FACTORORIZATION OF PERMUTATIONS INTO n-CYCLES*

Richard P. STANLEY

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Using the character theory of the symmetric group \( S_n \), an explicit formula is derived for the number \( g_k(\pi) \) of ways of writing a permutation \( \pi \in S_n \) as a product of \( k \) \( n \)-cycles. From this the asymptotic expansion for \( g_k(\pi) \) is derived, provided that when \( k = 2 \), \( \pi \) has \( O(\log n) \) fixed points. In particular, there follows a conjecture of Walkup that if \( \pi \in S_n \) is an even permutation with no fixed point, then \( \lim_{n \to \infty} g_2(\pi_n)/(n-2)! = 2 \).

1. Introduction

Let \( \pi \) be an element of the symmetric group \( S_n \) of all permutations of an \( n \)-element set. Let \( g_k(\pi) \) be the number of \( k \)-tuples \( (\sigma_1, \ldots, \sigma_k) \) of cycles \( \sigma_i \) of length \( n \) such that \( \pi = \sigma_1 \cdots \sigma_k \). Thus \( g_k(\pi) = 0 \) if either
(a) \( \pi \) is an odd permutation and \( n \) is an odd integer, or
(b) \( \pi \) is odd, \( n \) is even, and \( k \) is even, or
(c) \( \pi \) is even, \( n \) is even, and \( k \) is odd.

Husemoller [6, Proposition 4] attributes to Gleason the result that \( g_2(\pi) > 0 \) for any even \( \pi \). The function \( g_2(\pi) \) was subsequently considered in [1, 2, 9]. In particular, Walkup [9, p. 316] conjectured that \( \lim_{n \to \infty} g_2(\pi_n)/(n-2)! = 2 \) where \( \pi_1, \pi_2, \ldots \) is any sequence of even permutations without fixed points, with \( \pi_n \in S_n \). We will use the character theory of \( S_n \) to derive an explicit expression for \( g_k(\pi) \) from which Walkup's conjecture can be deduced. More generally, we can write down the entire asymptotic expansion of the function \( g_k(\pi) \) for fixed \( k \) (provided the number of fixed points of \( \pi \) remains small when \( k = 2 \)). The technique of character theory was also used in [1, Section 3], and some special cases of our results overlap with this paper. In [2, Corollary 4.8] an explicit expression for \( g_2(\pi) \) is derived, which is simpler than ours, and which can also be used to prove Walkup's conjecture. I am grateful to the referee for calling my attention to [2].

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2. Character theory

We first review the results from character theory that we will need. Let $G$ be any finite group and $\mathbb{C}G$ its group algebra over $\mathbb{C}$. If $C_i$, $1 \leq i \leq t$, is a conjugacy class of $G$, then let $K_i = \sum_{g \in C_i} g$ be the corresponding element of $\mathbb{C}G$. If $\chi^1, \ldots, \chi^t$ are the irreducible (ordinary) characters of $G$ with $\deg \chi^i = f^i$, then the elements

$$F_i = \frac{f^i}{|G|} \sum_{i=1}^t \chi^i K_i, \quad 1 \leq j \leq t,$$

are a set of orthogonal idempotents in the center of $\mathbb{C}G$, where $\chi^i$ denotes $\chi^i$ evaluated at any element of $C_i$. Inverting (1) yields

$$K_i = |C_i| \sum_{i=1}^t \frac{\chi^i}{f^i} F_i,$$

where $|C_i|$ is the number of elements of the class $C_i$. See, e.g., [3, Section 236]. Since the $F_i$'s are orthogonal idempotents, we have for any integer $k \geq 1$,

$$K_i^k = |C_i|^k \sum_{i=1}^t \left(\frac{\chi^i}{f^i}\right)^k F_i = |C_i|^k \sum_{i=1}^t \left(\frac{\chi^i}{f^i}\right)^k \frac{f^i}{|G|} \sum_{i=1}^t \chi^i K_i$$

$$= \frac{|C_i|^k}{|G|} \sum_{i=1}^t K_i \sum_{i=1}^t \left(\frac{\chi^i}{f^i}\right)^k f^i \chi^i.$$

