A NOTE ON THE SYMMETRIC POWERS OF THE STANDARD REPRESENTATION OF $S_n$

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\textbf{Abstract}

In this paper, we prove that the dimension of the space spanned by the characters of the symmetric powers of the standard $n$-dimensional representation of $S_n$ is asymptotic to $n^2/2$. This is proved by using generating functions to obtain formulas for upper and lower bounds, both asymptotic to $n^2/2$, for this dimension. In particular, for $n \geq 7$, these characters do not span the full space of class functions on $S_n$.

\textbf{Notation}

Let $P(n)$ denote the number of (unordered) partitions of $n$ into positive integers, and let $\phi$ denote the Euler totient function. Let $V$ be the standard $n$-dimensional representation of $S_n$, so that $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$ with $\sigma(e_i) = e_{\sigma(i)}$ for $\sigma \in S_n$. Let $S^N V$ denote the $N^{th}$ symmetric power of $V$, and let $\chi_N : S_n \to \mathbb{Z}$ denote its character. Finally, let $D(n)$ denote the dimension of the space of class functions on $S_n$ spanned by all the $\chi_N$, $N \geq 0$.

\textbf{1. Preliminaries}

Our aim in this paper is to investigate the numbers $D(n)$. It is a fundamental problem of invariant theory to decompose the character of the symmetric powers of an irreducible representation of a finite group (or more generally a reductive group). A special case with a nice theory is the reflection representation of a finite Coxeter group. This is essentially what we are looking at. (The defining representation of $S_n$ consists of the direct sum of the reflection representation and the trivial representation. This trivial summand has no significant effect on the theory.) In this context

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it seems natural to ask: what is the dimension of the space spanned by the symmetric powers? Moreover, decomposing the symmetric powers of the character of an irreducible representation of $S_n$ is an example of the operation of inner plethysm [1, Exer. 7.74], so we are also obtaining some new information related to this operation.

We begin with:

**Lemma 1.1.** Let $\lambda=(\lambda_1,\ldots,\lambda_k)$ be a partition of $n$ (which we denote by $\lambda \vdash n$), and suppose $\sigma \in S_n$ is a $\lambda$-cycle. Then $\chi_\lambda(\sigma)$ is equal to the number of solutions $(x_1,\ldots,x_k)$ in nonnegative integers to the equation $\lambda_1 x_1 + \cdots + \lambda_k x_k = N$.

**Proof.** Suppose without loss of generality that $\sigma=(1 \ 2 \ \cdots \ \lambda_1)(\lambda_1 + 1 \ \cdots \ \lambda_2)\cdots(\lambda_1 + \cdots + \lambda_{k-1} + 1 \ \cdots \ n)$. Consider a basis vector $e_1^{\otimes \lambda_1} \otimes \cdots \otimes e_n^{\otimes \lambda_n}$ of $S^N V$, so that $c_1 + \cdots + c_n = N$ with each $c_i \geq 0$. This vector is fixed by $\sigma$ if and only if $c_1 = \cdots = c_{\lambda_1}$, $c_{\lambda_1+1} = \cdots = c_{\lambda_1+\lambda_2}$ and so on. Since $\chi_\lambda(\sigma)$ equals the number of basis vectors fixed by $\sigma$, the lemma follows.

It seems difficult to work directly with the $\chi_\lambda$’s; fortunately, it is not too hard to restate the problem in more concrete terms. Given a partition $\lambda=(\lambda_1,\ldots,\lambda_k)$ of $n$, define

$$f_\lambda(q) = \frac{1}{(1-q^{\lambda_1}) \cdots (1-q^{\lambda_k})}.$$  

Next, define $F_n \subset \mathbb{C}[[q]]$ to be the complex vector space spanned by all of these $f_\lambda(q)$’s. We have:

**Proposition 1.2.** $\dim F_n = D(n)$.

**Proof.** Consider the table of the characters $\chi_\lambda$: we are interested in the dimension of the row-span of this table. Since the dimension of the row-span of a matrix is equal to the dimension of its column-span, we can equally well study the dimension of the space spanned by the columns of the table. By the preceding lemma, the $N^\text{th}$ entry of the column corresponding to the $\lambda$-cycles is equal to the number of nonnegative integer solutions to the equation $\lambda_1 x_1 + \cdots + \lambda_k x_k = N$. Consequently, one easily verifies that $f_\lambda(q)$ is the generating function for the entries of the column corresponding to the $\lambda$-cycles. The dimension of the column-span of our table is therefore equal to $\dim F_n$, and the proposition is proved.

