A SHIFTED PARKING FUNCTION SYMMETRIC FUNCTION

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ABSTRACT. We define a "shifted analogue" SH_n of the parking function symmetric function PF_n. The expansion of SH_n in terms of three bases for shifted symmetric functions is explicitly described. We don't know a shifted analogue for parking functions themselves, but some desirable properties of such an analogue are discussed.

1. INTRODUCTION

We assume knowledge of symmetric functions such as may be found in Macdonald [1] or Stanley [5, Ch. 7]. Given a symmetric function $f(x) \in \Lambda_{\mathbb{Q}}$ (i.e., whose coefficients of monomials are rational numbers) in variables $x = (x_1, x_2, \dots)$, define the *superfication* $f(x/y) = \omega_y f(x, y)$, where $y = (y_1, y_2, \dots)$ is a new set of variables, and where ω_y denotes the involution ω acting on the y variables only. We then define the *shiftification* of f to be $f(x/x)$. Equivalently, regarding f as a polynomial in the power sums p_i, we have

$$f(x/x) = f(p_{2i+1} \to 2p_{2i+1}, p_{2i} \to 0). \tag{1.1}$$

Thus $f(x/x) \in \Gamma := \mathbb{Q}[p_1, p_3, p_5, \dots]$.

If λ is a partition of n into distinct parts, denoted $\lambda \models n$, then let P_λ denote Schur's P-function indexed by λ, a shifted analogue (but not the shiftification) of the ordinary Schur function s_λ [1, §III.8]. In particular,

$$2P_n(x) = h_n(x/x) = \sum_{k=0}^{n} e_k(x) h_{n-k}(x).$$

We also have the generating function

$$K(x,t) := 1 + 2\sum_{n\geq 1} P_n(x) t^n = \prod_i \frac{1+x_i t}{1-x_i t}. \tag{1.2}$$

Moreover, the P_{2n+1}'s are algebraically independent and generate Γ as a $\mathbb{Q}$-algebra.

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We now consider parking functions. A good survey of this topic was given by C. H. Yan [7]. A *parking function* of length n is a sequence $(a_1, \ldots, a_n)$ of positive integers whose increasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies $b_i \leq i$. Thus the symmetric group $\mathfrak{S}_n$ acts on the set $\mathcal{P}_n$ of all parking functions of length n (the number of which is $(n+1)^{n-1}$) by permuting coordinates. The Frobenius characteristic symmetric function of this action is denoted PF_n, the *parking function symmetric function*.

The symmetric function PF_n has many remarkable properties. There are simple product formulas for the coefficients when PF_n is expanded in terms any of the six bases $m_\lambda, \mathrm{fo}_\lambda, h_\lambda, e_\lambda, p_\lambda, s_\lambda$, where fo_λ denotes the forgotten symmetric function $\omega(m_\lambda)$. Moreover, we have the generating function

$$\sum_{n\geq 0} \mathrm{PF}_n(x) t^{n+1} = (tH(x,-t))^{\langle -1\rangle}, \tag{1.3}$$

where

$$H(x,t) = \sum_{n\geq 0} h_n(x) t^n = \prod_i (1-x_i t)^{-1}, \tag{1.4}$$

and $^{\langle -1\rangle}$ denotes compositional inverse with respect to t. See for instance [6].

The main object of study in this paper is the shiftification of PF_n, which we denote by SH_n. Thus

$$\mathrm{SH}_n(x) = \mathrm{PF}_n(x/x).$$

In Section 2 we give simple expansions of SH_n as linear combinations of power sums p_λ and the Schur P-functions P_λ. The proofs are analogous to those for the expansion of PF_n as a linear combination of power sums and Schur functions, respectively. In Section 3 give the expansion of SH_n as a polynomial in $P_1, P_3, P_5, \ldots$, the analogue of expanding PF_n as a polynomial in $h_1, h_2, h_3, \ldots$. The proof, however, is not analogous. It is first necessary to express P_{2n} in terms of the P_{2i+1}'s. The most succinct way of expressing this relationship is the formula

$$\frac{-1+\sqrt{1+4A^2}}{2} = P_2 t^2 + P_4 t^4 + P_6 t^6 + \cdots,$$

where $A = P_1 t + P_3 t^3 + P_5 t^5 + \cdots$.

What is currently missing from this development is a satisfactory shifted analogue of parking functions themselves, not just their corresponding symmetric functions. In the last section (Section 4) we discuss some desirable properties of such an interpretation, as well as a "naive shifted parking function" which does not seem as interesting.

