Alternating Permutations

Richard P. Stanley

M.I.T.
A sequence \( a_1, a_2, \ldots, a_k \) of distinct integers is **alternating** if

\[ a_1 > a_2 < a_3 > a_4 < \cdots , \]

and **reverse alternating** if

\[ a_1 < a_2 > a_3 < a_4 > \cdots . \]
Euler numbers

\( \mathfrak{S}_n \): symmetric group of all permutations of 1, 2, \ldots, n

E.g., \( E_4 = 5 : 2143; 3142; 3241; 4132; 4231 \)
Euler numbers

\( \mathcal{S}_n \): symmetric group of all permutations of 1, 2, \ldots, \( n \)

Euler number:

\[
E_n = \# \{ w \in \mathcal{S}_n : w \text{ is alternating} \}
\]

\[
= \# \{ w \in \mathcal{S}_n : w \text{ is reverse alternating} \}
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E.g., \( E_4 = 5 : 2143, 3142, 3241, 4132, 4231 \)
Theorem (Désiré André, 1879)

\[ y := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x \]
André’s theorem

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\( E_{2n+1} \) is a tangent number.
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⇒ combinatorial trigonometry
Example of combinatorial trig.

$$\sec^2 x = 1 + \tan^2 x$$
Example of combinatorial trig.

\[ \sec^2 x = 1 + \tan^2 x \]

Equate coefficients of \( x^{2n} / (2n)! \):

\[
\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} E_{2(n-k)}
\]

\[
= \sum_{k=0}^{n-1} \binom{2n}{2k + 1} E_{2k+1} E_{2(n-k) - 1}.
\]
Example of combinatorial trig.

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Equate coefficients of \( x^{2n} / (2n)! : \)

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= \sum_{k=0}^{n-1} \binom{2n}{2k + 1} E_{2k+1} E_{2(n-k)-1}.
\]

Prove combinatorially (exercise).
Proof of André’s theorem

\[ y := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x \]
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Naive proof.

\[ 2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \ n \geq 1 \]

\[ \Rightarrow 2y' = 1 + y^2, \ \text{etc.} \]

(details omitted)
Some occurrences of Euler numbers

(1) \( E_{2n-1} \) is the number of complete increasing binary trees on the vertex set \([2n + 1] = \{1, 2, \ldots, 2n + 1\} \).
Five vertices

Alternating Permutations – p. 8
Five vertices

Slightly more complicated for $E_{2n}$
(2) $b_1 b_2 \cdots b_k$ has a **double descent** if for some $1 < i < n$,

$$b_{i-1} > b_i > b_{i+1}.$$
Simsun permutations

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$w = a_1 a_2 \cdots a_n \in S_n$ is a **simsun** permutation if the subsequence with elements $1, 2, \ldots, k$ has no double descents, $1 \leq k \leq n$. 

Example. $3241$ is not simsun: the subsequence with $1, 2, 3$ is $321$. 

Theorem (R. Simion & S. Sundaram) The number of simsun permutations in $S_n$ is $E_{n+1}$. 

Alternating Permutations – p. 9
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**Example.** 3241 is **not** simsun: the subsequence with 1, 2, 3 is 321.
Simsun permutations

(2) $b_1 b_2 \cdots b_k$ has a **double descent** if for some $1 < i < n$,

$$b_{i-1} > b_i > b_{i+1}. \tag{2}$$

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**Theorem** *(R. Simion & S. Sundaram)* The number of simsun permutations in $\mathcal{S}_n$ is $E_{n+1}$. 
(3) Start with \( n \) one-element sets \( \{1\}, \ldots, \{n\} \).
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$1 - 2 - 3 - 4 - 5 - 6, \ 12 - 3 - 4 - 5 - 6, \ 12 - 34 - 5 - 6$

$125 - 34 - 6, \ 125 - 346, \ 123456$
Orbits of mergings

(3) Start with $n$ one-element sets $\{1\}, \ldots, \{n\}$. Merge together two at a time until reaching $\{1, 2, \ldots, n\}$.

\[1 - 2 - 3 - 4 - 5 - 6, \quad 12 - 3 - 4 - 5 - 6, \quad 12 - 34 - 5 - 6\]
\[125 - 34 - 6, \quad 125 - 346, \quad 123456\]

$S_n$ acts on these sequences.
(3) Start with \( n \) one-element sets \( \{1\}, \ldots, \{n\} \). Merge together two at a time until reaching \( \{1, 2, \ldots, n\} \).