Now let $G = \mathfrak{S}_n$. A partition of $n$ may be regarded as a sequence $\rho = \langle a_1, \ldots, a_n \rangle$ of non-negative integers such that $\sum ia_i = n$. We then write $\rho \rightarrow n$. We also write $\rho = (1^{a_1}, 2^{a_2}, \ldots, n^{a_n})$ where terms $i^{a_i}$ with $a_i = 0$ are omitted and where exponents $a_i = 1$ are omitted. For instance, $\langle 0, 1, 0, 0, 2 \rangle = (2, 5^2)$ is a partition of 12. For later convenience we also write $(1^n - 1, 1)$ for the partition $(1^n) = (n, 0, \ldots, 0)$, and we set $\lambda_i = (1^i, n-i)$ for $0 \leq i \leq n-1$. If $\rho = (a_1, \ldots, a_n) \rightarrow n$, then the set of elements of $\mathfrak{S}_n$ with $a_i$ cycles of length $i$ forms a conjugacy class $C_\rho$ of $\mathfrak{S}_n$. The class $C_{(n)}$ of $n$-cycles is abbreviated $C_n$, so $|C_n| = (n-1)!$. If $\phi : \mathfrak{S}_n \rightarrow \mathbb{C}$ is constant on conjugacy classes and if $\pi \in C_\rho$, then we write interchangeably $\phi(\pi)$ or $\phi(\rho)$ or $\phi(C_\rho)$. Note in particular that $g_k(\pi)$ has this property, so $g_k(\rho)$ denotes $g_k(\pi)$ for any $\pi \in C_\rho$. Recall that for each partition $\lambda$ of $n$ there is a natural way of associating an irreducible character $\chi^\lambda$ of $\mathfrak{S}_n$ [5, Chapter 7; 7, Chapter 5]. In particular, the partition $\lambda = (n)$ corresponds to the trivial character $\chi^n = 1$ for all $\rho \rightarrow n$.

We next state two crucial lemmas involving the characters $\chi^\lambda$. A proof of Lemma 2.1 is an immediate consequence of the 'graphical method' for determining the characters of $\mathfrak{S}_n$ [5, Chapter 7.4; 7, Chapter 5.3; 8, Chapter 4]. See [5, p. 205; 8, Lemma 4.11] in particular. A proof of Lemma 2.2 essentially appears in [7, p. 139].
Lemma 2.1. Let $0 \leq i \leq n-1$ and $\lambda 
rightarrow n$. Then

$$\chi_n^\lambda = \begin{cases} \frac{(-1)^i}{i!} \lambda = \lambda_i = (1^i, n-i), \\ 0, \quad \text{otherwise,} \end{cases}$$

where $\chi_n^\lambda$ is the value of the character $\chi^\lambda$ at any element of $C_n$.

Lemma 2.2. Let $0 \leq i \leq n-1$ and $\rho = \langle a_1, a_2, \ldots, a_n \rangle \nrightarrow n$. Then

$$\chi_n^\rho = \sum \binom{a_1 - 1}{r_1} \binom{a_2}{r_2} \binom{a_3}{r_3} \cdots \binom{a_i}{r_i} \frac{(-1)^i}{r_1 \cdots r_i},$$

where the sum is over all partitions $\langle r_1, r_2, \ldots, r_i \rangle$ of $i$. In particular, $\deg \chi_n^\lambda = f^\lambda = \binom{n-1}{i}.$

3. A formula for $g_k(\pi)$

It is now easy to give a formula for $g_k(\pi)$.

Theorem 3.1. Let $\rho = \langle a_1, \ldots, a_n \rangle \nrightarrow n$. Then

$$g_k(\rho) = \frac{(n-1)!^{k-1}}{n} \sum_{i=0}^{n-1} \binom{a_1 - 1}{r_1} \binom{a_2}{r_2} \binom{a_3}{r_3} \cdots \binom{a_i}{r_i} \frac{(-1)^i}{r_1 \cdots r_i},$$

where $\langle r_1, \ldots, r_i \rangle$ ranges over all solutions in non-negative integers to $\sum j_{r_i} = i$.

Proof. As above, let $C_n$ denote the class of $n$-cycles in $S_n$ and $K_n = \sum_{\pi \in C_n} \pi \in C \otimes S_n$. By definition of $C \otimes S_n$, we have

$$K_n^k = \sum_{\rho \nrightarrow n} g_k(\rho) K_\rho.$$ 

Hence by (3) and the fact that the characters of $S_n$ are real, there follows

$$g_k(\rho) = \frac{(n-1)!^k}{n!} \sum_{\mu \nrightarrow n} \binom{\chi_n^\mu}{f^\mu} \chi_n^\mu.$$ 

Then by Lemma 2.1,

$$g_k(\rho) = \frac{(n-1)!^{k-1}}{n} \sum_{i=0}^{n-1} \binom{\chi_n^\mu}{f^\mu} \chi_n^\mu = \frac{(n-1)!^{k-1}}{n} \sum_{i=0}^{n-1} \binom{-1}{f^\mu} \chi_n^\mu.$$ 