2. The bases p_λ and P_λ

In this section we discuss the expansion of SH_n in terms of the power sums p_λ (where λ has odd parts) and the Schur P-functions P_λ (where λ has distinct parts). As mentioned in the previous section, the proofs are straightforward variants of the corresponding results for PF_n.

Theorem 2.1. *For $n \geq 1$ we have*

$$\mathrm{SH}_n = \sum_{\substack{\lambda \vdash n \\ \lambda_i \text{ odd}}} z_\lambda^{-1} 2^{\ell(\lambda)} (n+1)^{\ell(\lambda)-1} p_\lambda,$$

where $\ell(\lambda)$ denotes the length (number of $\lambda_i > 0$) of λ.

Proof. We have [6, Prop. 1.1][5, Exer. 7.48(f)] (note that in this latter exercise, we have $\mathrm{PF}_n = \omega F_{\mathrm{NC}_{n+1}}$)

$$\mathrm{PF}_n = \sum_{\lambda \vdash n} z_\lambda^{-1} (n+1)^{\ell(\lambda)-1} p_\lambda.$$

Now use equation (1.1) and the definition $\mathrm{SH}_n(x) = \mathrm{PF}_n(x/x)$. □

If f is a homogeneous symmetric function of degree n, we define its *dimension* by $\dim f = \langle f, p_1^n \rangle$. If f is the Frobenius characteristic $\mathrm{ch}(\chi)$ of a character χ of $\mathfrak{S}_n$, then $\dim f = \dim \chi = \chi(\mathrm{id})$.

Corollary 2.2. *We have* $\dim \mathrm{SH}_n = 2^n (n+1)^{n-1}$.

Proof. Follows from Theorem 2.1 and the orthogonality relation $\langle p_\mu, p_\lambda \rangle = z_\lambda \delta_{\mu,\lambda}$ [5, Prop. 7.9.3]. □

For the P_λ expansion we need the following fundamental lemma, the shifted analogue of equation (1.3).

Lemma 2.3. *We have*

$$\sum_{n \geq 0} \mathrm{SH}_n(x) t^{n+1} = (tK(x,-t))^{\langle -1 \rangle}, \tag{2.1}$$

where $^{\langle -1 \rangle}$ denote compostional inverse with respect to t, and where $K(x,t)$ is defined in equation (1.2)*.*

Proof. Shiftification is an algebra homomorphism, from which it follows that it commutes with compositional inverse with respect to t. That is, if $F(x,t) = \sum_{n \geq 0} f_n(x) t^{n+1}$, where $f_n(x) \in \Lambda_\mathbb{Q}$ and $f_0(x) = 1$, and if $G(x,t) = F(x,t)^{\langle -1 \rangle}$ (taken with respect to t), then $F(x/x,t)^{\langle -1 \rangle} =$

$G(x/x,t)$. Now with $H(x,t)$ as in equation (1.4) we have

$$\begin{aligned} H(x/x,t) &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} e_k h_{n-k}\right) t^n \\ &= \left(\sum_{n\geq 0} e_n t^n\right)\left(\sum_{n\geq 0} h_n t^n\right) \\ &= \prod_i \frac{1-x_i t}{1+x_i t}, \end{aligned}$$

Now replace t by $-t$ and shiftify equation (1.3). □

The appearance of a compositional inverse in Lemma 2.3 suggests the Lagrange inversion formula could be useful. We use it in the following form [5, Thm. 5.4.2]. Let $F(t) = t + \sum_{n\geq 2} a_n t^n$ be a power series over a field of characteristic 0. Write $[t^m]G(t)$ for the coefficient of t^m in the power series $G(t)$. Then for any $n \in \mathbb{N} := \{0,1,2,\dots\}$,

$$[t^{n+1}]F^{\langle -1\rangle}(t) = \frac{1}{n+1}[t^n]\left(\frac{t}{F(t)}\right)^{n+1}. \tag{2.2}$$

The next result gives the P_λ expansion of SH_n. We use the notation $f(1^{n+1})$ to indicate that in $f(x)$ we set $x_1 = \cdots = x_{n+1} = 1$ and $x_i = 0$ for $i > n+1$.

