Polynomial Sequences of Binomial Type

Richard P. Stanley

M.I.T.
我今天的讲座要用中文说，
我今天的讲座要用中文说，除了接下来要说的以外。
Some motivation

Let \( D = \frac{d}{dn} \), acting on \( f(n) \in \mathbb{C}[n] \). Then

\[
D n^k = k n^{k-1}
\]

\[
f(n) = \sum_{k \geq 0} D^k f(0) \frac{n^k}{k!} \quad \text{(Taylor series)}.
\]
Some motivation

Let $D = \frac{d}{dn}$, acting on $f(n) \in \mathbb{C}[n]$. Then

$$Dn^k = kn^{k-1}$$

$$f(n) = \sum_{k \geq 0} D^k f(0) \frac{n^k}{k!} \quad \text{(Taylor series)}. $$

Let $\Delta f(n) = f(n + 1) - f(n)$ and $\binom{n}{k} = n(n-1) \cdots (n-k+1)$. Then

$$\Delta(n)_k = k(n)_{k-1}$$

$$f(n) = \sum_{k \geq 0} \Delta^k f(0) \frac{(n)_k}{k!}.$$
By Taylor’s theorem,

\[ f(n + x) = \sum_{k \geq 0} D^k f(n) \frac{x^k}{k!}. \]
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Put \( x = 1 \):
Connection (continued)

\[ f(n + 1) = \left( \sum_{k \geq 0} \frac{D^k}{k!} \right) f(n) = e^D f(n). \]
Connection (continued)

\[ f(n + 1) = \left( \sum_{k \geq 0} \frac{D^k}{k!} \right) f(n) \]
\[ = e^D f(n). \]

\[ \Rightarrow \Delta f(n) = (e^D - 1) f(n) \Rightarrow \Delta = e^D - 1. \]

Thus also \( D = \log(\Delta + 1). \)
Finite operator calculus

General theory developed by G.-C. Rota and collaborators, called finite operator calculus.
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Define $E : \mathbb{C}[n] \rightarrow \mathbb{C}[n]$ by

$$Ef(n) = f(n + 1).$$
Theorem. Let \( L : \mathbb{C}[n] \rightarrow \mathbb{C}[n] \) be linear (over \( \mathbb{C} \)) and satisfy \( L(n) = 1 \) and \( L(\text{deg } d) = \text{deg } d - 1 \). The following two conditions are equivalent.

\[ LE = EL \]
Main thm. of operator calculus

**Theorem.** Let $L : \mathbb{C}[n] \rightarrow \mathbb{C}[n]$ be linear (over $\mathbb{C}$) and satisfy $L(n) = 1$ and $L(\deg d) = \deg d - 1$. The following two conditions are equivalent.

- $LE = EL$
- There exist polynomials $p_k(n), k \geq 0$, such that $p_0(n) = 1$, $\deg p_k(n) = k$, and
  
  \[
  Lp_k(n) = kp_{k-1}(n)
  \]

  \[
  \sum_{k \geq 0} p_k(n) \frac{x^k}{k!} = \left( \sum_{k \geq 0} p_k(1) \frac{x^k}{k!} \right)^n.
  \]
If $p_0(n), p_1(n), \ldots$ is a sequence of polynomials satisfying

$$\sum_{k \geq 0} p_k(n) \frac{x^k}{k!} = \left( \sum_{k \geq 0} p_k(1) \frac{x^k}{k!} \right)^n,$$

then we call $p_0(n), p_1(n), \ldots$ a sequence of polynomials of binomial type, or just polynomials of binomial type.
A characterization

Note. The condition $\deg p_k(n) = k$ is then equivalent to $p_1(n) \neq 0$ (or just $p_1(1) \neq 0$). Sometimes this extra condition is part of the definition of binomial type.
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**Note.** The condition $\deg p_k(n) = k$ is then equivalent to $p_1(n) \neq 0$ (or just $p_1(1) \neq 0$). Sometimes this extra condition is part of the definition of binomial type.

