Alternating Permutations

Richard P. Stanley

M.I.T.
A sequence $a_1, a_2, \ldots, a_k$ of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \cdots,$$

and **reverse alternating** if

$$a_1 < a_2 > a_3 < a_4 > \cdots.$$
Euler numbers

$\mathfrak{S}_n$: symmetric group of all permutations of 1, 2, \ldots, n.
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Euler number:

$$E_n = \# \{ w \in \mathfrak{S}_n : w \text{ is alternating} \}$$

$$= \# \{ w \in \mathfrak{S}_n : w \text{ is reverse alternating} \}$$
Euler numbers

$\mathcal{S}_n$ : symmetric group of all permutations of 1, 2, \ldots , n

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$$= \# \{ w \in \mathcal{S}_n : w \text{ is reverse alternating} \}$$

E.g., $E_4 = 5 : 2143, 3142, 3241, 4132, 4231$
André’s theorem

**Theorem (Désiré André, 1879)**

\[ y := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x \]
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\( E_{2n} \) is a **secant number**.

\( E_{2n+1} \) is a **tangent number**.
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\( E_{2n} \) is a **secant number**.

\( E_{2n+1} \) is a **tangent number**.

\[ \Rightarrow \text{combinatorial trigonometry} \]
Example of combinatorial trig.

\[ \sec^2 x = 1 + \tan^2 x \]
Example of combinatorial trig.

\[ \sec^2 x = 1 + \tan^2 x \]

Equate coefficients of \( x^{2n} / (2n)! \):

\[
\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} E_{2(n-k)}
\]

\[
= \sum_{k=0}^{n-1} \binom{2n}{2k + 1} E_{2k+1} E_{2(n-k)-1}.
\]
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\]

Prove combinatorially (exercise).
Proof of André’s theorem

\[ y := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x \]
Proof of André’s theorem

\[ y := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x \]

Naive proof.

\[ 2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \quad n \geq 1 \]

\[ \Rightarrow 2y' = 1 + y^2, \text{ etc.} \]

(details omitted)
Some occurrences of Euler numbers

(1) $E_{2n-1}$ is the number of complete increasing binary trees on the vertex set $[2n + 1] = \{1, 2, \ldots, 2n + 1\}$. 
Five vertices

1 1 3 3 4 5 5
2 2

3 3

4 4

5 5

Slightly more complicated for $E_2^n$
Five vertices

Slightly more complicated for $E_{2n}$
(2) $b_1 b_2 \cdots b_k$ has a **double descent** if for some $1 < i < n$,

$$b_{i-1} > b_i > b_{i+1}.$$
Simsun permutations

$(2)$ $b_1 b_2 \cdots b_k$ has a **double descent** if for some $1 < i < n$,

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$w = a_1 a_2 \cdots a_n \in \mathcal{S}_n$ is a **simsun** permutation if the subsequence with elements $1, 2, \ldots, k$ has no double descents, $1 \leq k \leq n$. 

Example. $3241$ is not simsun: the subsequence with $1; 2; 3$ is $321$. 

Theorem (R. Simion & S. Sundaram) The number of simsun permutations in $\mathcal{S}_n$ is $\sum_{n=1}^{\infty} n! = T_0 + T_1 + T_2 + \cdots$. 

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(2) $b_1 b_2 \cdots b_k$ has a **double descent** if for some $1 < i < n$, 

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**Example.** 3241 is not simsun: the subsequence with 1, 2, 3 is 321.
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**Example.** 3241 is not simsun: the subsequence with 1, 2, 3 is 321.

**Theorem** *(R. Simion & S. Sundaram)* The number of simsun permutations in \(\mathcal{S}_n\) is \(E_{n+1}\).
(3) Start with $n$ one-element sets $\{1\}, \ldots, \{n\}$.
(3) Start with $n$ one-element sets $\{1\}, \ldots, \{n\}$. Merge together two at a time until reaching $\{1, 2, \ldots, n\}$. 
Orbits of mergings

(3) Start with \( n \) one-element sets \( \{1\}, \ldots, \{n\} \). Merge together two at a time until reaching \( \{1, 2, \ldots, n\} \).