Theorem 2.4. *We have*

$$\mathrm{SH}_n(x) = \frac{1}{n+1}\sum_{\lambda\models n} 2^{\ell(\lambda)} P_\lambda(1^{n+1}) P_\lambda(x).$$

Proof. By Lemma 2.3 we have

$$\mathrm{SH}_n(x) = [t^{n+1}]\left(t\prod_i \frac{1-x_i t}{1+x_i t}\right)^{\langle -1\rangle}.$$

By equation (2.2) it follows that

$$\mathrm{SH}_n(x) = \frac{1}{n+1}[t^n]\left(\prod_i \frac{1+x_i t}{1-x_i t}\right)^{n+1}. \tag{2.3}$$

The Cauchy identity for P_λ [1, (8.13)] states that

$$\sum_\lambda 2^{\ell(\lambda)} P_\lambda(x) P_\lambda(y) = \prod_i \frac{1+x_i y_j}{1-x_i y_j}, \tag{2.4}$$

where λ ranges over all partitions with distinct parts. Set $y_1 = \cdots = y_{n+1} = 1$ and $y_i = 0$ for $i > n+1$ and compare with equation (2.3) to complete the proof. □

3. The basis $P_{\lambda_1}P_{\lambda_2}\cdots$

In this section we write SH_n as a polynomial in the Schur P-functions $P_1, P_3, P_5, \dots$. Before doing so, let us note that it is very easy to write SH_n as a polynomial in $P_1, P_2, P_3, \dots$ (not unique since the P_i's are not algebraically independent). For any $\lambda \vdash n$ define

$$V_\lambda = P_{\lambda_1}P_{\lambda_2}\cdots.$$

Then we can shiftify the formula [5, Exer. 7.48(f)][6, (1.1)]

$$\mathrm{PF}_n = \sum_{\lambda\vdash n} \frac{n(n-1)\cdots(n-\ell(\lambda)+2)}{m_1!\cdots m_n!} h_\lambda,$$

where λ has m_i parts equal to i. Since $h_i(x/x) = 2P_i(x)$, we obtain

$$\mathrm{SH}_n = \sum_{\lambda\vdash n} \frac{2^{\ell(\lambda)}n(n-1)\cdots(n-\ell(\lambda)+2)}{m_1!\cdots m_n!} V_\lambda. \tag{3.1}$$

We now turn to the more difficult problem of writing SH_n as a polynomial in $P_1, P_3, P_5, \dots$. This representation is unique since the P_{2i+1}'s are algebraically independent. The first step is to write P_{2n} as a polynomial in $P_1, P_3, P_5, \dots$.

Lemma 3.1. *Let $A = P_1t + P_3t^3 + P_5t^5 + \cdots$. Then*

$$P_2t^2 + P_4t^4 + P_6t^6 + \cdots = \frac{-1+\sqrt{1+4A^2}}{2}.$$

Proof. We have $K(x,t) = 1+2A+2B$ and $K(x,-t) = 1-2A+2B$. Note also that

$$K(x,-t) = \frac{1}{K(x,t)} = \frac{1}{1+2A+2B}.$$

Hence

$$1-2A+2B = \frac{1}{1+2A+2B}.$$

Solving for B gives

$$B = \frac{-1\pm\sqrt{1+4A^2}}{2}.$$

The correct sign is plus. □

It is now easy to give an explicit formula for P_{2n} as a polynomial in $P_1, P_3, P_5, \dots$.

Corollary 3.2. *We have*

$$P_{2n} = \sum_{\substack{\lambda\vdash 2n\\ \lambda_i \text{ odd}}} (-1)^{\frac{1}{2}(\ell(\lambda)-2)} C_{\frac{1}{2}(\ell(\lambda-2))} \binom{\ell(\lambda)}{m_1, m_3, \dots} V_\lambda,$$

where C_k denotes a Catalan number, and where λ has m_i parts equal to i.

Proof. The well-known generating function for Catalan numbers (e.g., [5, §6.2]) implies

$$\frac{-1+\sqrt{1+4A^2}}{2} = \sum_{k\geq 0}(-1)^{k-1}C_k A^{2k+2}.$$

Now by the multinomial theorem,

$$\begin{aligned} A^{2k+2} &= (P_1 t + P_3 t^3 + P_5 t^5 + \cdots)^{2k+2} \\ &= \sum_{n\geq 1} \sum_{\substack{\lambda\vdash n \\ \ell(\lambda)=2k+2 \\ \lambda_i \text{ odd}}} \binom{2k+2}{m_1, m_2, \ldots} V_\lambda t^n, \end{aligned}$$

and the proof follows. □

Lemma 3.3. *We have the Taylor series expansion*

$$(2z+\sqrt{1+4z^2})^{n+1} = 1 + \sum_{k\geq 1}(n+1)g_k(n)\frac{z^k}{k!},$$

where

$$g_k(n) = 2^k(n+k-1)(n+k-3)(n+k-5)\cdots(n-k+3). \tag{3.2}$$

Proof. This result is surely well-known or equivalent to a well-known result, so we just give the idea of a proof. It is very similar to [4, Problem A32(a)]. The function $f(z) = z(2z+\sqrt{1+4z^2})$ has compositional inverse

$$f^{\langle -1\rangle}(z) = \frac{z}{\sqrt{1+4z}}.$$

The proof follows easily from equation (2.2). □

We can now prove the main result of this section.

