



Alternating Permutations

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M.I.T.

Basic definitions

A sequence a_1, a_2, \dots, a_k of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \dots ,$$

and **reverse alternating** if

$$a_1 < a_2 > a_3 < a_4 > \dots .$$

Euler numbers

\mathfrak{S}_n : symmetric group of all permutations of $1, 2, \dots, n$

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E.g., $E_4 = 5$: 2143, 3142, 3241, 4132, 4231

André's theorem

Theorem (Désiré André, 1879)

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E_{2n+1} is a **tangent number**.

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\Rightarrow **combinatorial trigonometry**

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Equate coefficients of $x^{2n}/(2n)!$:

$$\begin{aligned} & \sum_{k=0}^n \binom{2n}{2k} E_{2k} E_{2(n-k)} \\ &= \sum_{k=0}^{n-1} \binom{2n}{2k+1} E_{2k+1} E_{2(n-k)-1}. \end{aligned}$$

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Prove combinatorially (exercise).

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Naive proof.

$$2E_{n+1} = \sum_{k=0}^n \binom{n}{k} E_k E_{n-k}, \quad n \geq 1$$

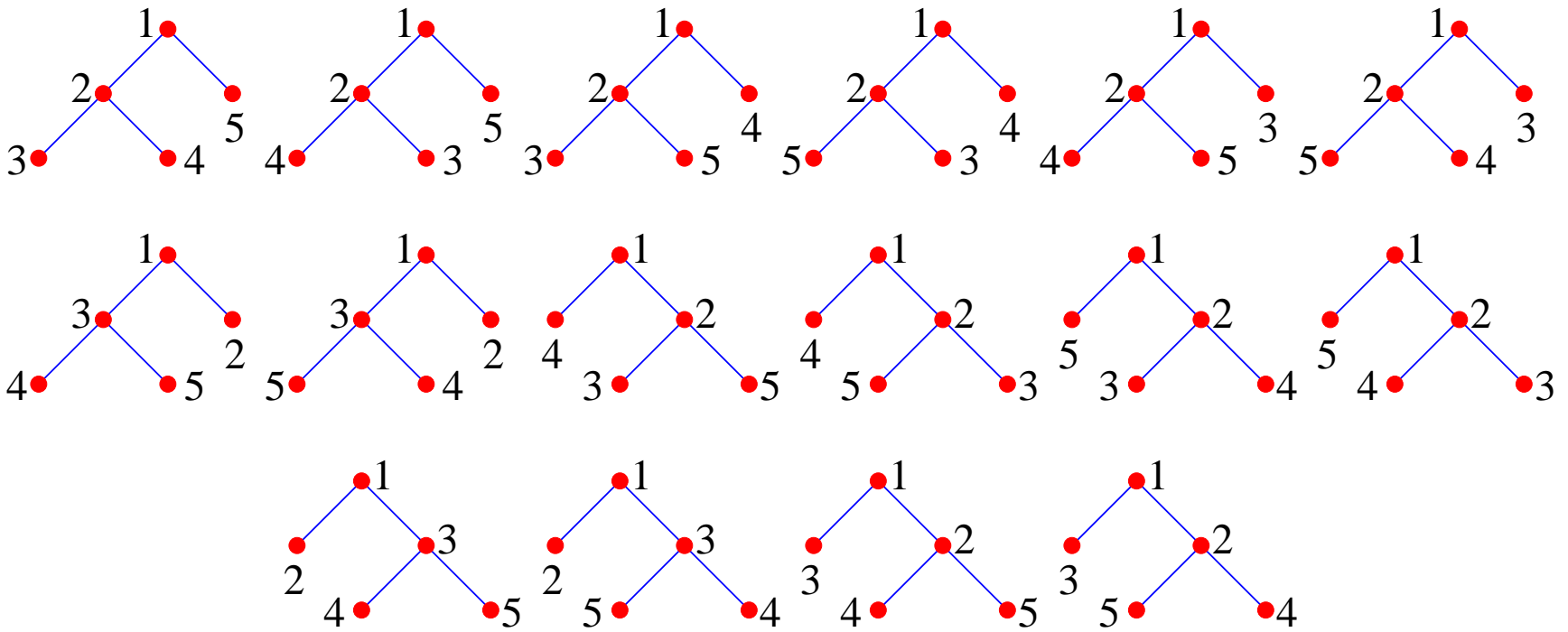
$$\Rightarrow 2y' = 1 + y^2, \text{ etc.}$$

