Alternating Permutations

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M.I.T.

Basic definitions

A sequence $a_1, a_2, \ldots, a_k$ of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \cdots ,$$

and **reverse alternating** if

$$a_1 < a_2 > a_3 < a_4 > \cdots .$$

Euler numbers

$\mathfrak{S}_n$: symmetric group of all permutations of $1, 2, \ldots, n$

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Euler number:

$$\begin{aligned} E_n &= \#\{w \in \mathfrak{S}_n \,:\, w \text{ is alternating}\} \\ &= \#\{w \in \mathfrak{S}_n \,:\, w \text{ is reverse alternating}\} \end{aligned}$$

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E.g., $E_4 = 5 : \ 2143, 3142, 3241, 4132, 4231$

André's theorem

Theorem (**Désiré André**, 1879)

$$\boldsymbol{y} := \sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

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E_{2n+1} is a **tangent number**.

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$\Rightarrow$ **combinatorial trigonometry**

Example of combinatorial trig.

$$\sec^2 x = 1 + \tan^2 x$$

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Equate coefficients of $x^{2n}/(2n)!$:

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} E_{2(n-k)}$$

$$= \sum_{k=0}^{n-1} \binom{2n}{2k+1} E_{2k+1} E_{2(n-k)-1}.$$

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Prove combinatorially (exercise).

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Naive proof.

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \ n \geq 1$$

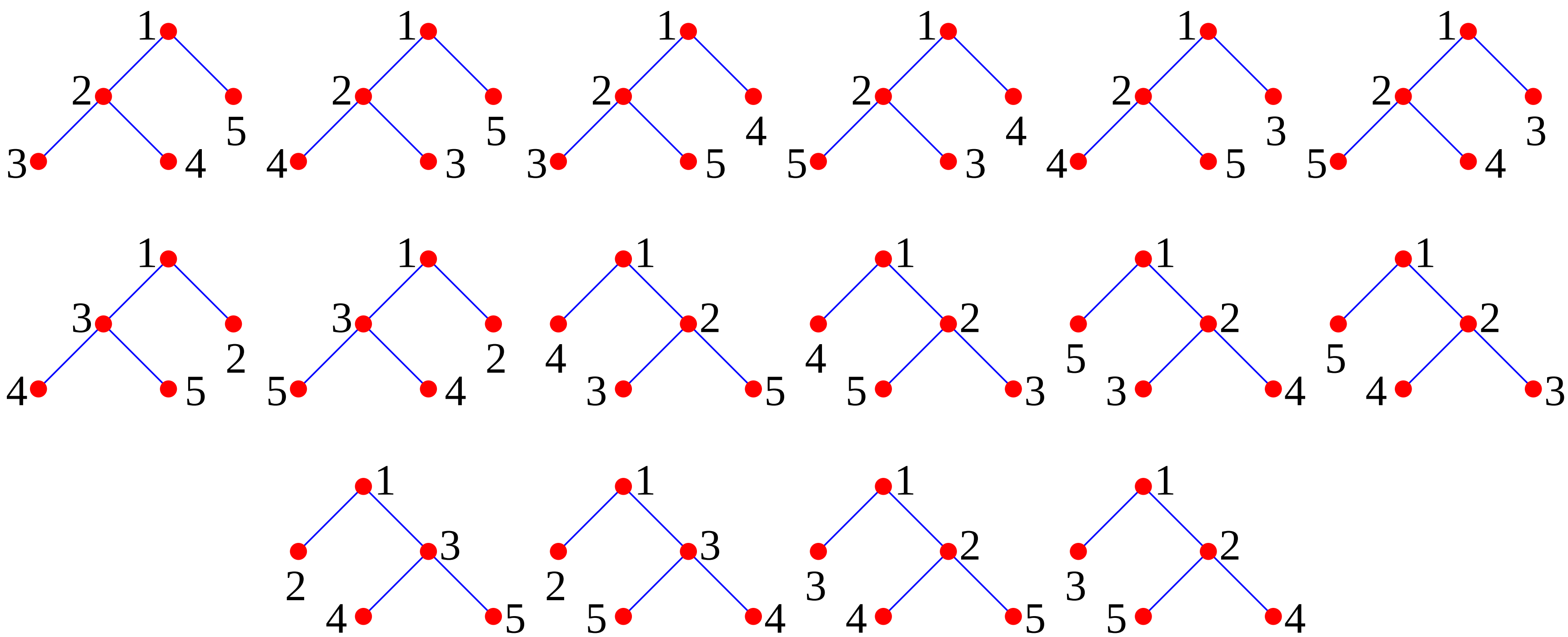
$$\Rightarrow 2y' = 1 + y^2, \text{ etc.}$$