**Theorem.** A sequence $p_0(n) = 1, p_1(n), \ldots$ of polynomials is of binomial type if and only if

$$p_k(m + n) = \sum_{i=0}^{k} \binom{k}{i} p_i(m) p_{k-i}(n), \quad k \geq 0.$$
Some classical examples

\[ p_k(n) = n^k \]
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\sum_{k \geq 0} n^k \frac{x^k}{k!} = \left( \sum_{k \geq 0} \frac{x^k}{k!} \right)^n = e^{nx}
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\]

- \( p_k(n) = (n)_k = n(n - 1) \cdots (n - k + 1) \)

\[
\sum_{k \geq 0} (n)_k \frac{x^k}{k!} = \sum_{k \geq 0} \binom{n}{k} x^k = (1 + x)^n
\]
**More classical examples**

\[ p_k(n) = n^k = n(n + 1) \cdots (n + k - 1) \]
More classical examples

\[ p_k(n) = n^{(k)} = n(n + 1) \cdots (n + k - 1) \]

\[ \sum_{k \geq 0} n^{(k)} \frac{x^k}{k!} = (1 - x)^{-n} \]
More classical examples

1. \( p_k(n) = n^{(k)} = n(n + 1) \cdots (n + k - 1) \)

\[
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\]

2. \( p_k(n) = n(n - ak)^{k-1}, \ a \in \mathbb{C} \)
   (Abel polynomials)
More classical examples

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- \( p_k(n) = n(n - a k)^{k-1}, \ a \in \mathbb{C} \) (Abel polynomials)

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\sum_{k \geq 0} n(n - a k)^{k-1} \frac{x^k}{k!} = \left( \sum_{k \geq 0} (1 - a k)^{k-1} \frac{x^k}{k!} \right)^n
\]
More on Abel polynomials

Binomial type is equivalent to \textit{Abel’s identity}:

\[(x + y)^k = \sum_{i=0}^{k} \binom{k}{i} x(x - iz)^{i-1}(y + iz)^{k-i}.\]
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\[(x + y)^k = \sum_{i=0}^{k} \binom{k}{i} x(x - iz)^{i-1}(y + iz)^{k-i}.\]

Closely related to tree enumeration.
Yet another example

\[ p_k(n) = \sum_{i=1}^{k} S(k, i) n^i \]

Stirling no. of 2nd kind

(exponential polynomials)
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Stirling no. of 2nd kind

(expoential polynomials)

\[
\sum_{k \geq 0} \left( \sum_{i} S(k, i) n^i \right) \frac{x^k}{k!} = \left( \sum_{k \geq 0} B(k) \frac{x^k}{k!} \right)^n
\]
One more

\[ p_k(n) = \sum_{i=1}^{k} \binom{k}{i} i^{k-i} n^i \]
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\[ \sum_{k \geq 0} \left( \sum_i \binom{k}{i} i^{k-i} n^i \right) \frac{x^k}{k!} = \exp nxe^x \]
Binomial posets

Are there interesting examples of polynomials of binomial type for which explicit formulas don’t exist?
Binomial posets

Are there interesting examples of polynomials of binomial type for which explicit formulas don’t exist?

\[ P = P_0 \cup P_1 \cup \cdots \] (disjoint union): a poset (partially ordered set) such that all maximal chains have the form \( t_0 < t_1 < \ldots \), where \( t_i \in P_i \).

Write \( \text{rank}(t_i) = i \). Then \( P \) is a binomial poset if for all \( s \leq t \), where \( k = \text{rank}(t) - \text{rank}(s) \), the number of (saturated) chains \( s = t_0 < t_1 < \cdots < t_k = t \) depends only on \( k \). Call this number \( B(k) \).
\[ P = \{0, 1, 2, \ldots \} \text{ (a chain)}: \ B(k) = 1. \]
Two further examples

$P$ is the set of all finite subsets of $\{1, 2, \ldots \}$, ordered by inclusion: $B(k) = k!$. 
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- $P$ is the set of all finite subsets of $\{1, 2, \ldots \}$, ordered by inclusion: $B(k) = k!$.