\[
1 - 2 - 3 - 4 - 5 - 6, \quad 12 - 3 - 4 - 5 - 6, \quad 12 - 34 - 5 - 6
\]

\[
125 - 34 - 6, \quad 125 - 346, \quad 123456
\]
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\[
1 - 2 - 3 - 4 - 5 - 6, \quad 12 - 3 - 4 - 5 - 6, \quad 12 - 34 - 5 - 6 \quad 125 - 34 - 6, \quad 125 - 346, \quad 123456
\]

$S_n$ acts on these sequences.
(3) Start with $n$ one-element sets $\{1\}, \ldots, \{n\}$. Merge together two at a time until reaching $\{1, 2, \ldots, n\}$.

$$1\,2\,3\,4\,5\,6, \hspace{1em} 12\,3\,4\,5\,6, \hspace{1em} 12\,34\,5\,6$$

$$125\,34\,6, \hspace{1em} 125\,346, \hspace{1em} 123456$$

$S_n$ acts on these sequences.

**Theorem.** The number of $S_n$-orbits is $E_{n-1}$. 
## Orbit representatives for $n = 5$

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(4) Let $\mathcal{E}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

\[
\begin{align*}
x_i & \geq 0, \quad 1 \leq i \leq n \\
x_i + x_{i+1} & \leq 1, \quad 1 \leq i \leq n - 1.
\end{align*}
\]
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**Theorem.** The volume of $\mathcal{E}_n$ is $E_n/n!$. 
Triangulate $E_n$ so that the maximal simplices $\sigma_w$ are indexed by alternating permutations $w \in S_n$. 

Show $\text{Vol}(w) = \frac{1}{n!}$. 

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The “nicest” proof

- Triangulate $\mathcal{E}_n$ so that the maximal simplices $\sigma_w$ are indexed by alternating permutations $w \in \mathfrak{S}_n$.

- Show $\text{Vol}(\sigma_w) = 1/n!$. 
An $n \times n$ matrix $M = (m_{ij})$ is **tridiagonal** if $m_{ij} = 0$ whenever $|i - j| \geq 2$.

**doubly-stochastic**: $m_{ij} \geq 0$, row and column sums equal 1

$\mathcal{T}_n$: set of $n \times n$ tridiagonal doubly stochastic matrices
Easy fact: the map

\[ \mathcal{T}_n \rightarrow \mathbb{R}^{n-1} \]

\[ M \mapsto (m_{12}, m_{23}, \ldots, m_{n-1,n}) \]

is a (linear) bijection from \( \mathcal{T}_n \) to \( \mathcal{E}_n \).
Polytope structure of $\mathcal{T}_n$

**Easy fact:** the map

$$\mathcal{T}_n \rightarrow \mathbb{R}^{n-1}$$

$$M \mapsto (m_{12}, m_{23}, \ldots, m_{n-1,n})$$

is a (linear) bijection from $\mathcal{T}_n$ to $\mathcal{E}_n$.

**Application** *(Diaconis et al.)*: random doubly stochastic tridiagonal matrices and random walks on $\mathcal{T}_n$
Yesterday: \( \text{is}(w) = \) length of longest increasing subsequence of \( w \in \mathfrak{S}_n \)

\[
E(n) \sim 2\sqrt{n}
\]
Yesterday: $\text{is}(w) = \text{length of longest increasing subsequence of } w \in \mathfrak{S}_n$

$$E(n) \sim 2\sqrt{n}$$

For fixed $t \in \mathbb{R}$,

$$\lim_{n \to \infty} \text{Prob} \left( \frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t),$$

the **Tracy-Widom distribution**.
Analogues of distribution of $\text{is}(w)$

- Length of longest alternating subsequence of $w \in S_n$
Analogues of distribution of is($w$)

- Length of longest alternating subsequence of $w \in \mathfrak{S}_n$

- Length of longest increasing subsequence of an alternating permutation $w \in \mathfrak{S}_n$. 

Analogues of distribution of $is(w)$

- Length of longest alternating subsequence of $w \in \mathcal{S}_n$
- Length of longest increasing subsequence of an alternating permutation $w \in \mathcal{S}_n$.