2. Upper Bounds on $D(n)$

Our basic strategy for computing upper bounds for $\dim F_n$ is to put all the generating functions $f_\lambda(q)$ over a common denominator; then the dimension of their span is bounded above by 1 plus the degree of their numerators. For example, one can see without much difficulty that $(1-q)(1-q^2)\cdots(1-q^n)$ is the least common multiple of the denominators of the $f_\lambda(q)$’s. Putting all of the $f_\lambda(q)$’s over this common
denominator, their numerators then have degree \(n(n + 1)/2 - n\), which proves

\[
D(n) \leq \frac{n(n - 1)}{2} + 1.
\]

By modifying this strategy carefully, it is possible to find a somewhat better bound. Observe that the denominator of each of our \(f_\lambda\)'s is (up to sign change) a product of cyclotomic polynomials. In fact, the power of the \(j\)th cyclotomic polynomial \(\Phi_j(q)\) dividing the denominator of \(f_\lambda(q)\) is precisely equal to the number of \(\lambda_i\)'s which are divisible by \(j\). It follows that \(\Phi_j(q)\) divides the denominator of \(f_\lambda(q)\) at most \(\left\lfloor \frac{n}{j} \right\rfloor\) times, and the partitions \(\lambda\) for which this upper bound is achieved are precisely the \(P\left(n-j\left\lfloor \frac{n}{j} \right\rfloor\right)\) partitions of \(n\) which contain \(\left\lfloor \frac{n}{j} \right\rfloor\) copies of \(j\). Let \(S_j\) be the collection of \(f_\lambda\)'s corresponding to these \(P\left(n-j\left\lfloor \frac{n}{j} \right\rfloor\right)\) partitions. One sees immediately that the dimension of the space spanned by the functions in \(S_j\) is just \(D\left(n-j\left\lfloor \frac{n}{j} \right\rfloor\right)\): in fact, the functions in this space are exactly \(1/(1-q^i)^{\left\lfloor \frac{n}{j} \right\rfloor}\) times the functions in \(F_{n-j\left\lfloor \frac{n}{j} \right\rfloor}\).

Now the power of \(\Phi_j(q)\) in the least common multiple of the denominators of all of the \(f_\lambda(q)\)'s excluding those in \(S_j\) is only \(\left\lfloor \frac{n}{j} \right\rfloor - 1\), so the degree of this common denominator is only \(n(n + 1)/2 - \phi(j)\). Therefore, as in the first paragraph of this section, the dimension of the space spanned by all of the \(f_\lambda\)'s except those in \(S_j\) is at most \(n(n - 1)/2 + 1 - \phi(j)\); since the dimension spanned by the functions in \(S_j\) is \(D\left(n-j\left\lfloor \frac{n}{j} \right\rfloor\right)\), we have proved the upper bound

\[
D(n) \leq \frac{n(n - 1)}{2} + 1 - \phi(j) + D\left(n-j\left\lfloor \frac{n}{j} \right\rfloor\right).
\]

If it happens that \(D\left(n-j\left\lfloor \frac{n}{j} \right\rfloor\right) < \phi(j)\), then this upper bound is an improvement on our original upper bound. If we repeat this process, this time simultaneously excluding the sets \(S_j\) for all of the \(j\)'s which gave us an improved upper bound in the above argument, we find that we have proved:

**Proposition 2.1.**

\[
D(n) \leq \frac{n(n - 1)}{2} + 1 - \sum_{j=1}^{n} \max \left(0, \phi(j) - D\left(n-j\left\lfloor \frac{n}{j} \right\rfloor\right)\right).
\]

Finally, we obtain an upper bound for \(D(n)\) which does not depend on other values of \(D(\cdot)\):

**Corollary 2.2.** Recursively define \(U(0) = 1\) and

\[
U(n) = \frac{n(n - 1)}{2} + 1 - \sum_{j=1}^{n} \max \left(0, \phi(j) - U\left(n-j\left\lfloor \frac{n}{j} \right\rfloor\right)\right).
\]

Then \(D(n) \leq U(n)\).
Proof. We proceed by induction on \( n \). Equality certainly holds for \( n = 0 \). For larger \( n \), the inductive hypothesis shows that \( D\left(n - j \left\lfloor \frac{n}{j} \right\rfloor \right) \leq U\left(n - j \left\lfloor \frac{n}{j} \right\rfloor \right) \) when \( j > 0 \), and so

\[
D(n) \leq \frac{n(n - 1)}{2} + 1 - \sum_{j=1}^{n} \max \left( 0, \phi(j) - D\left(n - j \left\lfloor \frac{n}{j} \right\rfloor \right) \right)
\]

\[
\leq \frac{n(n - 1)}{2} + 1 - \sum_{j=1}^{n} \max \left( 0, \phi(j) - U\left(n - j \left\lfloor \frac{n}{j} \right\rfloor \right) \right)
\]

\[
= U(n).
\]

Below is a table of values of \( D(n) \) and \( U(n) \) for \( n \leq 23 \), calculated in Maple, with \( P(n) \) and our first estimate \( \frac{n(n - 1)}{2} + 1 \) provided for contrast. Note that in the range \( 1 \leq n \leq 23 \), we have \( D(n) = U(n) \) except for \( n = 19, 20 \), when \( U(n) - D(n) = 1 \). Is it true, for instance, that

\[
-D(n) + \frac{n(n - 1)}{2} + 1 - \sum_{j=1}^{n} \max \left( 0, \phi(j) - D\left(n - j \left\lfloor \frac{n}{j} \right\rfloor \right) \right)
\]

is bounded as \( n \to \infty \)?