Substituting the values $\chi_n^\lambda$ and $f^\lambda$ from Lemma 2.2 completes the proof.
Some special cases of Theorem 3.1 are particularly simple. Putting \( \rho = (1^n) \) yields

\[
g_k(1^n) = \frac{(n-1)!k^{-1}}{n} \sum_{i=0}^{n-1} (-1)^{k} \binom{n-1}{i}^{-(k-2)},
\]

(4)

the number of ways of writing the identity permutation in \( S_n \) as a product of \( k \) \( n \)-cycles. When \( k = 3 \), the sum (4) can be evaluated [4, (2.1); 1. Section 3(ii)]. Namely,

\[
g_3(1^n) = \begin{cases} 
0, & n \text{ even,} \\
2(n-1)!^2/(n+1), & n \text{ odd.}
\end{cases}
\]

A more combinatorial proof of (5) is essentially given in [1, Corollary 2.2]. It is also clear that \((n-1)!g_k(C_n) = g_{k+1}(1^n)\), since \( \pi_1 \cdots \pi_k \in C_n \) if and only if there is a (unique) \( \pi_{k+1} \in C_n \) satisfying \( \pi_1 \cdots \pi_k \pi_{k+1} = \varepsilon \). Hence

\[
g_k(C_n) = \frac{(n-1)!k^{-1}}{n} \sum_{i=0}^{n-1} (-1)^{i(k+1)} \binom{n-1}{i}^{-(k-1)},
\]

\[
g_2(C_n) = 2(n-1)!/(n+1), \quad n \text{ odd.} \quad (6)
\]

This same formula is obtained by setting \( \rho = (n) \) in Theorem 3.1. More generally, we have

\[
g_k(1^{n-1}, j) = \frac{(n-1)!k^{-1}}{n} \sum_{i=0}^{n-1} \binom{n-1-j}{i}(-1)^{(n-j-1)} \binom{n-j-1}{i-j} \binom{n-1}{i}^{-(k-1)},
\]

for \( 2 \leq j \leq n \), where we set \( \binom{n-j-1}{i} = 0 \) if \( i < j \).

As a further special case, if \( n = mj + 1 \), then from Theorem 3.1 we obtain

\[
g_k(1, j^m) = \frac{(n-1)!k^{-1}}{n} \sum_{i=0}^{m} (-1)^{i} \binom{m}{i} \binom{n-1}{ij}^{-(k-1)}.
\]

In particular, when \( m = 1 \) we get \( g_k(1, n-1) = 2(n-1)!/n \). Walkup [9, Theorem 1] gives a combinatorial proof that \( ng_2(1^n, 2^n, \ldots, n^n) = g_2(1^{a_1+1}, 2^{a_2}, \ldots, n^{a_n}) \). Thus from \( g_2(1, n-1) = 2(n-1)!/n \) we get another proof of (6). In effect, we have another proof of the identity [4, (2.1)]. Some other explicit values of \( g_2(\rho) \) appear in [1, Corollary 2.2; 2, Example 4.9] and can be deduced from Theorem 3.1 using the appropriate binomial coefficient identity.
4. Asymptotics

We now derive an asymptotic expansion for \( g_k(\rho) \), where \( \rho = (a_1, a_2, \ldots, a_n) \). When \( k = 2 \), it will be necessary to assume that \( a_1 \) is not too large. First we dispose of the easy case \( k \geq 3 \).

**Theorem 4.1.** Fix \( k \geq 3 \). Let \( \rho = (a_1, a_2, \ldots, a_n) \). If \( (n-1)k + a_2 + a_4 + \cdots \) is odd, then \( g_k(\rho) = 0 \). If \( (n-1)k + a_2 + a_4 + \cdots \) is even, then for any fixed \( j \geq 0 \) we have

\[
g_k(\rho) = \frac{2(n-1)!^{k-1}}{n} \left[ \sum_{i=0}^{l} \frac{(-1)^i \chi_{\rho}^i}{(n-1)^{k-1}} + O(n^{-(i+1)(k-2)}) \right],
\]

uniformly in \( a_1, a_2, \ldots, a_n \) and \( n = \sum i a_i \).