(details omitted)

Some occurences of Euler numbers

(1) E_{2n-1} is the number of complete increasing binary trees on the vertex set $[2n+1] = \{1, 2, \ldots, 2n+1\}$.

Five vertices



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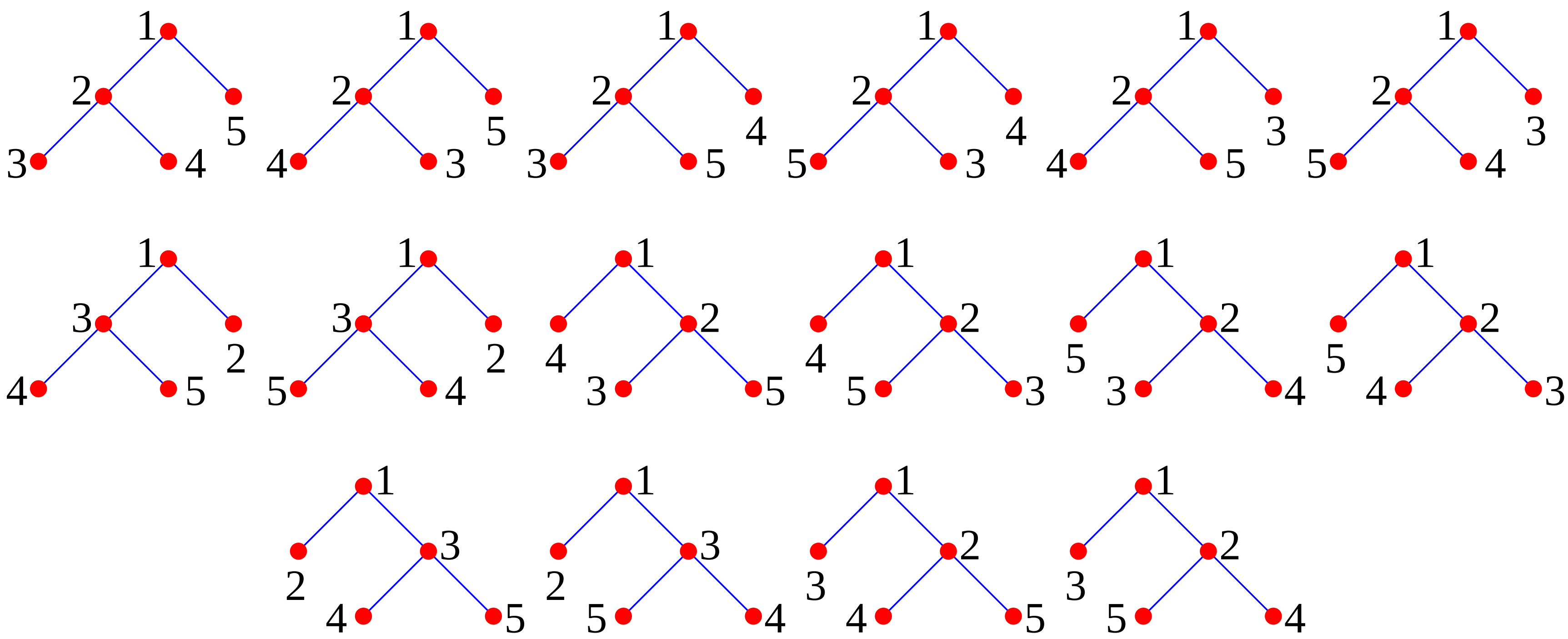

Slightly more complicated for E_{2n}

Simsun permutations

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Theorem (**R. Simion** & **S. Sundaram**) The number of simsun permutations in $\mathfrak{S}_n$ is E_{n+1}.