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Table 1. Values of \( D(n) \), \( U(n) \), \( n(n - 1)/2 + 1 \), \( P(n) \) for small \( n \)

Example 1. The first dimension where \( D(n) < P(n) \) is \( n = 7 \), and it is easy then to show that \( D(n) < P(n) \) for all \( n \geq 7 \). The difference \( P(7) - D(7) = 2 \) arises from the following two relations:

\[
\frac{4}{(1 - x^2)^2(1 - x)^3} = \frac{3}{(1 - x^3)(1 - x)^4} + \frac{1}{(1 - x^3)(1 - x^2)^2}
\]

and

\[
\frac{3}{(1 - x^3)(1 - x^2)(1 - x)^2} = \frac{2}{(1 - x^4)(1 - x)^3} + \frac{1}{(1 - x^4)(1 - x^3)}.
\]
The first relation, for example, says that if \( \chi \) is a linear combination of \( \chi_N \)'s, then
\[
4 \cdot \chi((2, 2)\text{-cycle}) = 3 \cdot \chi(3\text{-cycle}) + \chi((3, 2, 2)\text{-cycle}).
\]
Alternately, it tells us that for any \( N \geq 0 \), four times the number of nonnegative integral solutions to \( 2x_1 + 2x_2 + x_3 + x_4 + x_5 = N \) is equal to three times the number of such solutions to \( 3x_1 + 2x_2 + x_3 + x_4 + x_5 = N \) plus the number of such solutions to \( 3x_1 + 2x_2 + 2x_3 = N \).

3. Lower Bounds on \( D(n) \)

Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n \). The rational function \( f_\lambda(q) \) of equation (1) can be written as
\[
f_\lambda(q) = p_\lambda(1, q, q^2, \ldots),
\]
where \( p_\lambda \) denotes a power sum symmetric function. (See [1, Ch. 7] for the necessary background on symmetric functions.) Since the \( p_\lambda \) for \( \lambda \vdash n \) form a basis for the vector space (say over \( \mathbb{C} \)) \( \Lambda^n \) of all homogeneous symmetric functions of degree \( n \) [1, Cor. 7.7.2], it follows that if \( \{ u_\lambda \}_{\lambda \vdash n} \) is any basis for \( \Lambda^n \) then
\[
D(n) = \dim \text{span}_\mathbb{C} \{ u_\lambda(1, q, q^2, \ldots) : \lambda \vdash n \}.
\]
In particular, let \( u_\lambda = e_\lambda \), the elementary symmetric function indexed by \( \lambda \). Define
\[
d(\lambda) = \sum_i \binom{\lambda_i}{2}.
\]
According to [1, Prop. 7.8.3], we have
\[
e_\lambda(1, q, q^2, \ldots) = \frac{q^{d(\lambda)}}{\prod (1 - q)(1 - q^2) \cdots (1 - q^{\lambda_k})}.
\]
Since power series of different degrees (where the degree of a power series is the exponent of its first nonzero term) are linearly independent, we obtain from Proposition 1.2 the following result.

**Proposition 3.1.** Let \( E(n) \) denote the number of distinct integers \( d(\lambda) \), where \( \lambda \) ranges over all partitions of \( n \). Then \( D(n) \geq E(n) \).

**Note.** We could also use the basis \( s_\lambda \) of Schur functions instead of \( e_\lambda \), since by [1, Cor. 7.21.3] the degree of the power series \( s_\lambda(1, q, q^2, \ldots) = d(\lambda') \), where \( \lambda' \) denotes the conjugate partition to \( \lambda \).

