem. Let $Ef(q) = f(q-1)$. Then

$$\chi_{\mathcal{L}_n}(q) = \frac{q}{2^n}(E-1)^n q^{n-1}.$$

Lemma. *Let $f(q) \in \mathbb{C}[q]$, such that every zero of f has real part m. Let $|s| = 1$ and*

$$h(q) = (E-s)f(q) = f(q-1) - sf(q).$$

Then every zero of $h(q)$ has real part $m + \frac{1}{2}$.

Theorem (Postnikov). *Let*

$$\psi_n(q) = (-2i)^{n-1} \frac{\chi_{\mathcal{L}_n}((iq+n)/2)}{(iq+n)/2},$$

so all zeros are real. Then

$$\lim_{m\to\infty} \frac{\psi_{2m}(q)}{\psi'_{2m}(1)} \to \frac{\sin(hq)}{q},$$

where

$$\cosh(h) = \frac{h}{\sqrt{h^2-1}}, \quad h > 1,$$

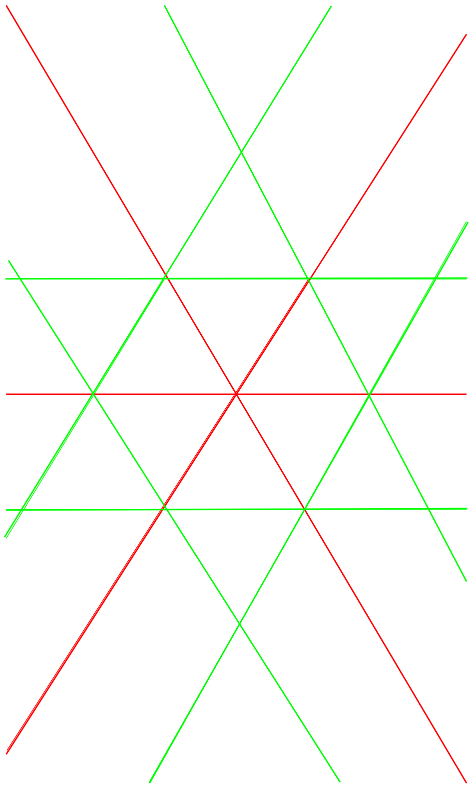
$$h \approx 1.199678640 \cdots .$$

Corollary. *Every zero of* $\sin z$ *is real.*

SOME OTHER ARRANGEMENTS

Catalan arrangment $\mathcal{C}_n$:

$$x_i - x_j = -1, 0, 1, \text{ for } 1 \leq i < j \leq n$$

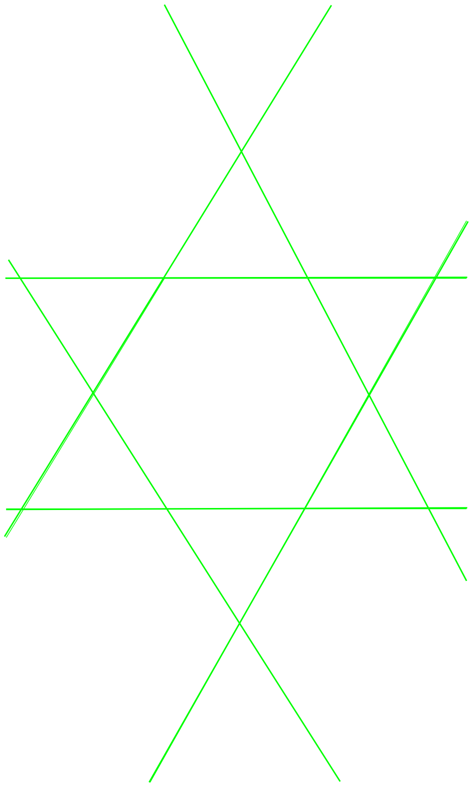


$$r(\mathcal{C}_n) = n!\, C_n = n! \frac{1}{n+1}\binom{2n}{n}$$

$$\chi_{\mathcal{C}_n}(q) =$$
$$q(q-n-1)(q-n-2)\cdots(q-2n+1)$$

semiorder arrangement $\mathcal{I}_n$:

$$x_i - x_j = \pm 1, \text{ for } 1 \le i < j \le n$$



A **semiorder** is a partial ordering $\prec$ of a set S obtained by choosing $f: S \to \mathbb{R}$ and defining