The first is much easier!
Longest alternating subsequences

\[ \text{as}(w) = \text{length longest alt. subseq. of } w \]

\[ D(n) = \frac{1}{n!} \sum_{w \in S_n} \text{as}(w) \sim ? \]

\[ w = 56218347 \Rightarrow \text{as}(w) = 5 \]
Definitions of $a_k(n)$ and $b_k(n)$

\[
\begin{align*}
    a_k(n) & = \# \{ w \in \mathfrak{S}_n : \text{as}(w) = k \} \\
    b_k(n) & = a_1(n) + a_2(n) + \cdots + a_k(n) \\
             & = \# \{ w \in \mathfrak{S}_n : \text{as}(w) \leq k \}.
\end{align*}
\]
The case $n = 3$

<table>
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<tr>
<th>$w$</th>
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<tr>
<td>123</td>
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<td>132</td>
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<td>213</td>
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$a_1(3) = 1, \ a_2(3) = 3, \ a_3(3) = 2$

$b_1(3) = 1, \ b_2(3) = 4, \ b_3(3) = 6$
Lemma. \( \forall w \in \mathcal{S}_n \exists \) alternating subsequence of maximal length that contains \( n \).
Lemma. \( \forall w \in S_n \exists \) alternating subsequence of maximal length that contains \( n \).

Corollary.

\[
\Rightarrow a_k(n) = \sum_{j=1}^{n} \binom{n-1}{j-1}
\]

\[
\sum_{2r+s=k-1} (a_{2r}(j - 1) + a_{2r+1}(j - 1)) a_s(n - j)
\]
The main generating function

\[ B(x, t) = \sum_{k,n \geq 0} b_k(n) t^k \frac{x^n}{n!} \]

**Theorem.**

\[ B(x, t) = 2^{t \rho} \frac{1}{1 - \frac{1-\rho}{t} e^{\rho x} - \frac{1}{\rho}}, \]

where \( \rho = \sqrt{1 - t^2} \).
Corollary.

\[ b_1(n) = 1 \]
\[ b_2(n) = n \]
\[ b_3(n) = \frac{1}{4}(3^n - 2n + 3) \]
\[ b_4(n) = \frac{1}{8}(4^n - (2n - 4)2^n) \]
\[ \vdots \]
Corollary.

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\[ \vdots \]

no such formulas for longest increasing subsequences
Mean (expectation) of $\text{as}(w)$

$$D(n) = \frac{1}{n!} \sum_{w \in \mathcal{S}_n} \text{as}(w) = \frac{1}{n!} \sum_{k=1}^{n} k \cdot a_k(n),$$

the expectation of $\text{as}(w)$ for $w \in \mathcal{S}_n$
Mean (expectation) of \( as(w) \)

\[
D(n) = \frac{1}{n!} \sum_{w \in S_n} as(w) = \frac{1}{n!} \sum_{k=1}^{n} k \cdot a_k(n),
\]

the expectation of \( as(w) \) for \( w \in S_n \)

Let

\[
A(x, t) = \sum_{k, n \geq 0} a_k(n) t^k \frac{x^n}{n!} = (1 - t) B(x, t)
\]

\[
= (1 - t) \left( \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho} \right).
\]
Formula for $D(n)$

$$\sum_{n \geq 0} D(n)x^n = \frac{\partial}{\partial t} A(x, 1)$$

$$= \frac{6x - 3x^2 + x^3}{6(1 - x)^2}$$

$$= x + \sum_{n \geq 2} \frac{4n + 1}{6} x^n.$$
Formula for $D(n)$

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\[ \Rightarrow D(n) = \frac{4n + 1}{6}, \ n \geq 2 \quad (\text{why?}) \]
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$$\Rightarrow D(n) = \frac{4n + 1}{6}, \ n \geq 2 \quad \text{why?}$$

Compare $E(n) \sim 2\sqrt{n}$. 
Variance of \( \text{as}(w) \)

\[ V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left( \text{as}(w) - \frac{4n + 1}{6} \right)^2, \quad n \geq 2 \]

the variance of \( \text{as}(n) \) for \( w \in \mathfrak{S}_n \)
Variance of $\text{as}(w)$

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the variance of $\text{as}(n)$ for $w \in \mathfrak{S}_n$

Corollary.

\[ V(n) = \frac{8}{45} n - \frac{13}{180}, \quad n \geq 4 \]
Variance of $\text{as}(w)$

\[ V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left( \text{as}(w) - \frac{4n + 1}{6} \right)^2, \quad n \geq 2 \]

the variance of $\text{as}(n)$ for $w \in \mathfrak{S}_n$

Corollary.

\[ V(n) = \frac{8}{45} n - \frac{13}{180}, \quad n \geq 4 \]

similar results for higher moments
A new distribution?