Theorem 3.4. *We have*

$$\mathrm{SH}_n =$$

$$\sum_{\substack{\lambda\vdash n \\ \lambda_i \text{ odd}}} \frac{2^\ell}{\ell!}\binom{\ell}{m_1, m_3, \ldots}(n+\ell-1)(n+\ell-3)\cdots(n-\ell+3)V_\lambda,$$

where $\ell = \ell(\lambda)$ and λ has m_i parts equal to i.

Proof. By Lemma 3.1 we have

$$K(x,t) = 1 + 2A + 2\left(\frac{-1+\sqrt{1+4A^2}}{2}\right) = 2A + \sqrt{1+4A^2}.$$

Hence by equation (2.3) and Lemma 3.3,

$$\begin{aligned} \mathrm{SH}_n(x) &= \frac{1}{n+1}[t^n](2A+\sqrt{1+4A^2})^{n+1} \\ &= [t^n]\sum_{k\geq 1} g_k(n)\frac{A^k}{k!}, \end{aligned}$$

where $g_k(n)$ is given by equation (3.2). Now expand each A^k by the multinomial theorem and extract the coefficient of t^n, □

Let r_n denote a large Schröder number (see e.g. [2, A006318][5, Exer. 6.39]). They play a role for SH_n similar to Catalan numbers for PF_n. Corollary 3.6 below gives some occurrences of r_n, but first we need a lemma.

Lemma 3.5. *Let $f \in \Gamma$, and assume that f is homogeneous of degree n. Consider the expansions*

$$f = \sum_{\substack{\lambda\vdash n \\ \lambda_i \text{ odd}}} a_\lambda V_\lambda = \sum_{\lambda \models n} b_\lambda P_\lambda.$$

Then $\sum_\lambda a_\lambda = b_n$, where b_n is short for $b_{(n)}$.

Proof. Let $f(1) = f(x_1 = 1, x_2 = x_3 = \cdots = 0)$. Thus the map $f(x) \mapsto f(1)$ is an algebra homomorphism $\Gamma \to \mathbb{Q}$. Putting $x = (1,0,0,\dots)$ in equation (1.2) shows that $P_n(1) = 1$ for all $n \geq 1$. Hence $V_\lambda(1) = 1$ for all partitions λ. It follows that $f(1)$ is the sum of the coefficients when $f(x)$ is written as a polynomial in $P_1, P_3, P_5, \dots$.

Let λ be a partition with distinct parts. We claim that

$$P_\lambda(1) = \left\{\begin{array}{ll} 1, & \lambda = (n) \\ 0, & \text{otherwise.} \end{array}\right\}$$

One way to see this is to put $x = (1,0,0,\dots)$ in the shifted Cauchy identity (equation (2.4)) and compare with equation (1.2). Another proof follows from the combinatorial interpretion of P_λ in terms of shifted Young tableaux (e.g., [1, 8.16]). It follows that $f(1)$ is the coefficient of P_n when $f(x)$ is written as a linear combination of P_λ's, where $\lambda \models n$, and the proof follows. □

Corollary 3.6. *The following numbers are equal to r_n.*

- $\frac{2}{n+1}P_n(1^{n+1})$

- *the coefficient of* P_n *when* SH_n *is written as a linear combination of the* P_λ*'s, where* $\lambda \vDash n$
- *the sum of the coefficients of* SH_n *when written as a polynomial in the* P_k*'s (*k *odd)*

Proof. From equation (1.2) we have

$$1+2\sum_{n\geq 1} P_n(1^{n+1})t^n = \left(\frac{1+t}{1-t}\right)^{n+1}. \tag{3.3}$$

In equation (2.2) set $F(t)=t(1-t)/(1+t)$. By equation (3.3) the right-hand side of equation (2.2) becomes $2P_n(1^{n+1})/(n+1)$. Now

$$\begin{aligned}\left(\frac{t(1-t)}{1+t}\right)^{\langle -1\rangle} &= \frac{1-t-\sqrt{1-6t+t^2}}{2}\\ &= \sum_{n\geq 0} r_n t^{n+1},\end{aligned}$$

and the proof of the first item follows.

