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The Stable Behavior of Some Characters of SL(n, ℂ)

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1. INTRODUCTION

Let $\mathfrak{g} = \mathrm{sl}(n, \mathbb{C})$ denote the Lie algebra of all $n \times n$ complex matrices of trace 0. Let $\mathrm{ad}: \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(\mathfrak{g})$ denote the adjoint representation of $\mathrm{SL}(n, \mathbb{C})$, defined by

$$(\mathrm{ad}\, X)(A) = XAX^{-1},$$

where $X \in \mathrm{SL}(n, \mathbb{C})$ and $A \in \mathfrak{g}$. Introduce two infinite sets $u = (u_1, u_2, \ldots)$ and $v = (v_1, v_2, \ldots)$ of variables, and the following function on $\mathrm{SL}(n, \mathbb{C})$ with values in the formal power series ring $\mathbb{Q}[[u, v]]$:

$$\det \prod_k \frac{1 - u_k ad\, X}{1 - v_k ad\, X}. \tag{1}$$

This function is a virtual character of $\mathrm{SL}(n, \mathbb{C})$, and a wide variety of problems in combinatorics and representation theory involve its decomposition into irreducibles. In particular, the q-Dyson conjecture for equal exponents and the computation of the generalized exponents of $\mathrm{SL}(n, \mathbb{C})$ are special cases of this problem, discussed in more detail in Sections 8 and 9. We will study the behavior of (1) as $n \to \infty$, and obtain what amounts to a decomposition into irreducibles in this limiting case.

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The virtual character (1) is a symmetric function of the eigenvalues of X. Our approach will be based on the theory of symmetric functions, which we now briefly review. Two basic references on symmetric functions are [14] and [20]. In general we will adhere to the notation and terminology of [14].

2. SYMMETRIC FUNCTIONS AND THE CHARACTERS OF $\mathrm{SL}(n,\mathbb{C})$

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a *partition*, i.e., a decreasing sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ of nonnegative integers with only finitely many λ_i unequal to zero. If $\lambda_{n+1} = \lambda_{n+2} = \cdots = 0$, then we also write $\lambda = (\lambda_1, \ldots, \lambda_n)$. The number of nonzero λ_i is the *length* of λ, denoted $l(\lambda)$. If $m = \lambda_1 + \lambda_2 + \cdots$ then we write $\lambda \vdash m$ or $|\lambda| = m$ and say that λ is a *partition* of m. The *conjugate* partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ to λ has $\lambda_i - \lambda_{i+1}$ parts equal to i. We also let $m_k(\lambda)$ denote the number of parts of λ which are equal to k, so $|\lambda| = \sum k \cdot m_k(\lambda)$.

Let $\Lambda_n = \Lambda_n(x)$ denote the ring of all symmetric polynomials with rational coefficients in the variables $x = (x_1, \ldots, x_n)$, and let Ω_n denote Λ_n modulo the ideal generated by $x_1 x_2 \cdots x_n - 1$. A vector space basis for Ω_n consists of all *Schur functions* $s_\lambda(x) = s_\lambda(x_1, \ldots, x_n)$, where λ ranges over all partitions of length $\leqslant n-1$. For the definition and basic properties of Schur functions, see [14] or [20].

A *polynomial representation* of $\mathrm{SL}(n,\mathbb{C})$ of dimension N is a homomorphism $\phi : \mathrm{SL}(n,\mathbb{C}) \to \mathrm{GL}(N,\mathbb{C})$ such that for $X \in \mathrm{SL}(n,\mathbb{C})$, the entries of the matrix $\phi(X)$ are polynomial functions of the entries of X. For instance,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix} \tag{2}$$

is a polynomial representation of $\mathrm{SL}(2,\mathbb{C})$ of dimension 3. Every continuous representation $\phi : \mathrm{SL}(n,\mathbb{C}) \to \mathrm{GL}(N,\mathbb{C})$ is equivalent to a polynomial representation, so for all practical purposes it costs us nothing to consider only polynomial representations. If ϕ is a polynomial representation of $\mathrm{SL}(n,\mathbb{C})$, then there is a unique polynomial in Ω_n, denoted $\operatorname{char}\phi$ and called the *character* of ϕ, satisfying $(\operatorname{char}\phi)(x_1, \ldots, x_n) = \operatorname{tr}\phi(X)$ for any $X \in \mathrm{SL}(n,\mathbb{C})$ with eigenvalues