Orbits of mergings

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$$1\text{--}2\text{--}3\text{--}4\text{--}5\text{--}6,\quad 12\text{--}3\text{--}4\text{--}5\text{--}6,\quad 12\text{--}34\text{--}5\text{--}6$$

$$125\text{--}34\text{--}6,\quad 125\text{--}346,\quad 123456$$

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$\mathfrak{S}_n$ acts on these sequences.

Theorem. *The number of $\mathfrak{S}_n$-orbits is E_{n-1}.*

Orbit representatives for $n = 5$

$12-3-4-5$	$123-4-5$	$1234-5$
$12-3-4-5$	$123-4-5$	$123-45$
$12-3-4-5$	$12-34-5$	$125-34$
$12-3-4-5$	$12-34-5$	$12-345$
$12-3-4-5$	$12-34-5$	$1234-5$

Volume of a polytope

(4) Let $\mathcal{E}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

$$\begin{aligned} x_i &\geq 0, \quad 1 \leq i \leq n \\ x_i + x_{i+1} &\leq 1, \quad 1 \leq i \leq n-1. \end{aligned}$$

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Theorem. *The volume of $\mathcal{E}_n$ is $E_n/n!$.*

The “nicest” proof

- Triangulate $\mathcal{E}_n$ so that the maximal simplices σ_w are indexed by alternating permutations $w \in \mathfrak{S}_n$.

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- Show Vol$(\sigma_w) = 1/n!$.

Tridiagonal matrices

An $n \times n$ matrix $\boldsymbol{M} = (m_{ij})$ is **tridiagonal** if $m_{ij} = 0$ whenever $|i - j| \geq 2$.

doubly-stochastic: $m_{ij} \geq 0$, row and column sums equal 1

$\boldsymbol{\mathcal{T}_n}$: set of $n \times n$ tridiagonal doubly stochastic matrices

Polytope structure of $\mathcal{T}_n$

Easy fact: the map

$$\begin{aligned} \mathcal{T}_n &\rightarrow \mathbb{R}^{n-1} \\ M &\mapsto (m_{12}, m_{23}, \ldots, m_{n-1,n}) \end{aligned}$$

is a (linear) bijection from $\mathcal{T}_n$ to $\mathcal{E}_n$.

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Application (**Diaconis** et al.): random doubly stochastic tridiagonal matrices and random walks on $\mathcal{T}_n$

Distribution of $\mathrm{is}(w)$

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$$E(n) \sim 2\sqrt{n}$$

For fixed $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t\right) = F(t),$$

the **Tracy-Widom distribution**.

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- Length of longest increasing subsequence of an alternating permutation $w \in \mathfrak{S}_n$.

The first is much easier!

Longest alternating subsequences

$\mathbf{as}(\boldsymbol{w})$= length longest alt. subseq. of w

$$\boldsymbol{D(n)} = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\mathrm{as}(w) \sim\ ?$$

$$w = \mathbf{5}6\mathbf{2}1\mathbf{834}7 \Rightarrow \mathrm{as}(w) = 5$$

Definitions of $a_k(n)$ and $b_k(n)$

$$\begin{aligned} \boldsymbol{a_k(n)} &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) = k\} \\ \\ \boldsymbol{b_k(n)} &= a_1(n) + a_2(n) + \cdots + a_k(n) \\ &= \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}(w) \leq k\}. \end{aligned}$$

The case $n = 3$

w	$\mathrm{as}(w)$
$\mathbf{1}23$	1
$1\mathbf{32}$	2
$\mathbf{213}$	3
$\mathbf{2}3\mathbf{1}$	2
$\mathbf{312}$	3
$\mathbf{3}2\mathbf{1}$	2

The case $n = 3$

w	$\text{as}(w)$
123	1
132	2
213	3
231	2
312	3
321	2

$$a_1(3) = 1,\ a_2(3) = 3,\ a_3(3) = 2$$

$$b_1(3) = 1,\ b_2(3) = 4,\ b_3(3) = 6$$

The main lemma

Lemma. $\forall\, w \in \mathfrak{S}_n \;\exists$ *alternating subsequence of maximal length that contains* n.