(details omitted)

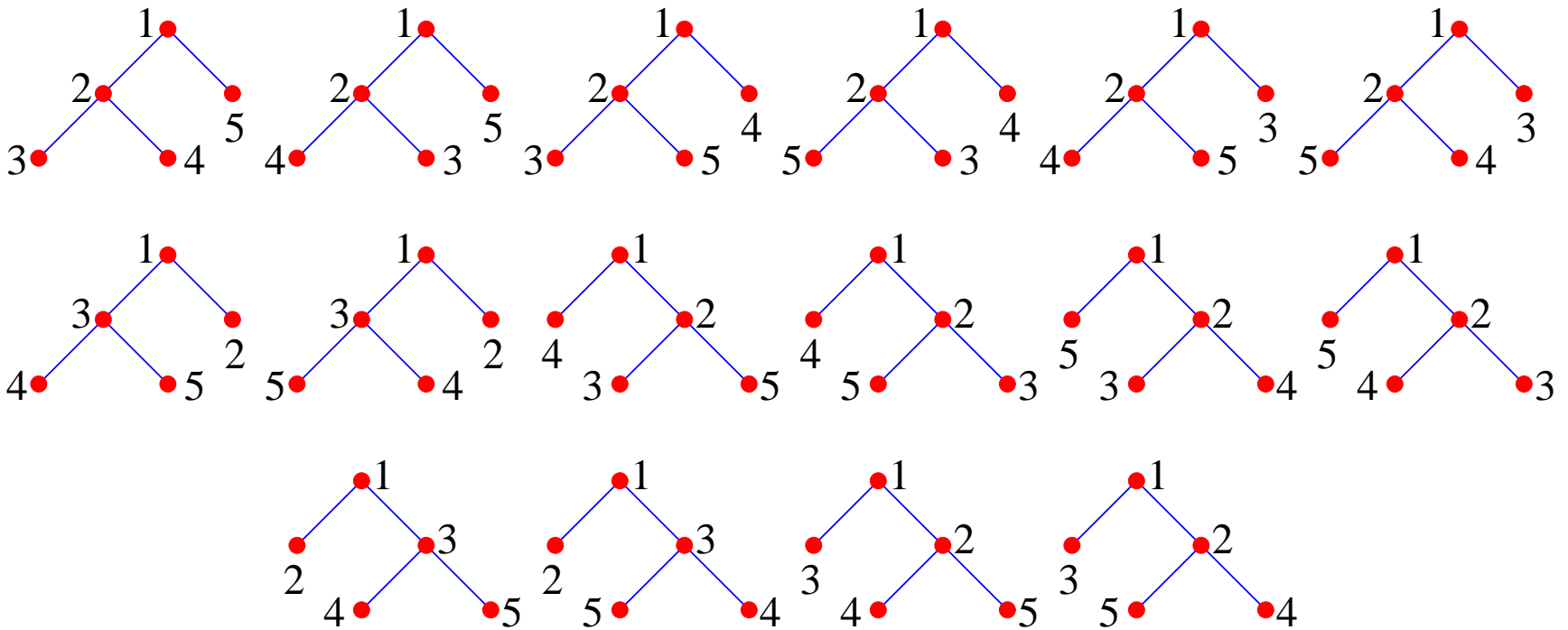
Some occurrences of Euler numbers

(1) E_{2n-1} is the number of complete increasing binary trees on the vertex set $[2n + 1] = \{1, 2, \dots, 2n + 1\}$.

Five vertices



Five vertices



Slightly more complicated for E_{2n}

Simsun permutations

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Theorem (R. Simion & S. Sundaram) The number of simsun permutations in \mathfrak{S}_n is E_{n+1} .

Orbits of mergings

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Theorem. *The number of \mathfrak{S}_n -orbits is E_{n-1} .*

Orbit representatives for $n = 5$

12-3-4-5

123-4-5

1234-5

12-3-4-5

123-4-5

123-45

12-3-4-5

12-34-5

125-34

12-3-4-5

12-34-5

12-345

12-3-4-5

12-34-5

1234-5

Volume of a polytope

(4) Let \mathcal{E}_n be the convex polytope in \mathbb{R}^n defined by

$$\begin{aligned}x_i &\geq 0, & 1 \leq i \leq n \\x_i + x_{i+1} &\leq 1, & 1 \leq i \leq n - 1.\end{aligned}$$

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Theorem. *The volume of \mathcal{E}_n is $E_n/n!$.*

The “nicest” proof

- Triangulate \mathcal{E}_n so that the maximal simplices σ_w are indexed by alternating permutations $w \in \mathfrak{S}_n$.