\[ P(t) = \lim_{n \to \infty} \text{Prob}_{w \in \mathfrak{S}_n} \left( \frac{as(w) - 2n/3}{\sqrt{n}} \leq t \right) \]
A new distribution?

\[ P(t) = \lim_{n \to \infty} \text{Prob}_{w \in S_n} \left( \frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right) \]

Stanley distribution?
Theorem (Pemantle, Widom, (Wilf)).

$$\lim_{n \to \infty} \text{Prob}_{w \in \mathfrak{S}_n} \left( \frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45/4}} e^{-s^2} ds$$

(Gaussian distribution)
Limiting distribution

**Theorem** (Pemantle, Widom, (Wilf)).

\[
\lim_{n \to \infty} \mathbb{P}_{w \in S_n} \left( \frac{as(w) - 2n/3}{\sqrt{n}} \leq t \right)
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45}/4} e^{-s^2} \, ds
\]

(Gaussian distribution)
Umbral enumeration

**Umbral formula:** involves $E^k$, where $E$ is an indeterminate (the **umbra**). Replace $E^k$ with the Euler number $E_k$. (Technique from 19th century, modernized by Rota et al.)

Example. 

$$(1 + E_2)^3 = 1 + 3E_2 + 3E_4 + E_6 = 1 + 3 \cdot 1 + 3 \cdot 5 + 61 = 80$$
Umbral enumeration

**Umbral formula:** involves $E^k$, where $E$ is an indeterminate (the umbra). Replace $E^k$ with the Euler number $E_k$. (Technique from 19th century, modernized by Rota et al.)

**Example.**

\[
(1 + E^2)^3 = 1 + 3E^2 + 3E^4 + E^6 \\
= 1 + 3E_2 + 3E_4 + E_6 \\
= 1 + 3 \cdot 1 + 3 \cdot 5 + 61 \\
= 80
\]
Another example

\[(1 + t)^E = 1 + Et + \binom{E}{2} t^2 + \binom{E}{3} t^3 + \cdots\]

\[= 1 + Et + \frac{1}{2} E(E - 1)t^2 + \cdots\]

\[= 1 + E_1 t + \frac{1}{2} (E_2 - E_1))t^2 + \cdots\]

\[= 1 + t + \frac{1}{2}(1 - 1)t^2 + \cdots\]

\[= 1 + t + O(t^3).\]
fixed-point free involution $w \in S_{2n}$: all cycles of length two

Let $f(n)$ be the number of alternating fixed-point free involutions in $S_{2n}$. For example, for $n = 3$,

\[ f(3) = 2 \]

where $214365 = (1; 2)(3; 4)(5; 6)$ and $645231 = (1; 6)(2; 4)(3; 5)$. The text continues with more details about the enumeration of such permutations.
fixed-point free involution \( w \in \mathfrak{S}_{2n} \): all cycles of length two

Let \( f(n) \) be the number of alternating fixed-point free involutions in \( \mathfrak{S}_{2n} \).
fixed-point free involution \( w \in S_{2n} \): all cycles of length two

Let \( f(n) \) be the number of alternating fixed-point free involutions in \( S_{2n} \).

\[
\begin{align*}
n = 3 : & \quad 214365 = (1, 2)(3, 4)(5, 6) \\
& \quad 645231 = (1, 6)(2, 4)(3, 5) \\
f(3) &= 2
\end{align*}
\]
An umbral theorem

Theorem.

\[ F(x) := \sum_{n \geq 0} f(n) x^n \]
An umbral theorem

**Theorem.**

\[ F(x) := \sum_{n \geq 0} f(n) x^n \]

\[ = \left( \frac{1 + x}{1 - x} \right)^{(E^2 + 1)/4} \]
Proof. Uses representation theory of the symmetric group $S_n$. 
Proof idea

Proof. Uses representation theory of the symmetric group $\mathfrak{S}_n$.