- $P$ is the set of all finite-dimensional subspaces of an infinite-dimensional vector space over the finite field $\mathbb{F}_q$:

$$B(k) = (k)! = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{k-1}).$$
Two further examples

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- $P$ is the set of all finite-dimensional subspaces of an infinite-dimensional vector space over the finite field $\mathbb{F}_q$:

  $$B(k) = (k)! = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{k-1}).$$

Many other examples . . .
**Theorem.** Let $P$ be a binomial poset. Let $p_k(n)$ be the number of multichains

$$s = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = t,$$

where $\text{rank}(t) - \text{rank}(s) = k$. Then

$$\sum_{k \geq 0} p_k(n) \frac{x^k}{B(k)} = \left( \sum_{k \geq 0} \frac{x^k}{B(k)} \right)^n.$$
**Theorem.** Let $P$ be a binomial poset. Let $p_k(n)$ be the number of multichains $s = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = t$, where $\text{rank}(t) - \text{rank}(s) = k$. Then

$$\sum_{k \geq 0} p_k(n) \frac{x^k}{B(k)} = \left( \sum_{k \geq 0} \frac{x^k}{B(k)} \right)^n.$$

**Corollary.** $k! \frac{p_k(n)}{B(k)}$, $k \geq 0$, is a sequence of polynomials of binomial type.
$\mathbb{Z}^d_n$: the $n \times n \times \cdots \times n$ ($d$ times) $d$-dimensional toroidal graph.
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$\mathbb{Z}^d_n$: the $n \times n \times \cdots \times n$ ($d$ times) $d$-dimensional toroidal graph.

The green squares are the vertices of $\mathbb{Z}^2_4$. 
$\mathbb{Z}_n$: integers modulo $n$

$\mathbb{Z}_n^d: \{(a_1, \ldots, a_d) : a_i \in \mathbb{Z}_n\}$ (vertex set)

$\alpha = (a_1, \ldots, a_d)$ and $\beta = (b_1, \ldots, b_d)$ are adjacent if $\alpha - \beta$ has one nonzero coordinate, which is equal to $\pm 1$ (modulo $n$).
Figures

A set $S$ of figures:

\{ \begin{array}{ccc}
\text{figure 1} & \text{figure 2} & \text{figure 3} \\
\end{array} \}
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A placement of $S$ on $\mathbb{Z}_4^2$:
The function $f_k(n^d)$

Fix $d$ and a finite set $S$ of tiles.

$f_k(n^d)$: number of placements of $S$ on $\mathbb{Z}_n^d$ using a total of $k$ $1 \times 1 \times \cdots \times 1$ boxes.
The function $f_k(n^d)$

Fix $d$ and a finite set $S$ of tiles.

$f_k(n^d)$: number of placements of $S$ on $\mathbb{Z}_n^d$ using a total of $k \ 1 \times 1 \times \cdots \times 1$ boxes.

Example. $S = \{\Box\}$. Then $f_k(n^2) = \binom{n^2}{k}$.
Example. \( S = \{ \Box \Box \} \)

\[
\begin{align*}
  f_{2j+1}(n^2) &= 0 \\
  f_2(n^2) &= n^2 \\
  f_4(n^2) &= \frac{1}{2}n^2(n^2 - 3)
\end{align*}
\]
Example. \( S = \{ \square \quad \square \} \)

\[
\begin{align*}
  f_1(n^2) &= n^2 \\
  f_2(n^2) &= \binom{n^2}{2} + n^2 \\
  f_3(n^2) &= \binom{n^2}{3} + n^2(n^2 - 2)
\end{align*}
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\end{align*}
\]

Note that these are **polynomials** in \( n^2 \).
A main result of Schneider Theorem.

(a) For $n \gg 0$ (so all tiles fit on $\mathbb{Z}_d^n$), there is a polynomial $p_k$ for which $p_k(n) = f_k(n^d)$. 

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A main result of Schneider

Theorem.

(a) For \( n \gg 0 \) (so all tiles fit on \( \mathbb{Z}_d^n \)), there is a polynomial \( p_k \) for which \( p_k(n) = f_k(n^d) \).