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Corollary.

$$\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} \left(a_{2r}(j-1) + a_{2r+1}(j-1)\right) a_s(n-j)$$

The main generating function

$$\boldsymbol{B(x,t)} = \sum_{k,n\geq 0} b_k(n) t^k \frac{x^n}{n!}$$

Theorem.

$$B(x,t) = \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho},$$

where $\boldsymbol{\rho} = \sqrt{1-t^2}$.

Formulas for $b_k(n)$

Corollary.

$$
\begin{aligned}
\Rightarrow \quad b_1(n) &= 1 \\
b_2(n) &= n \\
b_3(n) &= \frac{1}{4}(3^n - 2n + 3) \\
b_4(n) &= \frac{1}{8}(4^n - (2n-4)2^n) \\
&\vdots
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no such formulas for longest increasing subsequences

Mean (expectation) of $\mathrm{as}(w)$

$$\boldsymbol{D(n)} = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\mathrm{as}(w) = \frac{1}{n!}\sum_{k=1}^{n} k\cdot a_k(n),$$

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Let

$$\begin{aligned}\boldsymbol{A(x,t)} &= \sum_{k,n\geq 0} a_k(n)t^k\frac{x^n}{n!} = (1-t)B(x,t) \\ &= (1-t)\left(\frac{2/\rho}{1-\frac{1-\rho}{t}e^{\rho x}}-\frac{1}{\rho}\right).\end{aligned}$$

Formula for $D(n)$

$$\begin{aligned}\sum_{n\geq 0} D(n)x^n &= \frac{\partial}{\partial t}A(x,1)\\ &= \frac{6x-3x^2+x^3}{6(1-x)^2}\\ &= x+\sum_{n\geq 2}\frac{4n+1}{6}x^n.\end{aligned}$$

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Compare $E(n)\sim 2\sqrt{n}$.

Variance of $\mathrm{as}(w)$

$$\boldsymbol{V(n)} = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n}\left(\mathrm{as}(w) - \frac{4n+1}{6}\right)^2,\ \ n \geq 2$$

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Corollary.

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Corollary.

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similar results for higher moments

A new distribution?

$$\boldsymbol{P(t)} = \lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w)-2n/3}{\sqrt{n}} \leq t\right)$$

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Stanley distribution?

Limiting distribution

Theorem (Pemantle, Widom, (Wilf)).

$$\lim_{n\to\infty} \mathrm{Prob}_{w\in\mathfrak{S}_n}\left(\frac{\mathrm{as}(w)-2n/3}{\sqrt{n}} \le t\right)$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t\sqrt{45}/4} e^{-s^2}\, ds$$

(Gaussian distribution)

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Umbral enumeration

Umbral formula: involves E^k, where E is an indeterminate (the **umbra**). Replace E^k with the Euler number E_k. (Technique from 19th century, modernized by Rota et al.)

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Example.

$$\begin{aligned}(1+E^2)^3 &= 1+3E^2+3E^4+E^6 \\ &= 1+3E_2+3E_4+E_6 \\ &= 1+3\cdot 1+3\cdot 5+61 \\ &= 80\end{aligned}$$

Another example

$$
\begin{aligned}
(1+t)^E &= 1 + Et + \binom{E}{2} t^2 + \binom{E}{3} t^3 + \cdots \\
&= 1 + Et + \frac{1}{2} E(E-1) t^2 + \cdots \\
&= 1 + E_1 t + \frac{1}{2}(E_2 - E_1)) t^2 + \cdots \\
&= 1 + t + \frac{1}{2}(1-1) t^2 + \cdots \\
&= 1 + t + O(t^3).
\end{aligned}
$$