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- Show $\text{Vol}(\sigma_w) = 1/n!$,

Tridiagonal matrices

An $n \times n$ matrix $M = (m_{ij})$ is **tridiagonal** if $m_{ij} = 0$ whenever $|i - j| \geq 2$.

doubly-stochastic: $m_{ij} \geq 0$, row and column sums equal 1

\mathcal{T}_n : set of $n \times n$ tridiagonal doubly stochastic matrices

Polytope structure of \mathcal{T}_n

Easy fact: the map

$$\begin{aligned}\mathcal{T}_n &\rightarrow \mathbb{R}^{n-1} \\ M &\mapsto (m_{12}, m_{23}, \dots, m_{n-1,n})\end{aligned}$$

is a (linear) bijection from \mathcal{T}_n to \mathcal{E}_n .

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Application (**Diaconis** et al.): random doubly stochastic tridiagonal matrices and random walks on \mathcal{T}_n

Distribution of $is(w)$

Yesterday: $is(w)$ = length of longest increasing subsequence of $w \in \mathfrak{S}_n$

$$E(n) \sim 2\sqrt{n}$$

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For fixed $t \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t),$$

the **Tracy-Widom distribution**.

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The first is much easier!

Longest alternating subsequences

$as(w)$ = length longest alt. subseq. of w

$$D(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} as(w) \sim ?$$

$$w = \mathbf{56218347} \Rightarrow as(w) = 5$$

Definitions of $a_k(n)$ and $b_k(n)$

$$a_k(n) = \#\{w \in \mathfrak{S}_n : \text{as}(w) = k\}$$

$$\begin{aligned} b_k(n) &= a_1(n) + a_2(n) + \cdots + a_k(n) \\ &= \#\{w \in \mathfrak{S}_n : \text{as}(w) \leq k\}. \end{aligned}$$

The case $n = 3$

w	$as(w)$
123	1
132	2
213	3
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$$a_1(3) = 1, a_2(3) = 3, a_3(3) = 2$$

$$b_1(3) = 1, b_2(3) = 4, b_3(3) = 6$$

The main lemma

Lemma. $\forall w \in \mathfrak{S}_n \exists$ *alternating subsequence of maximal length that contains n .*

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Corollary.

$$\Rightarrow a_k(n) = \sum_{j=1}^n \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} (a_{2r}(j-1) + a_{2r+1}(j-1)) a_s(n-j)$$

The main generating function

$$B(x, t) = \sum_{k, n \geq 0} b_k(n) t^k \frac{x^n}{n!}$$

Theorem.

$$B(x, t) = \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho},$$

where $\rho = \sqrt{1 - t^2}$.

Formulas for $b_k(n)$

Corollary.

$$\Rightarrow b_1(n) = 1$$

$$b_2(n) = n$$

$$b_3(n) = \frac{1}{4}(3^n - 2n + 3)$$

$$b_4(n) = \frac{1}{8}(4^n - (2n - 4)2^n)$$

⋮

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no such formulas for longest **increasing** subsequences

Mean (expectation) of $\text{as}(w)$

$$D(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{as}(w) = \frac{1}{n!} \sum_{k=1}^n k \cdot a_k(n),$$

the **expectation** of $\text{as}(w)$ for $w \in \mathfrak{S}_n$

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the **expectation** of $as(w)$ for $w \in \mathfrak{S}_n$

Let

$$\begin{aligned} A(x, t) &= \sum_{k, n \geq 0} a_k(n) t^k \frac{x^n}{n!} = (1-t)B(x, t) \\ &= (1-t) \left(\frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho} \right). \end{aligned}$$

Formula for $D(n)$

$$\begin{aligned}\sum_{n \geq 0} D(n)x^n &= \frac{\partial}{\partial t} A(x, 1) \\ &= \frac{6x - 3x^2 + x^3}{6(1-x)^2} \\ &= x + \sum_{n \geq 2} \frac{4n+1}{6} x^n.\end{aligned}$$

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Compare $E(n) \sim 2\sqrt{n}$.