(b) \( p_0, 1! p_1, 2! p_2, \ldots \) is a sequence of polynomials of binomial type.
An example

Recall: $S = \{ \square \quad \square \square \}$

\[
f_1(n^2) = n^2, \quad f_2(n^2) = \binom{n^2}{2} + n^2
\]

\[
f_3(n^2) = \binom{n^2}{3} + n^2(n^2 - 2)
\]
Recall: $S = \{ \square \ \square \ \square \}$

\[
\begin{align*}
  f_1(n^2) &= n^2, \\
  f_2(n^2) &= \binom{n^2}{2} + n^2 \\
  f_3(n^2) &= \binom{n^2}{3} + n^2(n^2 - 2) \\
  1 + nx + \left( \binom{n}{2} + n \right)x^2 + \left( \binom{n}{3} + n(n - 2) \right)x^3 &+ \cdots = (1 + x + x^2 - x^3 + \cdots)^n
\end{align*}
\]
Chromatic polynomials

$G$: finite graph with vertex set $V$, $q \geq 1$

$\chi_G(q)$: number of proper colorings

$f : V \rightarrow \{1, \ldots, q\}$,

i.e., adjacent vertices get different colors
Recall $\mathbb{Z}_n^d$ is a graph:

Much interest from physicists in the chromatic polynomial $\chi_{\mathbb{Z}_n^d}(q)$. 

The graph $\mathbb{Z}_n^d$
A trivial and nontrivial result

\[ \chi_{Z_n^d}(2) = \begin{cases} 
2, & n \text{ even} \\
0, & n \text{ odd}
\end{cases} \]
A trivial and nontrivial result

\[ \chi_{\mathbb{Z}_n^d}(2) = \begin{cases} 
2, & n \text{ even} \\
0, & n \text{ odd} 
\end{cases} \]

**Theorem (E. Lieb, 1967)**

\[
\lim_{n \to \infty} \chi_{\mathbb{Z}_n^2}(3)^{1/n^2} = \left(\frac{4}{3}\right)^{3/2} = 1.5396 \ldots
\]
A trivial and nontrivial result

$$\chi_{\mathbb{Z}_n^d}(2) = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

**Theorem (E. Lieb, 1967)**

$$\lim_{n \to \infty} \chi_{\mathbb{Z}_n^2}(3)^{1/n^2} = \left(\frac{4}{3}\right)^{3/2} = 1.5396 \ldots$$

$$\lim_{n \to \infty} \chi_{\mathbb{Z}_n^2}(4)^{1/n^2} : \text{not known}$$

$$\lim_{n \to \infty} \chi_{\mathbb{Z}_n^3}(3)^{1/n^3} : \text{not known}$$
Label the edges of the graph $G$ as $1, 2, \ldots, m$.

**broken circuit**: a circuit with its largest edge removed
The broken circuit theorem

**Theorem (H. Whitney, 1932)** Let $G$ have $N$ vertices. Write

$$
\chi_G(q) = a_0 q^N - a_1 q^{N-1} + a_2 q^{N-2} - \cdots .
$$

Then $a_i$ is the number of $i$-element sets of edges of $G$ that contain no broken circuit.
Example. If $G$ is a 4-cycle, then no 0-element, 1-element, or 2-element set of edges contains a broken circuit. One 3-element set contains (in fact, is) a broken circuit, and all four edges contain a broken circuit. Hence

$$
\chi_G(q) = q^4 - \binom{4}{1}q^3 + \binom{4}{2}q^2 - \left(\binom{4}{3} - 1\right)q
$$

$$
= q^4 - 4q^3 + 6q^2 - 3.
$$
Let $G = \mathbb{Z}_n^2$ and $N = n^2$ (number of vertices), so $2N$ edges. The smallest cycle in $G$ has length four. There are $N$ such cycles, so $N$ 3-element sets of edges containing (in fact, equal to) a broken circuit. Hence

$$\chi_{\mathbb{Z}_n^2}(q) = q^N - \binom{2N}{1} q^{N-1} + \binom{2N}{2} q^{N-2}$$

$$- \left( \binom{2N}{3} - N \right) q^{N-3} + \cdots$$
Theorem (J. Schneider). Let $N = n^d$, the number of vertices of $\mathbb{Z}^d_n$. Write

$$\chi_{\mathbb{Z}^d_n}(q) = c_0(N)q^N - c_1(N)q^{N-1} + c_2(N)q^{N-2} - \cdots.$$ 