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1-2-3-4-5-6, \quad 12-3-4-5-6, \quad 12-34-5-6
\]
\[
125-34-6, \quad 125-346, \quad 123456
\]

\( S_n \) acts on these sequences.

**Theorem.** *The number of \( S_n \)-orbits is \( E_{n-1} \).*
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(4) Let $\mathcal{E}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

$$
x_i \geq 0, \ 1 \leq i \leq n$$

$$
x_i + x_{i+1} \leq 1, \ 1 \leq i \leq n - 1.
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(4) Let $\mathcal{E}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

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$$x_i + x_{i+1} \leq 1, \ 1 \leq i \leq n - 1.$$ 

**Theorem.** The volume of $\mathcal{E}_n$ is $E_n/n!$. 
The “nicest” proof

Triangulate $\mathcal{E}_n$ so that the maximal simplices $\sigma_w$ are indexed by alternating permutations $w \in S_n$. 

$\text{Show } \text{Vol}(w) = 1 = n!$
The “nicest” proof

- Triangulate $\mathcal{E}_n$ so that the maximal simplices $\sigma_w$ are indexed by alternating permutations $w \in \mathcal{S}_n$.

- Show $\text{Vol}(\sigma_w) = 1/n!$, 
An $n \times n$ matrix $M = (m_{ij})$ is tridiagonal if $m_{ij} = 0$ whenever $|i - j| \geq 2$.

**doubly-stochastic:** $m_{ij} \geq 0$, row and column sums equal 1

$\mathcal{T}_n$: set of $n \times n$ tridiagonal doubly stochastic matrices
Polytope structure of $\mathcal{T}_n$

**Easy fact:** the map

$$\mathcal{T}_n \rightarrow \mathbb{R}^{n-1}$$

$$M \mapsto (m_{12}, m_{23}, \ldots, m_{n-1,n})$$

is a (linear) bijection from $\mathcal{T}_n$ to $\mathcal{E}_n$. 
Polytope structure of $\mathcal{T}_n$

Easy fact: the map

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Application (Diaconis et al.): random doubly stochastic tridiagonal matrices and random walks on $\mathcal{T}_n$
Yesterday: \( \text{is}(w) = \text{length of longest increasing subsequence of } w \in \mathcal{S}_n \)

\[
E(n) \sim 2\sqrt{n}
\]
Yesterday: $is(w) = \text{length of longest increasing subsequence of } w \in S_n$

$$E(n) \sim 2\sqrt{n}$$

For fixed $t \in \mathbb{R}$,

$$\lim_{n \to \infty} \text{Prob} \left( \frac{is_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t),$$

the Tracy-Widom distribution.
Analogues of distribution of $\text{is}(w)$

- Length of longest alternating subsequence of $w \in S_n$
Analogues of distribution of $\text{is}(w)$

- Length of longest alternating subsequence of $w \in S_n$
- Length of longest increasing subsequence of an alternating permutation $w \in S_n$. 

The first is much easier!
Analogues of distribution of $\text{is}(w)$

- Length of longest alternating subsequence of $w \in \mathcal{S}_n$

- Length of longest increasing subsequence of an alternating permutation $w \in \mathcal{S}_n$.

The first is much easier!
Longest alternating subsequences

$$\text{as}(w) = \text{length longest alt. subseq. of } w$$

$$D(n) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} \text{as}(w) \sim ?$$

$$w = 56218347 \Rightarrow \text{as}(w) = 5$$
Definitions of $a_k(n)$ and $b_k(n)$

\[
a_k(n) = \#\{w \in S_n : \text{as}(w) = k\}
\]

\[
b_k(n) = a_1(n) + a_2(n) + \cdots + a_k(n)
\]

\[
= \#\{w \in S_n : \text{as}(w) \leq k\}.
\]
The case $n = 3$

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$a_1(3) = 1$, $a_2(3) = 3$, $a_3(3) = 2$

$b_1(3) = 1$, $b_2(3) = 4$, $b_3(3) = 6$
Lemma. $\forall w \in S_n \exists$ alternating subsequence of maximal length that contains $n$. 
The main lemma

Lemma. \( \forall \, w \in \mathfrak{S}_n \, \exists \) alternating subsequence of maximal length that contains \( n \).