Variance of $as(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left(as(w) - \frac{4n+1}{6} \right)^2, \quad n \geq 2$$

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similar results for higher moments

A new distribution?

$$P(t) = \lim_{n \rightarrow \infty} \text{Prob}_{w \in \mathfrak{S}_n} \left(\frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right)$$

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Stanley distribution?

Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

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$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45}/4} e^{-s^2} ds$$

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Umbral enumeration

Umbral formula: involves E^k , where E is an indeterminate (the **umbra**). Replace E^k with the Euler number E_k . (Technique from 19th century, modernized by Rota et al.)

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Example.

$$\begin{aligned}(1 + E^2)^3 &= 1 + 3E^2 + 3E^4 + E^6 \\ &= 1 + 3E_2 + 3E_4 + E_6 \\ &= 1 + 3 \cdot 1 + 3 \cdot 5 + 61 \\ &= 80\end{aligned}$$

Another example

$$\begin{aligned}(1+t)^E &= 1 + Et + \binom{E}{2}t^2 + \binom{E}{3}t^3 + \dots \\ &= 1 + Et + \frac{1}{2}E(E-1)t^2 + \dots \\ &= 1 + E_1t + \frac{1}{2}(E_2 - E_1)t^2 + \dots \\ &= 1 + t + \frac{1}{2}(1-1)t^2 + \dots \\ &= 1 + t + O(t^3).\end{aligned}$$

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Let $f(n)$ be the number of **alternating** fixed-point free involutions in \mathfrak{S}_{2n} .

$$n = 3 : \quad 214365 = (1, 2)(3, 4)(5, 6)$$

$$645231 = (1, 6)(2, 4)(3, 5)$$

$$f(3) = 2$$

An umbral theorem

Theorem.

$$F(x) = \sum_{n \geq 0} f(n)x^n$$

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$$\begin{aligned} F(x) &= \sum_{n \geq 0} f(n)x^n \\ &= \left(\frac{1+x}{1-x} \right)^{(E^2+1)/4} \end{aligned}$$

Proof idea

Proof. Uses representation theory of the symmetric group \mathfrak{S}_n .

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$$\chi(w) = 0 \text{ or } \pm E_k.$$

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Now use known results on combinatorial properties of characters of \mathfrak{S}_n .

Ramanujan's Lost Notebook

Theorem (Ramanujan, Berndt, implicitly) As $x \rightarrow 0+$,

$$2 \sum_{n \geq 0} \left(\frac{1-x}{1+x} \right)^{n(n+1)} \sim \sum_{k \geq 0} f(k) x^k = F(x),$$

an **analytic** (non-formal) identity.

A formal identity

Corollary (via Ramanujan, Andrews).

$$F(x) = 2 \sum_{n \geq 0} q^n \frac{\prod_{j=1}^n (1 - q^{2j-1})}{\prod_{j=1}^{2n+1} (1 + q^j)},$$

where $q = \left(\frac{1-x}{1+x}\right)^{2/3}$, a **formal** identity.

Simple result, hard proof

Recall: number of n -cycles in \mathfrak{S}_n is $(n - 1)!$.

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Theorem. Let $b(n)$ be the number of **alternating** n -cycles in \mathfrak{S}_n . Then if n is odd,

$$b(n) = \frac{1}{n} \sum_{d|n} \mu(d) (-1)^{(d-1)/2} E_{n/d}.$$

Special case

Corollary. *Let p be an odd prime. Then*

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Combinatorial proof?

Inc. subsequences of alt. perms.

Recall: $\text{is}(w)$ = length of longest increasing subsequence of $w \in \mathfrak{S}_n$. Define

$$C(n) = \frac{1}{E_n} \sum_w \text{is}(w),$$

where w ranges over all E_n alternating permutations in \mathfrak{S}_n .

β

Little is known, e.g., what is

$$\beta = \lim_{n \rightarrow \infty} \frac{\log C(n)}{\log n} ?$$

I.e., $C(n) = n^{\beta + o(1)}$.

Compare $\lim_{n \rightarrow \infty} \frac{\log E(n)}{\log n} = 1/2$.

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Easy: $\beta \geq \frac{1}{2}$.

Limiting distribution?

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Possible tool: \exists “umbral analogue” of Gessel’s determinantal formula.