There is a character $\chi$ of $\mathfrak{S}_n$ (due to H. O. Foulkes) such that for all $w \in \mathfrak{S}_n$,

$$\chi(w) = 0 \text{ or } \pm E_k.$$
Proof idea

**Proof.** Uses representation theory of the symmetric group $\mathcal{S}_n$.

There is a character $\chi$ of $\mathcal{S}_n$ (due to H. O. Foulkes) such that for all $w \in \mathcal{S}_n$,

$$\chi(w) = 0 \text{ or } \pm E_k.$$ 

Now use known results on combinatorial properties of characters of $\mathcal{S}_n$. 
Theorem (Ramanujan, Berndt, implicitly) As $x \to 0+$,

$$2 \sum_{n \geq 0} \left( \frac{1 - x}{1 + x} \right)^{n(n+1)} \sim \sum_{k \geq 0} f(k) x^k = F(x),$$

an analytic (non-formal) identity.
A formal identity

Corollary (via Ramanujan, Andrews).

\[ F(x) = 2 \sum_{n \geq 0} q^n \frac{\prod_{j=1}^{n} (1 - q^{2j-1})}{\prod_{j=1}^{2n+1} (1 + q^j)}, \]

where \( q = \left( \frac{1-x}{1+x} \right)^{2/3} \), a formal identity.
Recall: number of \( n \)-cycles in \( S_n \) is \((n - 1)!\).
Simple result, hard proof

Recall: number of $n$-cycles in $\mathfrak{S}_n$ is $(n - 1)!$.

**Theorem.** Let $b(n)$ be the number of alternating $n$-cycles in $\mathfrak{S}_n$. Then if $n$ is odd,

$$b(n) = \frac{1}{n} \sum_{d|n} \mu(d)(-1)^{(d-1)/2} E_{n/d}$$

$$\sim E_{n}/n.$$
Corollary. Let $p$ be an odd prime. Then

$$
b(p) = \frac{1}{p} \left( E_p - (-1)^{(p-1)/2} \right).$$
Corollary. Let $p$ be an odd prime. Then

$$b(p) = \frac{1}{p} \left( E_p - (-1)^{(p-1)/2} \right).$$

Combinatorial proof?
Recall: $is(w) = \text{length of longest increasing subsequence of } w \in S_n$. Define

$$C(n) = \frac{1}{E_n} \sum_{w} is(w),$$

where $w$ ranges over all $E_n$ alternating permutations in $S_n$. 
Little is known, e.g., what is

$$\beta = \lim_{n \to \infty} \frac{\log C(n)}{\log n}?$$

I.e., $C(n) = n^{\beta + o(1)}$.

Compare $\lim_{n \to \infty} \frac{\log E(n)}{\log n} = 1/2$. 
Little is known, e.g., what is
\[ \beta = \lim_{n \to \infty} \frac{\log C(n)}{\log n} \]?

I.e., \( C(n) = n^{\beta + o(1)} \).

Compare \( \lim_{n \to \infty} \frac{\log E(n)}{\log n} = 1/2 \).

**Easy:** \( \beta \geq \frac{1}{2} \).
Limiting distribution?

What is the (suitably scaled) limiting distribution of $is(w)$, where $w$ ranges over all alternating permutations in $\mathfrak{S}_n$?
What is the (suitably scaled) limiting distribution of $\text{is}(w)$, where $w$ ranges over all alternating permutations in $\mathfrak{S}_n$?

Is it the Tracy-Widom distribution?
What is the (suitably scaled) limiting distribution of $\text{is}(w)$, where $w$ ranges over all alternating permutations in $\mathfrak{S}_n$?

Is it the Tracy-Widom distribution?

**Possible tool:** ∃ “umbral analogue” of Gessel’s determinantal formula.
Descent sets

Let $w = a_1 a_2 \cdots a_n \in S_n$. **Descent set of** $w$:

$$D(w) = \{ i : a_i > a_{i+1} \} \subseteq \{ 1, \ldots, n - 1 \}$$
Descent sets

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D(w) = \{ i : a_i > a_{i+1} \} \subseteq \{1, \ldots, n - 1\}
\]

\[
D(\mathbf{4157623}) = \{1, 4, 5\}
\]
\[
D(\mathbf{4152736}) = \{1, 3, 5\} \text{ (alternating)}
\]
\[
D(\mathbf{4736152}) = \{2, 4, 6\} \text{ (reverse alternating)}
\]
\[ \beta_n(S) = \# \{ w \in S_n : D(w) = S \} \]
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<table>
<thead>
<tr>
<th>( w )</th>
<th>( D(w) )</th>
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</thead>
<tbody>
<tr>
<td>123</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>213</td>
<td>( {1} )</td>
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<tr>
<td>312</td>
<td>( {1} )</td>
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<tr>
<td>132</td>
<td>( {2} )</td>
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<tr>
<td>231</td>
<td>( {2} )</td>
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<tr>
<td>321</td>
<td>( {1, 2} )</td>
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</table>

\[ \beta_3(\emptyset) = 1, \quad \beta_3(1) = 2 \]

\[ \beta_3(2) = 2, \quad \beta_3(1, 2) = 1 \]
Fix $n$. Let $S \subseteq \{1, \cdots, n - 1\}$. Let $u_S = t_1 \cdots t_{n-1}$, where
\[
t_i = \begin{cases} 
a, & i \notin S \\
b, & i \in S.
\end{cases}
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  b, & i \in S.
\end{cases}$$