Corollary.

\[
\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}
\]

\[
\sum_{2r+s=k-1} (a_{2r}(j-1) + a_{2r+1}(j-1)) a_s(n-j)
\]
The main generating function

\[ B(x, t) = \sum_{k,n \geq 0} b_k(n) t^k \frac{x^n}{n!} \]

Theorem.

\[ B(x, t) = \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho}, \]

where \( \rho = \sqrt{1 - t^2} \).
Corollary.

\[ \Rightarrow b_1(n) = 1 \]
\[ b_2(n) = n \]
\[ b_3(n) = \frac{1}{4}(3^n - 2n + 3) \]
\[ b_4(n) = \frac{1}{8}(4^n - (2n - 4)2^n) \]
\[ \vdots \]
Corollary.

\[\Rightarrow \quad b_1(n) = 1\]
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\[\vdots\]

no such formulas for longest increasing subsequences
Mean (expectation) of \( \text{as}(w) \)

\[
D(n) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} \text{as}(w) = \frac{1}{n!} \sum_{k=1}^{n} k \cdot a_k(n),
\]

the expectation of \( \text{as}(w) \) for \( w \in \mathcal{S}_n \).
Mean (expectation) of $as(w)$

$$D(n) = \frac{1}{n!} \sum_{w \in S_n} as(w) = \frac{1}{n!} \sum_{k=1}^{n} k \cdot a_k(n),$$

the expectation of $as(w)$ for $w \in S_n$

Let

$$A(x, t) = \sum_{k, n \geq 0} a_k(n) t^k \frac{x^n}{n!} = (1 - t) B(x, t)$$

$$= (1 - t) \left( \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho} \right).$$
Formula for $D(n)$

$$\sum_{n \geq 0} D(n) x^n = \frac{\partial}{\partial t} A(x, 1)$$

$$= \frac{6x - 3x^2 + x^3}{6(1 - x)^2}$$

$$= x + \sum_{n \geq 2} \frac{4n + 1}{6} x^n.$$
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$$\Rightarrow D(n) = \frac{4n + 1}{6}, \quad n \geq 2$$
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= x + \sum_{n \geq 2} \frac{4n + 1}{6} x^n.
\]

\[
\Rightarrow D(n) = \frac{4n + 1}{6}, \quad n \geq 2
\]

Compare $E(n) \sim 2\sqrt{n}$. 
Variance of $\text{as}(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} \left( \text{as}(w) - \frac{4n + 1}{6} \right)^2, \quad n \geq 2$$

the variance of $\text{as}(n)$ for $w \in \mathcal{S}_n$
Variance of $\text{as}(w)$

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the variance of $\text{as}(n)$ for $w \in \mathcal{S}_n$

Corollary.

$$V(n) = \frac{8}{45} n - \frac{13}{180}, \ n \geq 4$$
Variance of $\text{as}(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} \left( \text{as}(w) - \frac{4n + 1}{6} \right)^2, \quad n \geq 2$$

the variance of $\text{as}(n)$ for $w \in \mathcal{S}_n$

Corollary.

$$V(n) = \frac{8}{45}n - \frac{13}{180}, \quad n \geq 4$$

similar results for higher moments
A new distribution?

\[ P(t) = \lim_{n \to \infty} \text{Prob}_{w \in \mathcal{S}_n} \left( \frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right) \]
A new distribution?

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Stanley distribution?
Limiting distribution

**Theorem** (Pemantle, Widom, (Wilf)).

$$\lim_{n \to \infty} \text{Prob}_{w \in \mathcal{S}_n} \left( \frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45/4}} e^{-s^2} ds$$

(Gaussian distribution)
Limiting distribution

**Theorem** (Pemantle, Widom, (Wilf)).

\[
\lim_{n \to \infty} \text{Prob}_{w \in S_n} \left( \frac{\alpha_s(w) - 2n/3}{\sqrt{n}} \leq t \right)
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t \sqrt{45}/4} e^{-s^2} \, ds
\]

(Gaussian distribution)

😊😊
Umbral formula: involves $E^k$, where $E$ is an indeterminate (the umbra). Replace $E^k$ with the Euler number $E_k$. (Technique from 19th century, modernized by Rota et al.)
Umbral enumeration

**Umbral formula:** involves $E^k$, where $E$ is an indeterminate (the *umbra*). Replace $E^k$ with the Euler number $E_k$. (Technique from 19th century, modernized by Rota et al.)