Define \( G(n) + 1 \) to be the least positive integer that cannot be written in the form \( \sum_i \binom{\lambda_i}{2} \), where \( \lambda \vdash n \). Thus all integers \( 1, 2, \ldots, G(n) \) can be so represented, so \( D(n) \geq E(n) \geq G(n) \). We can obtain a relatively tractable lower bound for \( G(n) \), as follows. For a positive integer \( m \), write (uniquely)
\[
m = \binom{k_1}{2} + \binom{k_2}{2} + \cdots + \binom{k_r}{2},
\]
where
where \( k_1 \geq k_2 \geq \cdots \geq k_r \geq 2 \) and \( k_1, k_2, \ldots \) are chosen successively as large as possible so that

\[
m - \binom{k_1}{2} - \binom{k_2}{2} - \cdots - \binom{k_i}{2} \geq 0
\]

for all \( 1 \leq i \leq r \). For instance, \( 26 = \binom{6}{2} + \binom{3}{2} + \binom{2}{2} \). Define \( \nu(m) = k_1 + k_2 + \cdots + k_r \). Suppose that \( \nu(m) \leq n \) for all \( m \leq N \). Then if \( m \leq N \) we can write \( m = \binom{k_1}{2} + \cdots + \binom{k_r}{2} \) so that \( k_1 + \cdots + k_r \leq n \). Hence if \( \lambda = (k_1, \ldots, k_r, 1^{n-\sum k_i}) \) (where \( 1^s \) denotes \( s \) parts equal to 1), then \( \lambda \) is a partition of \( n \) for which \( \sum_i \binom{k_i}{2} = m \). It follows that if \( \nu(m) \leq n \) for all \( m \leq N \) then \( G(n) \geq N \). Hence if we define \( H(n) \) to be the largest integer \( N \) for which \( \nu(m) \leq n \) whenever \( m \leq N \), then we have established the string of inequalities

\[
(4) \quad D(n) \geq E(n) \geq G(n) \geq H(n).
\]

Here is a table of values of these numbers for \( 1 \leq n \leq 23 \). Note that \( D(n) \) appears to be close to \( E(n + 1) \). We don’t have any theoretical explanation of this observation.

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Table 2. Values of \( D(n) \), \( E(n) \), \( G(n) \), \( H(n) \) for small \( n \)

**Proposition 3.2.** We have

\[
(5) \quad \nu(m) \leq \sqrt{2m + 3m^{1/4}}
\]

for all \( m \geq 405 \).

**Proof.** The proof is by induction on \( m \). It can be checked with a computer that equation (5) is true for \( 405 \leq m \leq 50000 \). Now assume that \( M > 50000 \) and that (5) holds for \( 405 \leq m < M \). Let \( p = p_M \) be the unique positive integer satisfying

\[
\binom{p}{2} \leq M < \binom{p + 1}{2}.
\]

Thus \( p \) is just the integer \( k_1 \) of equation (3). Explicitly we have

\[
p_M = \left\lfloor \frac{1 + \sqrt{8M + 1}}{2} \right\rfloor.
\]
By the definition of $\nu(M)$ we have
\[
\nu(M) = p_M + \nu\left(M - \left(\frac{p_M}{2}\right)\right).
\]
It can be checked that the maximum value of $\nu(m)$ for $m < 405$ is $\nu(404) = 42$. Set $q_M = (1 + \sqrt{8M + 1})/2$. Since $M - \left(\frac{p_M}{2}\right) \leq p_M \leq q_M$, by the induction hypothesis we have
\[
\nu(M) \leq q_M + \max(42, \sqrt{2q_M + 3q_M^{1/4}}).
\]
It is routine to check that when $M > 50000$ the right hand side is less than $\sqrt{2M + 3M^{1/4}}$, and the proof follows. \qed

**Proposition 3.3.** There exists a constant $c > 0$ such that
\[
H(n) \geq \frac{n^2}{2} - cn^{3/2}
\]
for all $n \geq 1$.

**Proof.** From the definition of $H(n)$ and Proposition 3.2 (and the fact that the right-hand side of equation (5) is increasing), along with the inequality $\nu(m) \leq 42 = \lfloor \sqrt{2 \cdot 405 + 3 \cdot 405^{1/4}} \rfloor$ for $m \leq 404$, it follows that
\[
H\left(\lfloor \sqrt{2m + 3m^{1/4}} \rfloor\right) \geq m
\]
for $m > 404$. For $n$ sufficiently large, we can evidently choose $m$ such that $n = \lfloor \sqrt{2m + 3m^{1/4}} \rfloor$, so $H(n) \geq m$. Since $\sqrt{2m + 3m^{1/4}} + 1 > n$, an application of the quadratic formula (again for $n$ sufficiently large) shows
\[
m^{1/4} \geq \frac{-3 + \sqrt{9 + 4\sqrt{2(n - 1)}}}{2\sqrt{2}},
\]
from which the result follows without difficulty. \qed

Since we have established both upper bounds (equation (2)) and lower bounds (equation (4) and Proposition 3.3) for $D(n)$ asymptotic to $n^2/2$, we obtain the following corollary.

**Corollary 3.4.** There holds the asymptotic formula $D(n) \sim \frac{1}{2}n^2$.

**References**