**Example.** $n = 8$, $S = \{2, 5, 6\} \subseteq \{1, \ldots, 7\}$

$$u_S = abaabba$$
A noncommutative gen. function

\[ \Psi_n(a, b) = \sum_{S \subseteq \{1, \ldots, n-1\}} \beta_n(S) u_S. \]
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Example. Recall

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Thus

\[ \Psi_3(a, b) = aa + 2ab + 2ba + bb \]
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\[ = (a + b)^2 + (ab + ba) \]
The \textit{cd-index}

\textbf{Theorem.} There exists a noncommutative polynomial $\Phi_n(c, d)$, called the \textbf{cd-index} of $\mathfrak{S}_n$, with \textbf{nonnegative} integer coefficients, such that

$$\Psi_n(a, b) = \Phi_n(a + b, ab + ba).$$
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Example. Recall

$$\Psi_3(a, b) = aa + 2ab + 2ba + b^2 = (a + b)^2 + (ab + ba).$$

Therefore

$$\Phi_3(c, d) = c^2 + d.$$
Small values of $\Phi_n(c, d)$

\[
\begin{align*}
\Phi_1 &= 1 \\
\Phi_2 &= c \\
\Phi_3 &= c^2 + d \\
\Phi_4 &= c^3 + 2cd + 2dc \\
\Phi_5 &= c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2 \\
\Phi_6 &= c^5 + 4c^3d + 9c^2dc + 9cdc^2 + 4dc^3 \\
&\quad + 12cd^2 + 10dcd + 12d^2c.
\end{align*}
\]
Let \( \deg c = 1 \), \( \deg d = 2 \).
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Replace each $c$ in $\mu$ with 0, each $d$ with 10, and remove final 0. Get the characteristic vector of a set $S_\mu \subseteq [n - 2]$. 
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**Example.** $n = 10$:

$$
\mu = cd^2 c^2 d \rightarrow 0 \cdot 10 \cdot 10 \cdot 0 \cdot 0 \cdot 1 \times = 01010001,
$$

the characteristic vector of $S_\mu = \{2, 4, 8\} \subseteq [8]$
Recall: $w = a_1 a_2 \cdots a_n \in \mathcal{S}_n$ is a simsun permutation if the subsequence with elements $1, 2, \ldots, k$ has no double descents, $1 \leq k \leq n$. 
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**Theorem (Simion-Sundaram, variant of Foata-Schützenberger)** The coefficient of $\mu$ in $\Phi(c, d)$ is equal to the number of simsun permutations in $\mathcal{S}_{n-1}$ with descent set $S_\mu$. 
Example. $\Phi_6 = c^5 + 4c^3d + 9c^2dc + 9cdc^2 + 4dc^3 + 12cd^2 + 10dcd + 12d^2c$,

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The ten simsun permutations $w \in S_5$ with $D(w) = \{1, 4\}$:

21354, 21453, 31254, 31452, 41253

41352, 42351, 51243, 51342, 52341,
An example

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The ten simsun permutations $w \in \mathcal{S}_5$ with $D(w) = \{1, 4\}$:

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but not 32451.
Two consequences

**Theorem.** (a) $\Phi_n(1, 1) = E_n$ (*the number of simsum permutations* $w \in \mathfrak{S}_n$).
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**Theorem.** (a) $\Phi_n(1, 1) = E_n$ *(the number of simsum permutations $w \in S_n$).*

(b) *(Niven, de Bruijn)* For all $S \subseteq \{1, \ldots, n - 1\}$,

$$\beta_n(S) \leq E_n,$$

with equality if and only if $S = \{1, 3, 5, \ldots \}$ or $S = \{2, 4, 6, \ldots \}$
An example

\[ \Phi_5 = 1c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2 \]

\[ 1 + 3 + 5 + 3 + 4 = 16 = E_5 \]
Darn!
That’s the end...