**Example.**

\[
(1 + E^2)^3 = 1 + 3E^2 + 3E^4 + E^6 \\
= 1 + 3E_2 + 3E_4 + E_6 \\
= 1 + 3 \cdot 1 + 3 \cdot 5 + 61 \\
= 80
\]
Another example

\[(1 + t)^E = 1 + Et + \left(\frac{E}{2}\right)t^2 + \left(\frac{E}{3}\right)t^3 + \cdots\]

\[= 1 + Et + \frac{1}{2}E(E - 1)t^2 + \cdots\]

\[= 1 + E_1t + \frac{1}{2}(E_2 - E_1))t^2 + \cdots\]

\[= 1 + t + \frac{1}{2}(1 - 1)t^2 + \cdots\]

\[= 1 + t + O(t^3).\]
fixed point free involution $w \in S_{2n}$: all cycles of length two
fixed point free involution \( w \in S_{2n} \): all cycles of length two

Let \( f(n) \) be the number of alternating fixed-point free involutions in \( S_{2n} \).
fixed point free involution $w \in S_{2n}$: all cycles of length two

Let $f(n)$ be the number of alternating fixed-point free involutions in $S_{2n}$.

$n = 3$:

- $214365 = (1, 2)(3, 4)(5, 6)$
- $645231 = (1, 6)(2, 4)(3, 5)$

$f(3) = 2$
An umbral theorem

Theorem.

\[ F(x) = \sum_{n \geq 0} f(n)x^n \]
An umbral theorem

Theorem.

\[ F(x) = \sum_{n \geq 0} f(n)x^n \]

\[ = \left( \frac{1 + x}{1 - x} \right)^{(E^2 + 1)/4} \]
Proof idea

**Proof.** Uses representation theory of the symmetric group $\mathfrak{S}_n$. 
Proof. Uses representation theory of the symmetric group $\mathfrak{S}_n$.

There is a character $\chi$ of $\mathfrak{S}_n$ (due to H. O. Foulkes) such that for all $w \in \mathfrak{S}_n$,

$$\chi(w) = 0 \text{ or } \pm E_k.$$
Proof idea

Proof. Uses representation theory of the symmetric group $\mathfrak{S}_n$.

There is a character $\chi$ of $\mathfrak{S}_n$ (due to H. O. Foulkes) such that for all $w \in \mathfrak{S}_n$,

$$\chi(w) = 0 \text{ or } \pm E_k.$$ 

Now use known results on combinatorial properties of characters of $\mathfrak{S}_n$. 
Theorem (Ramanujan, Berndt, implicitly) As $x \to 0+$,

\[
2 \sum_{n \geq 0} \left( \frac{1 - x}{1 + x} \right)^n (n+1) \sim \sum_{k \geq 0} f(k)x^k = F(x),
\]

an analytic (non-formal) identity.
A formal identity

**Corollary** (via Ramanujan, Andrews).

\[
F(x) = 2 \sum_{n \geq 0} q^n \frac{\prod_{j=1}^{n} (1 - q^{2j-1})}{\prod_{j=1}^{2n+1} (1 + q^j)},
\]

where \( q = \left( \frac{1-x}{1+x} \right)^{2/3} \), a formal identity.
Simple result, hard proof

**Recall:** number of $n$-cycles in $\mathfrak{S}_n$ is $(n - 1)!$. 
Recall: number of \( n \)-cycles in \( S_n \) is \((n - 1)!\).

**Theorem.** Let \( b(n) \) be the number of **alternating** \( n \)-cycles in \( S_n \). Then if \( n \) is odd,

\[
b(n) = \frac{1}{n} \sum_{d|n} \mu(d) (-1)^{(d-1)/2} E_{n/d}.
\]
**Corollary.** Let $p$ be an odd prime. Then

$$b(p) = \frac{1}{p} \left( E_p - (-1)^{(p-1)/2} \right).$$
Corollary. Let $p$ be an odd prime. Then

$$b(p) = \frac{1}{p} \left( E_p - (-1)^{(p-1)/2} \right).$$

Combinatorial proof?
Recall: $is(w)$ = length of longest increasing subsequence of $w \in S_n$. Define

$$C(n) = \frac{1}{E_n} \sum w is(w),$$

where $w$ ranges over all $E_n$ alternating permutations in $S_n$. 
Little is known, e.g., what is
\[ \beta = \lim_{n \to \infty} \frac{\log C(n)}{\log n} \]?