Descent sets

Let $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. **Descent set** of w :

$$D(w) = \{i : a_i > a_{i+1}\} \subseteq \{1, \dots, n-1\}$$

Descent sets

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$$D(\mathbf{4}15\mathbf{76}23) = \{1, 4, 5\}$$

$$D(\mathbf{4}1\mathbf{5}2\mathbf{7}36) = \{1, 3, 5\} \text{ (alternating)}$$

$$D(4\mathbf{7}3\mathbf{6}1\mathbf{5}2) = \{2, 4, 6\} \text{ (reverse alternating)}$$

$\beta_n(S)$

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w	$D(w)$
123	\emptyset
213	$\{1\}$
312	$\{1\}$
132	$\{2\}$
231	$\{2\}$
321	$\{1, 2\}$

$$\beta_3(\emptyset) = 1, \quad \beta_3(1) = 2$$

$$\beta_3(2) = 2, \quad \beta_3(1, 2) = 1$$

u_S

Fix n . Let $S \subseteq \{1, \dots, n-1\}$. Let $u_S = t_1 \cdots t_{n-1}$,
where

$$t_i = \begin{cases} a, & i \notin S \\ b, & i \in S. \end{cases}$$

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Example. $n = 8$, $S = \{2, 5, 6\} \subseteq \{1, \dots, 7\}$

$$u_S = abaabba$$

A noncommutative gen. function

$$\Psi_n(a, b) = \sum_{S \subseteq \{1, \dots, n-1\}} \beta_n(S) u_S.$$

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Thus

$$\Psi_3(a, b) = aa + 2ab + 2ba + bb$$

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$$\beta_3(\emptyset) = 1, \quad \beta_3(1) = 2, \quad \beta_3(2) = 2, \quad \beta_3(1, 2) = 1$$

Thus

$$\begin{aligned} \Psi_3(a, b) &= aa + 2ab + 2ba + bb \\ &= (a + b)^2 + (ab + ba) \end{aligned}$$

The cd -index

Theorem. *There exists a noncommutative polynomial $\Phi_n(c, d)$, called the **cd -index** of \mathfrak{S}_n , with **nonnegative** integer coefficients, such that*

$$\Psi_n(a, b) = \Phi_n(a + b, ab + ba).$$

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Example. Recall

$$\Psi_3(a, b) = aa + 2ab + 2ba + b^2 = (a + b)^2 + (ab + ba).$$

Therefore

$$\Phi_3(c, d) = c^2 + d.$$

Small values of $\Phi_n(c, d)$

$$\Phi_1 = 1$$

$$\Phi_2 = c$$

$$\Phi_3 = c^2 + d$$

$$\Phi_4 = c^3 + 2cd + 2dc$$

$$\Phi_5 = c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2$$

$$\Phi_6 = c^5 + 4c^3d + 9c^2dc + 9cdc^2 + 4dc^3 \\ + 12cd^2 + 10dcd + 12d^2c.$$

S_μ

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Example. $n = 10$:

$$\mu = cd^2c^2d \rightarrow 0 \cdot 10 \cdot 10 \cdot 0 \cdot 0 \cdot 10 \times = 01010001,$$

the characteristic vector of $S_\mu = \{2, 4, 8\} \subseteq [8]$

cd-index coefficients

Recall: $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is a **simsum** permutation if the subsequence with elements $1, 2, \dots, k$ has no double descents, $1 \leq k \leq n$.

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Theorem (Simion-Sundaram, variant of Foata-Schützenberger) *The coefficient of μ in $\Phi(c, d)$ is equal to the number of simsum permutations in \mathfrak{S}_{n-1} with descent set S_μ .*

An example

Example. $\Phi_6 =$

$$c^5 + 4c^3d + 9c^2dc + 9cdc^2 + 4dc^3 + 12cd^2 + 10dcd + 12d^2c,$$

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The ten simsun permutations $w \in \mathfrak{S}_5$ with $D(w) = \{1, 4\}$:

21354, 21453, 31254, 31452, 41253

41352, 42351, 51243, 51342, 52341,

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but **not** 32451.

Two consequences

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(b) (**Niven, de Bruijn**) For all $S \subseteq \{1, \dots, n-1\}$,

$$\beta_n(S) \leq E_n,$$

with equality if and only if $S = \{1, 3, 5, \dots\}$ or $S = \{2, 4, 6, \dots\}$

An example

$$\Phi_5 = 1c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2$$

$$1 + 3 + 5 + 3 + 4 = 16 = E_5$$

Darn!

That's
the
end...