$$i \prec j \quad \text{if} \quad f(i) + 1 < f(j).$$

$r(\mathcal{I}_n)$ = number of semiorders on an n-set

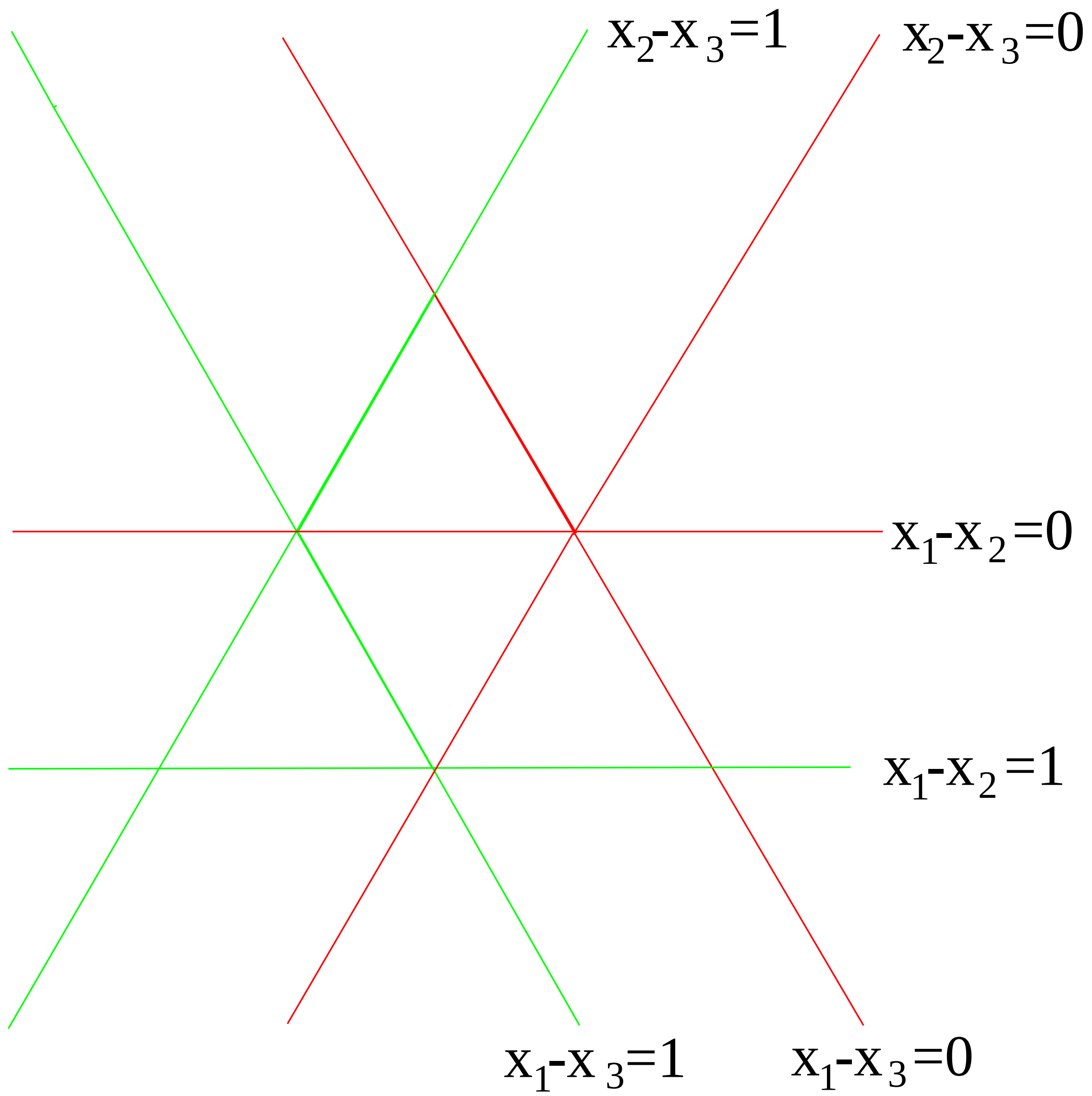
$$\sum_{n \geq 0} r(\mathcal{I}_n) \frac{x^n}{n!} = C(1 - e^{-x}),$$

where

$$C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}$$

Shi arrangement $\mathcal{S}_n$:

$$x_i - x_j = 0, 1, \text{ for } 1 \le i < j \le n$$



$$\begin{aligned} r(\mathcal{S}_n) &= (n+1)^{n-1} \\ \chi_{\mathcal{S}_n}(q) &= q(q-n)^{n-1} \end{aligned}$$

threshold arrangement $\mathcal{T}_n$:

$$x_i + x_j = 0, \text{for } 1 \leq i < j \leq n$$

$$\sum_{n \geq 0} r(\mathcal{T}_n) \frac{x^n}{n!} = \frac{e^x (1-x)}{2 - e^x}$$

$$\sum_{m \geq 0} (-1)^n \chi_{\mathcal{T}_n}(-q) \frac{x^n}{n!} = (1-x) \left(\frac{e^x}{2 - e^x} \right)^{\frac{q+1}{2}}$$

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