**Proof.** The assertion for \( (n-1)k + a_2 + a_4 + \cdots \) odd is equivalent to (a)–(c) of Section 4. Hence assume \( (n-1)k + a_2 + a_4 + \cdots \) is even. Since the partitions \( \lambda_i \) and \( \lambda_{n-i-1} \) are conjugate, we have e.g. by [7, p. 711] that \( \chi_{\rho}^i = (-1)^i a_2 + a_4 + \cdots \chi_{\rho}^{\lambda_{n-i-1}} \). Thus if we set \( T_i = (-1)^i \chi_{\rho}^i / (\binom{n}{i})^{k-1} \), then \( T_i = T_{n-i-1} \). Hence

\[
g_k(\rho) = \begin{cases} \frac{2(n-1)!^{k-1}}{n} \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} T_i, & \text{if } n \text{ is even,} \\ \frac{2(n-1)!^{k-1}}{n} \left( \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} T_i + \frac{1}{2} T_{(n-1)/2} \right), & \text{if } n \text{ is odd.} \end{cases}
\]

Thus

\[
\left| \frac{ng_k(\rho)}{2(n-1)!^{k-1}} - \sum_{i=0}^{\lfloor n/2 \rfloor} T_i \right| \leq \sum_{i=\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} |T_i| \leq \sum_{i=\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} \left| \frac{\chi_{\rho}^i}{(n-1)^{k-1}} \right| \leq \sum_{i=\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} \frac{n a_i}{(n-1)^{k-1}} \leq \sum_{i=\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} \frac{1}{(n-1)^{k-2}} \left( \frac{n}{i+1} \right) + \frac{n}{(n-1)^{k-2}} = O(n^{-(i+1)(k-2)}).
\]

This completes the proof.

Using Lemma 2.2, we can give the asymptotic expansion of \( g_k(\rho) \) as a function of \( a_1, a_2, \ldots, a_n \). We expect the \( (n-1)!^k \) products \( \pi_1 \pi_2 \cdots \pi_k \) to be approximately equidistributed through the \( \frac{1}{2} n! \) allowable elements of \( \mathcal{E}_n \). Indeed.
Theorem 4.1. say for $j = 2$, asserts that when $k \geq 3$,

$$\frac{\binom{n}{k} g_k(\rho)}{(n-1)!^k} = 1 + \frac{\binom{a_1 - 1}{2}}{(n-1)^{k-1}} + \frac{a_2}{(n-1)^{k-1}} + O(n^{-3(k-2)}).$$

When $k = 2$, we need a more delicate estimate than $\binom{n}{j} \leq (\binom{n}{2}^{-1})$. If $F(x) = \sum_{i=0}^n f_i x^i$ and $G(x) = \sum_{i=0}^n g_i x^i$ are power series with real coefficients, write $F(x) \geq G(x)$ if $f_i \geq g_i$ for all $i \geq 0$.

**Lemma 4.2.** If $F(x) \geq 0$, then

$$\frac{F(x)(1+x^j)}{1-x} \geq \frac{F(x)(1+x^{j+1})}{1-x}$$

for all $j \geq 0$.

**Proof.** We have

$$\frac{F(x)(1+x^j)}{1-x} - \frac{F(x)(1+x^{j+1})}{1-x} = x^j F(x) \geq 0,$$

as desired.

**Lemma 4.3.** Let $\rho = (a_1, \ldots, a_n) \vdash n$, and let $0 \leq i \leq \lfloor n/2 \rfloor$. Then

$$|\chi^\rho_i| \leq 2a_i \left(\frac{n/2}{[i/2]}\right).$$

**Proof.** According to Lemma 2.2, we have

$$\sum_{i=0}^{n-1} \chi^\rho_i x^i = (1+x)^{a_i-1}(1-x^2)^{a_2}(1+x^3)^{a_3} \cdots (1-(-1)^n x^n)^{a_n}.$$

Hence

$$\sum_{i=0}^{n-1} |\chi^\rho_i| x^i \leq \frac{(1+x)^{a_1}(1+x^2)^{a_2} \cdots (1+x^n)^{a_n}}{1-x}.$$

By successive applications of Lemma 4.2, we obtain

$$\sum_{i=0}^{n-1} |\chi^\rho_i| x^i \leq \frac{(1+x)^{a_1}(1+x^2)^{a_2+\ldots+a_n}}{1-x} \leq \frac{2a_1(1+x^2)^{\lfloor n/2 \rfloor}}{1-x}.$$

Since $\binom{n/2}{j} \leq \binom{n/2}{i}$ when $j \leq \lfloor n/2 \rfloor$, the proof follows.