The second item follows immediately from the first item and Theorem 2.4.

Lemma 3.5 shows that items two and three are equal, so the proof follows. □

The third item in Corollary 3.6 suggests the following question. By Theorem 3.4 the coefficients of SH_n, when written as a polynomial in the P_k's (k odd), are nonnegative. Do these coefficients have a combinatorial interpretation refining some combinatorial interpretation of r_n? Here is a table of some of these polynomials:

$$\begin{aligned}\mathrm{SH}_1 &= 2P_1\\ \mathrm{SH}_2 &= 6P_1^2\\ \mathrm{SH}_3 &= 20P_1^3+2P_3\\ \mathrm{SH}_4 &= 70P_1^4+20P_3P_1\\ \mathrm{SH}_5 &= 252P_1^5+140P_3P_1^2+2P_5\\ \mathrm{SH}_6 &= 924P_1^6+840P_3P_1^3+28P_5P_1+14P_3^2.\end{aligned}$$

Note that the coefficient of P_1^n in SH_n is $\binom{2n}{n}$, a special case of Theorem 3.4.

4. A combinatorial interpretation?

As mentioned in the Introduction, the one feature missing from this development is the concept of a shifted parking function. We can give a "naive" definition based on equation (3.1), but, as we discuss below, what is really wanted is a definition based on Theorem 3.4.

The naive definition is the following. A *naive shifted parking function* (NSPF) π of length n is an ordinary parking function of length n with each term colored red or blue. Since there are $(n+1)^{n-1}$ ordinary parking functions of length n, there are $2^n(n+1)^{n-1}$ NSPF's of length n, which is what we want by Corollary 2.2.

By equation (3.1) we would like to partition the set $\mathcal{N}_n$ of NSPF's of length n into r_n blocks such that (a) each block B is associated with a partition $\lambda = \lambda(B) \vdash n$, (b) the size of B is given by

$$\#B = \dim V_\lambda = 2^{n-\ell(\lambda)}\binom{n}{\lambda_1, \lambda_2, \dots},$$

and (c) the number of blocks B which correspond to λ is equal to the coefficient of V_λ in the right-hand side of equation (3.1). We can do this as follows: a block B consists of all NSPF's σ with specified part multiplicities (i.e., we specify for each i the number a_i of i's in σ) and specified colors for the first (leftmost) occurrence of each part equal to i $(1 \le i \le n)$. The partition λ consists of the a_i's arranged in weakly decreasing order.

Example 4.1. Let $n = 2$. There are 12 NSPF's of length two. The blocks are listed below. We use an overline for the color red and no overline for blue.

$$\{11, 1\bar{1}\},\ \{\bar{1}1, \bar{1}\bar{1}\},\ \{12, 21\},\ \{\bar{1}2, 2\bar{1}\},\ \{1\bar{2}, \bar{2}1\},\ \{\bar{1}\bar{2}, \bar{2}\bar{1}\}$$

All the blocks have size two since $\dim P_1^2 = \dim P_2 = 2$.

We now consider "serious" shifted parking functions. Let $\mathcal{S}_n$ be the (putative) set of all shifted parking functions of "size" n. It should have the following property: there is a partition ψ of $\mathcal{S}_n$ such that each block B corresponds in a natural way to a partition λ of n into odd parts. The number of elements of B is the coefficient of $P_{\lambda_1}P_{\lambda_2}\cdots$ when SH_n is written as a polynomial in the P_i's for i odd (given by Theorem 3.4). The size $\#B$ of B is equal to

$$\dim V_\lambda = 2^{n-\ell(\lambda)}\binom{n}{\lambda_1, \lambda_2, \dots}.$$

This implies that the total number of blocks is r_n and that $\#\mathcal{S}_n = 2^n(n+1)^{n-1}$. The fundamental difference with NSPF's σ is that the partition λ associated with a shifted parking function has only odd parts.

Also desirable would be a naive shifted analogue or shifted analogue of the action of $\mathfrak{S}_n$ on parking functions. Ideally we would have a suitable action of the double cover (for $n \geq 4$) $\widetilde{\mathfrak{S}}_n$ of $\mathfrak{S}_n$ on the complex vector space with basis $\mathcal{N}_n$ or $\mathcal{S}_n$. Probably this is too much to hope for.

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