I.e., \( C(n) = n^{\beta + o(1)} \).

Compare \( \lim_{n \to \infty} \frac{\log E(n)}{\log n} = 1/2 \).
Little is known, e.g., what is

$$\beta = \lim_{n \to \infty} \frac{\log C(n)}{\log n}?$$

I.e., $C(n) = n^{\beta + o(1)}$.

Compare $\lim_{n \to \infty} \frac{\log E(n)}{\log n} = 1/2$.

**Easy:** $\beta \geq \frac{1}{2}$. 
What is the (suitably scaled) limiting distribution of $is(w)$, where $w$ ranges over all alternating permutations in $\mathfrak{S}_n$?
What is the (suitably scaled) limiting distribution of $is(w)$, where $w$ ranges over all alternating permutations in $\mathfrak{S}_n$?

Is it the Tracy-Widom distribution?
Limiting distribution?

What is the (suitably scaled) limiting distribution of is(\(w\)), where \(w\) ranges over all alternating permutations in \(\mathfrak{S}_n\)?

Is it the Tracy-Widom distribution?

**Possible tool:** ∃ “umbral analogue” of Gessel’s determinantal formula.
Let $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. Descent set of $w$:

$$D(w) = \{ i : a_i > a_{i+1} \} \subseteq \{1, \ldots, n - 1 \}$$
Descent sets

Let $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. **Descent set of** $w$:

$$D(w) = \{ i : a_i > a_{i+1} \} \subseteq \{1, \ldots, n - 1\}$$

\[
\begin{align*}
D(4157623) &= \{1, 4, 5\} \\
D(4152736) &= \{1, 3, 5\} \text{ (alternating)} \\
D(4736152) &= \{2, 4, 6\} \text{ (reverse alternating)}
\end{align*}
\]
$\beta_n(S) = \#\{w \in \mathfrak{S}_n : D(w) = S\}$
\[ \beta_n(S) = \# \{ w \in \mathfrak{S}_n : D(w) = S \} \]

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<tr>
<th>( w )</th>
<th>( D(w) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>213</td>
<td>{1}</td>
</tr>
<tr>
<td>312</td>
<td>{1}</td>
</tr>
<tr>
<td>132</td>
<td>{2}</td>
</tr>
<tr>
<td>231</td>
<td>{2}</td>
</tr>
<tr>
<td>321</td>
<td>{1, 2}</td>
</tr>
</tbody>
</table>
Fix $n$. Let $S \subseteq \{1, \cdots, n-1\}$. Let $u_S = t_1 \cdots t_{n-1}$, where

$$t_i = \begin{cases} a, & i \notin S \\ b, & i \in S. \end{cases}$$
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\[
t_i = \begin{cases} 
a, & i \notin S \\
b, & i \in S.
\end{cases}
\]

**Example.** \( n = 8, \ S = \{2, 5, 6\} \subseteq \{1, \ldots, 7\} \)

\[
u_S = abaabba
\]
A noncommutative gen. function

\[ \Psi_n(a, b) = \sum_{S \subseteq \{1, \ldots, n-1\}} \beta_n(S) a_S. \]
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Example. Recall

\[ \beta_3(\emptyset) = 1, \quad \beta_3(1) = 2, \quad \beta_3(2) = 2, \quad \beta_3(1, 2) = 1 \]

Thus

\[ \Psi_3(a, b) = aa + 2ab + 2ba + bb \]
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\[ = (a + b)^2 + (ab + ba) \]
The cd-index Theorem. There exists a noncommutative polynomial $\Phi_n(c, d)$, called the cd-index of $S_n$, with nonnegative integer coefficients, such that

$$\Psi_n(a, b) = \Phi_n(a + b, ab + ba).$$
Theorem. There exists a noncommutative polynomial $\Phi_n(c, d)$, called the $cd$-index of $\mathfrak{S}_n$, with nonnegative integer coefficients, such that