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fixed-point free involution $w \in \mathfrak{S}_{2n}$: all cycles of length two

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Let $\boldsymbol{f(n)}$ be the number of **alternating** fixed-point free involutions in $\mathfrak{S}_{2n}$.

$$\begin{aligned} n = 3: \quad 214365 &= (1,2)(3,4)(5,6) \\ 645231 &= (1,6)(2,4)(3,5) \end{aligned}$$

$$f(3) = 2$$

An umbral theorem

Theorem.

$$\boldsymbol{F(x)} := \sum_{n\geq 0} f(n)x^n$$

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$$= \left(\frac{1+x}{1-x}\right)^{(E^2+1)/4}$$

Proof idea

Proof. Uses representation theory of the symmetric group $\mathfrak{S}_n$.

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$$\chi(w) = 0 \text{ or } \pm E_k.$$

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Now use known results on combinatorial properties of characters of $\mathfrak{S}_n$.

Ramanujan's Lost Notebook

Theorem (Ramanujan, Berndt, implicitly) *As* $x \to 0+$,

$$2\sum_{n\geq 0}\left(\frac{1-x}{1+x}\right)^{n(n+1)} \sim \sum_{k\geq 0} f(k)x^k = F(x),$$

an **analytic** (non-formal) identity.

A formal identity

Corollary (via Ramanujan, Andrews).

$$F(x) = 2\sum_{n\geq 0} q^n \frac{\prod_{j=1}^{n}(1-q^{2j-1})}{\prod_{j=1}^{2n+1}(1+q^j)},$$

where $\boldsymbol{q} = \left(\frac{1-x}{1+x}\right)^{2/3}$*, a* ***formal*** *identity.*

Simple result, hard proof

Recall: number of n-cycles in $\mathfrak{S}_n$ is $(n-1)!$.

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Theorem. *Let* $\boldsymbol{b(n)}$ *be the number of* ***alternating*** n*-cycles in* $\mathfrak{S}_n$*. Then if* n *is odd,*

$$\begin{aligned} b(n) &= \frac{1}{n}\sum_{d|n}\mu(d)(-1)^{(d-1)/2}E_{n/d} \\ &\sim E_n/n. \end{aligned}$$

Special case

Corollary. *Let p be an odd prime. Then*

$$b(p) = \frac{1}{p}\left(E_p - (-1)^{(p-1)/2}\right).$$

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Combinatorial proof?

Inc. subsequences of alt. perms.

Recall: $\mathrm{is}(w)$ = length of longest increasing subsequence of $w \in \mathfrak{S}_n$. Define

$$C(n) = \frac{1}{E_n} \sum_{w} \mathrm{is}(w),$$

where w ranges over all E_n alternating permutations in $\mathfrak{S}_n$.

$\boldsymbol{\beta}$

Little is known, e.g., what is

$$\boldsymbol{\beta} = \lim_{n\to\infty} \frac{\log C(n)}{\log n}?$$

I.e., $C(n) = n^{\beta+o(1)}$.

Compare $\lim_{n\to\infty} \frac{\log E(n)}{\log n} = 1/2$.

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Easy: $\beta \geq \frac{1}{2}$.

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Possible tool: $\exists$ "umbral analogue" of Gessel's determinantal formula.