Then for $N \gg 0$, $c_k(N)$ agrees with a polynomial $p_k(N)$. Moreover, $p_0, 1! p_1, 2! p_2, \ldots$ is a sequence of polynomials of binomial type.
Theorem (J. Schneider). Let $N = n^d$, the number of vertices of $\mathbb{Z}_n^d$. Write

$$\chi_{\mathbb{Z}_n^d}(q) = c_0(N)q^N - c_1(N)q^{N-1} + c_2(N)q^{N-2} - \cdots.$$ 

Then for $N \gg 0$, $c_k(N)$ agrees with a polynomial $p_k(N)$. Moreover, $p_0, 1! p_1, 2! p_2, \ldots$ is a sequence of polynomials of binomial type.

Proof uses a variant of Schneider’s previous result on placing tiles on $\mathbb{Z}_n^d$. 
Let $d = 2$. D. Kim and I. G. Enting made a computation (1979) equivalent to

$$\sum_{k \geq 0} p_k(N) x^k = (1 + 2x + x^2 - x^3 + x^4 - x^5 + x^6 - 2x^7 + 9x^8 - 38x^9 + 130x^{10} - 378x^{11} + 987x^{12} - 2436x^{13} + 5927x^{14} - 14438x^{15} + 34359x^{16} - 75058x^{17} + 134146x^{18} + \cdots)^N.$$
Let \( d = 2 \). D. Kim and I. G. Enting made a computation (1979) equivalent to

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\sum_{k \geq 0} p_k(N) x^k = (1 + 2x + x^2 - x^3 + x^4 - x^5 + x^6 - 2x^7 + 9x^8 - 38x^9 + 130x^{10} - 378x^{11} + 987x^{12} - 2436x^{13} + 5927x^{14} - 14438x^{15} + 34359x^{16} - 75058x^{17} + 134146x^{18} + \cdots)^N.
\]

Can anything be said about these numbers? Does the series converge for small \( x \)?
Some small values

\[ p_1(N) = 2N \]
\[ p_2(N) = 2N(2N - 1) \]
\[ p_3(N) = 2N(4N^2 - 6N - 1) \]
\[ p_4(N) = 4N(N + 1)(2N - 3)(2N - 5) \]
\[ p_5(N) = 8N(N + 2)(N - 2)(2N - 3)(2N - 7) \]
\[ p_6(N) = 8N(8N^5 - 60N^4 + 50N^3 + 495N^2 - 1228N + 825) \]
\[ p_7(N) = 8N(16N^6 - 168N^5 + 280N^4 + 2310N^3 - 10241N^2 + 14553N - 8010) \]
Further directions

What about $n_1 \times n_2 \times \cdots \times n_d$ tori?
Further directions

What about $n_1 \times n_2 \times \cdots \times n_d$ tori?

Nothing new: simply replace $N = n^d$ with $N = n_1 n_2 \cdots n_d$. 
What about replacing chromatic polynomials with Tutte polynomials?
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Currently under investigation. No known satisfactory generalization of broken circuit theorem.
What about replacing $p_k(n)$ with $p_{j,k}(n)$, say?
Multi-indexed polynomials

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To be investigated.
Multi-indexed polynomials

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To be investigated.

**Interesting example.** $K_{j,k}$: complete bipartite graph

**Theorem** (EC2, Exercise 5.6).

$$
\sum_{j,k \geq 0} \chi_{K_{j,k}}(n) \frac{x^j y^k}{j! k!} = (e^x + e^y - 1)^n
$$
Theorem (Schneider). Let $S$ be a bounded measurable set in $d$-dimensional Euclidean space. Let $P_k(n^d)$ be the probability that no two copies intersect when we place $k$ copies of $S$ independently and uniformly at random inside a $d$-dimensional torus of side length $n$. Then $n^{dk} P_k(n^d)$ is eventually a polynomial $p_k(n)$ for each $k$, and these polynomials form a sequence of binomial type.
arXiv:1206.6174
The last slide