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Example. Recall

$$\Psi_3(a, b) = aa + 2ab + 2ba + b^2 = (a + b)^2 + (ab + ba).$$

Therefore

$$\Phi_3(c, d) = c^2 + d.$$
Small values of $\Phi_n(c, d)$

\[
\begin{align*}
\Phi_1 &= 1 \\
\Phi_2 &= c \\
\Phi_3 &= c^2 + d \\
\Phi_4 &= c^3 + 2cd + 2dc \\
\Phi_5 &= c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2 \\
\Phi_6 &= c^5 + 4c^3d + 9c^2dc + 9cdc^2 + 4dc^3 \\
&\quad + 12cd^2 + 10dcd + 12d^2c.
\end{align*}
\]
Let $\deg c = 1$, $\deg d = 2$. 
Let \( \deg c = 1 \), \( \deg d = 2 \).

\( \mu \): \( cd \)-monomial of degree \( n - 1 \)
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$\mu$: $cd$-monomial of degree $n - 1$

Replace each $c$ in $\mu$ with $0$, each $d$ with $10$, and remove final $0$. Get the characteristic vector of a set $S_\mu \subseteq [n - 2]$. 
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\( \mu \): \( cd \)-monomial of degree \( n - 1 \)

Replace each \( c \) in \( \mu \) with 0, each \( d \) with 10, and remove final 0. Get the characteristic vector of a set \( S_\mu \subseteq [n - 2] \).

**Example.** \( n = 10 \):

\[
\mu = cd^2 c^2 d \rightarrow 0 \cdot 10 \cdot 10 \cdot 0 \cdot 0 \cdot 1\text{\cancel{0}} = 01010001,
\]

the characteristic vector of \( S_\mu = \{2, 4, 8\} \subseteq [8] \).
Recall: \( w = a_1 a_2 \cdots a_n \in S_n \) is a **simsun** permutation if the subsequence with elements 1, 2, \ldots, \( k \) has no double descents, 1 \( \leq k \leq n \).
Recall: \( w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n \) is a \textbf{simsun} permutation if the subsequence with elements \( 1, 2, \ldots, k \) has no double descents, \( 1 \leq k \leq n \).

**Theorem** (Simion-Sundaram, variant of Foata-Schützenberger) \textit{The coefficient of} \( \mu \) \textit{in} \( \Phi(c, d) \) \textit{is equal to the number of simsun permutations in} \( \mathfrak{S}_{n-1} \) \textit{with descent set} \( S_\mu \).
Example. \( \Phi_6 = c^5 + 4c^3d + 9c^2dc + 9cdc^2 + 4dc^3 + 12cd^2 + 10dcd + 12d^2c, \)

\[ dcd \rightarrow 10 \cdot 0 \cdot 1 \times \Rightarrow S_{dcd} = \{1, 4\} \]
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The ten simsun permutations $w \in S_5$ with $D(w) = \{1, 4\}$:

21354, 21453, 31254, 31452, 41253
41352, 42351, 51243, 51342, 52341,
**Example.** \( \Phi_6 = c^5 + 4c^3d + 9c^2dc + 9dcd^2 + 4dc^3 + 12cd^2 + 10dcd + 12d^2c \),

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The ten simsun permutations \( w \in S_5 \) with \( D(w) = \{1, 4\} \):

21354, 21453, 31254, 31452, 41253

41352, 42351, 51243, 51342, 52341,

but **not** 32451.
Two consequences

**Theorem.** (a) $\Phi_n(1, 1) = E_n$ *(the number of simsum permutations $w \in \mathfrak{S}_n$).*
Two consequences

Theorem. (a) \( \Phi_n(1, 1) = E_n \) (the number of simsum permutations \( w \in \mathcal{S}_n \)).

(b) \textbf{(Niven, de Bruijn)} For all \( S \subseteq \{1, \ldots, n-1\} \),

\[
\beta(n) \leq E_n,
\]

with equality if and only if \( S = \{1, 3, 5, \ldots\} \) or \( S = \{2, 4, 6 \ldots\} \)
An example

\[ \Phi_5 = 1c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2 \]

\[ 1 + 3 + 5 + 3 + 4 = 16 = E_5 \]
Darn!
That's the end...