Descent sets

Let $\boldsymbol{w} = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. **Descent set** of w:

$$\boldsymbol{D(w)} = \{i \,:\, a_i > a_{i+1}\} \subseteq \{1, \ldots, n-1\}$$

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$$\begin{aligned}
D(\mathbf{4}15\mathbf{76}23) &= \{1,4,5\} \\
D(\mathbf{4}1\mathbf{5}2\mathbf{7}36) &= \{1,3,5\} \text{ (alternating)} \\
D(4\mathbf{7}3\mathbf{6}1\mathbf{5}2) &= \{2,4,6\} \text{ (reverse alternating)}
\end{aligned}$$

$\beta_n(S)$

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w	$D(w)$
123	$\emptyset$
213	$\{1\}$
312	$\{1\}$
132	$\{2\}$
231	$\{2\}$
321	$\{1,2\}$

$$\beta_3(\emptyset) = 1, \quad \beta_3(1) = 2$$
$$\beta_3(2) = 2, \quad \beta_3(1,2) = 1$$

$\boldsymbol{u_S}$

Fix n. Let $S \subseteq \{1, \cdots, n-1\}$. Let $\boldsymbol{u_S} = t_1 \cdots t_{n-1}$, where

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Example. $n = 8,\ S = \{2,5,6\} \subseteq \{1, \ldots, 7\}$

$$\boldsymbol{u_S} = ababba$$

A noncommutative gen. function

$$\Psi_n(a,b) = \sum_{S \subseteq \{1,\dots,n-1\}} \beta_n(S) u_S.$$

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$$= (a+b)^2 + (ab+ba)$$

The cd-index

Theorem. *There exists a noncommutative polynomial $\Phi_n(c,d)$, called the **cd-index** of $\mathfrak{S}_n$, with **nonnegative** integer coefficients, such that*

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Example. Recall

$$\Psi_3(a,b) = aa + 2ab + 2ba + b^2 = (a+b)^2 + (ab+ba).$$

Therefore

$$\Phi_3(c,d) = c^2 + d.$$

Small values of $\Phi_n(c,d)$

$$
\begin{aligned}
\Phi_1 &= 1\\
\Phi_2 &= c\\
\Phi_3 &= c^2+d\\
\Phi_4 &= c^3+2cd+2dc\\
\Phi_5 &= c^4+3c^2d+5cdc+3dc^2+4d^2\\
\Phi_6 &= c^5+4c^3d+9c^2dc+9cdc^2+4dc^3\\
&\quad +12cd^2+10dcd+12d^2c.
\end{aligned}
$$

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Example. $n = 10$:

$$\mu = cd^2c^2d \to 0 \cdot 10 \cdot 10 \cdot 0 \cdot 0 \cdot 1\cancel{0} = 01010001,$$

the characteristic vector of $S_\mu = \{2,4,8\} \subseteq [8]$

cd-index coefficients

Recall: $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ is a **simsun** permutation if the subsequence with elements $1, 2, \ldots, k$ has no double descents, $1 \leq k \leq n$.

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Theorem (**Simion-Sundaram**, variant of **Foata-Schützenberger**) *The coefficient of μ in $\Phi(c,d)$ is equal to the number of simsun permutations in $\mathfrak{S}_{n-1}$ with descent set S_μ.*

An example

Example. $\Phi_6 =$
$c^5+4c^3d+9c^2dc+9cdc^2+4dc^3+12cd^2+10dcd+12d^2c$,

$$dcd \longrightarrow 10 \cdot 0 \cdot 1\cancel{0} \Rightarrow S_{dcd} = \{1,4\}$$

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The ten simsun permutations $w \in \mathfrak{S}_5$ with $D(w) = \{1, 4\}$:

$$21354, 21453, 31254, 31452, 41253$$

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but **not** 32451.

Two consequences

Theorem. (a) $\Phi_n(1,1) = E_n$ *(the number of simsum permutations* $w \in \mathfrak{S}_n$*).*

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(b) (**Niven, de Bruijn**) *For all* $S \subseteq \{1,\ldots,n-1\}$,

$$\beta_n(S) \leq E_n,$$

with equality if and only if $S = \{1,3,5,\ldots\}$ or $S = \{2,4,6\ldots\}$

An example

$$\Phi_5 = \mathbf{1}c^4 + \mathbf{3}c^2d + \mathbf{5}cdc + \mathbf{3}dc^2 + \mathbf{4}d^2$$

$$1 + 3 + 5 + 3 + 4 = 16 = E_5$$


Darn!
That's the end...
H