**Theorem 4.3.** Let $\rho = (a_1, a_2, \ldots, a_n) \vdash n$. If $a_1 + a_4 + \cdots$ is odd (i.e., if $\rho$ is odd), then $g_2(\rho) = 0$. If $a_1 + a_4 + \cdots$ is even (i.e., if $\rho$ is even), then for any fixed $j \geq 0$ we
have
\[
\zeta_2(\rho) = \frac{2(n-1)!}{n} \left[ \sum_{i=0}^{\frac{n}{2}} \frac{\chi_i^\lambda}{(n-1)^i} + O(2^an^{-(i+1)/2}) \right]
\]
uniformly in \(a_1, a_2, \ldots, a_n\) and \(n = \sum ia_i\).

**Proof.** As in Theorem 4.1, we may assume \(a_3 + a_4 + \cdots\) is even. Setting \(T_i = \chi_i^\lambda/(\binom{n}{i})\), then as in (7) we obtain
\[
\left| \frac{n\zeta_2(\rho)}{2(n-1)!} - \sum_{i=0}^{\frac{n}{2}} T_i \right| \leq \frac{n}{\sum_{i=0}^{\frac{n}{2}} \frac{\chi_i^\lambda}{(n-1)^i}}.
\]
Thus by Lemma 4.3,
\[
\left| \frac{n\zeta_2(\rho)}{2(n-1)!} - \sum_{i=0}^{\frac{n}{2}} T_i \right| \leq \sum_{i=\frac{n}{2}+1}^{\frac{n}{2}} \frac{2^\lambda i \binom{n}{i}}{\binom{n}{i}}.
\]
(9)

Denote the left-hand side of (9) by \(E_i\), and let \(t_i = \binom{n}{i}/(\binom{n}{i})\). Then \(t_i = O(n^{-(i+1)/2})\) for \(i = j + 1, j + 2, j + 3, j + 4\) and \([n/2]\). Hence
\[
E_i \leq 2^\lambda n \sum_{i=\frac{n}{2}+1}^{\frac{n}{2}+5} t_i + O(2^\lambda n^{-(i+1)/2}).
\]
(10)

We claim that \(t_i \geq t_{i+2}\) provided \(0 \leq i \leq [n/2] - 2\). We will prove only the case \(n = 2m, i = 2k - 1\) here. The three remaining cases are handled similarly. When \(n = 2m\) and \(i = 2k - 1\), we have by direct calculation
\[
t_i - t_{i+2} = \frac{2m!(2k-1)!(2m-2k-2)!(2m^2-(6k+2)m+4k^2-1)}{(k-1)!(m-k+1)!(2m-1)!}.
\]
The largest root of the equation \(2x^2 -(6k+2)x + 4k^2 - 1 = 0\) is given by
\[
x = \frac{1}{2}(3k + 1 + \sqrt{k^2 + 6k + 3}) < \frac{1}{2}(3k + 1 + k + 3) = 2k + 2.
\]
Hence if \(m > 2k + 1\), then \(2m^2 -(6k+2)m + 4k^2 - 1 > 0\). Since \(m > 2k + 1\) is equivalent to \(i < \frac{1}{2} n - 2\), the claim is proved.

It follows from (9) and the inequality \(t_i \geq t_{i+2}\) that
\[
E_i \leq 2^\lambda n^2 (t_{i+5} + t_{i+6}) + O(2^\lambda n^{-(i+1)/2}) = O(2^\lambda n^{-(i+1)/2}),
\]
completing the proof.
Thus for instance taking $j = 2$ in Theorem 4.3, we obtain that for even $p$,

$$g_2(p) = \frac{2(n-1)!}{n} \left[ 1 + \frac{a_1 - 1}{n-1} + \frac{(a_1 - 1) - a_2}{(n-1)^2} + O(2^{a_1 n^{-3/2}}) \right].$$

Since $a_1 = O(n)$, it follows that if $a_1 = 0$ (or in fact $a_1 = O(\log n)$), then $g_2(p)/(n-2)! \to 2$ as $n \to \infty$, which is Walkup's conjecture [9, p. 316]. In fact, it suffices to assume only $a_1 = O(\log n)$. For assume $a_1 = O(\log n)$ for all $n$. Take $j > 2B(\log 2) - 1$ in Theorem 4.3 to obtain

$$g_2(p) = \frac{2(n-1)!}{n} \left[ \sum_{i=0}^{\infty} \frac{\lambda^i}{(n-1)_i} + o(1) \right] = \frac{2(n-1)!}{n} [1 + o(1)],$$

By a more careful analysis, Kleitman has shown (private communication) that $g_2(p)$ has the asymptotic expansion

$$g_2(p) \sim \frac{2(n-1)!}{n} \sum_{i=0}^{\infty} \frac{\lambda^i}{(n-1)_i}$$

provided only $a_1 = o(n)$. The key step is an improved version of Lemma 4.3, but we will not enter into the details here.

References