

Separation probabilities for products of permutations

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Abstract

We study the mixing properties of permutations obtained as product of two uniformly random permutations of fixed types. For instance, we give an exact formula for the probability that elements $1, 2, \dots, k$ are in distinct cycles of the random permutation of $\{1, 2, \dots, n\}$ obtained as product of two uniformly random n -cycles.

1 Introduction

We study certain *separation probabilities* for products of permutations. The archetypal question can be stated as follows: *in the symmetric group \mathfrak{S}_n , what is the probability that the elements $1, 2, \dots, k$ are in distinct cycles of the product of two n -cycles chosen uniformly randomly?* The answer is surprisingly elegant: the probability is $\frac{1}{k!}$ if $n - k$ is odd and $\frac{1}{k!} + \frac{2}{(k-2)!(n-k+1)(n+k)}$ if $n - k$ is even. This result was originally conjectured by Bóna [3] for $k = 2$. Subsequently, Du and Stanley proved it for all k and proposed additional conjectures [11]. The goal of this paper is to prove these conjectures, and establish generalizations of the above result.

We first define a larger class of problems. Given a tuple $A = (A_1, \dots, A_k)$ of k disjoint non-empty subsets of $\{1, \dots, n\}$, we say that a permutation π is *A-separated* if no cycle of π contains elements of more than one of the subsets A_i . Now, given two integer partitions λ, μ of n , one can wonder about the probability $P_{\lambda, \mu}(A)$ that the product of two uniformly random permutations of cycle type λ and μ is *A-separated*. The example presented above corresponds to $A = (\{1\}, \dots, \{k\})$ and $\lambda = \mu = (n)$. Clearly, the separation probabilities $P_{\lambda, \mu}(A)$ only depend on A through the size of the subsets $\#A_1, \dots, \#A_k$, and we shall denote $\sigma_{\lambda, \mu}^\alpha = P_{\lambda, \mu}(A)$, where $\alpha = (\#A_1, \dots, \#A_k)$ is a composition (of size $m \leq n$). Note also that $\sigma_{\lambda, \mu}^\alpha = \sigma_{\lambda, \mu}^{\alpha'}$ whenever the composition α' is a permutation of the composition α . Below, we focus on the case $\mu = (n)$ and we further denote $\sigma_\lambda^\alpha := \sigma_{\lambda, (n)}^\alpha$.

In this paper, we establish a general formula the separation probabilities σ_λ^α . Using this expression we first prove a the following symmetry property: if $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_k)$ are compositions of the same size $m \leq n$ and of the same length k , then

$$\frac{\sigma_\lambda^\alpha}{\prod_{i=1}^k \alpha_i!} = \frac{\sigma_\lambda^\beta}{\prod_{i=1}^k \beta_i!}. \quad (1)$$

Moreover, we obtain explicit formulas for the separation probabilities σ_λ^α for certain partitions λ including $\lambda = (n)$ and $\lambda = 2^N$. For instance, the separation probability $\sigma_{(n)}^\alpha$ for the product of two

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n -cycles is found to be

$$\sigma_{(n)}^\alpha = \frac{(n-m)! \prod_{i=1}^k \alpha_i!}{(n+k)(n-1)!} \left(\frac{(-1)^{n-m} \binom{n-1}{k-2}}{\binom{n+m}{m-k}} + \sum_{r=0}^{m-k} \frac{(-1)^r \binom{m-k}{r} \binom{n+r+1}{m}}{\binom{n+k+r}{r}} \right) \quad (2)$$

This includes the case $\alpha = 1^k$ proved by Du and Stanley [11].

Our approach differs from the one used in [11]. Our starting point is a formula obtained in [8] about *colored* factorizations of the n -cycles. This formula displays a symmetry which turns out to be of crucial importance for our method. Our approach can in fact be made mostly bijective as explained in Section 5. Indeed, the formula obtained in [8] builds on a bijection established in [9]. An alternative bijective proof was given in [2] (using ideas developed in [1, Sec. 3]). We explain how to concatenate this bijective proof with the constructions of the present paper in Section 5.

Outline. In Section 2 we present our strategy for computing the separation probabilities. This involves counting certain colored factorizations of the n -cycles. We then gather our main results in Section 3. In particular we prove the symmetry property (1) and obtain formulas for the separation probabilities σ_λ^α for certain partitions λ including $\lambda = (n)$ or when $\lambda = 2^N$. In Section 4, we give formulas relating the separation probabilities σ_λ^α and $\sigma_{\lambda'}^\alpha$ when λ' is a partition obtained from another partition λ by adding some parts of size 1. In Section 5, we indicate how our proofs could be made bijective. We gather a few additional remarks in Section 6.

Notation. We denote $[n] := \{1, 2, \dots, n\}$. We denote by $\#S$ the cardinality of a set S .

A *composition* of an integer n is a tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of positive integers summing to n . We then say that α has *size* n and *length* $\ell(\alpha) = k$. An *integer partition* is a composition such that the *parts* α_i are in weakly decreasing order. We use the notation $\lambda \models n$ (resp. $\lambda \vdash n$) to indicate that λ is a composition (resp. integer partition) of n . We sometime write integer partitions in multiset notation: writing $\lambda = 1^{n_1}, 2^{n_2}, \dots, j^{n_j}$ means that λ has n_i parts equal to i .

We denote by \mathfrak{S}_n the symmetric group on $[n]$. Given a partition λ of n , we denote by \mathcal{C}_λ the set of permutations in \mathfrak{S}_n with cycle type λ . It is well known that $\#\mathcal{C}_\lambda = n!/z_\lambda$ where $z_\lambda = \prod_i i^{n_i(\lambda)} n_i(\lambda)!$ and $n_i(\lambda)$ is the number of parts equal to i in λ .

We shall consider symmetric functions in an infinite number of variables $\mathbf{x} = \{x_1, x_2, \dots\}$. For any sequence of non-negative integers, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ we denote $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$. For an integer partition $\lambda = (\lambda_1, \dots, \lambda_k)$ we denote by $p_\lambda(\mathbf{x})$ and $m_\lambda(\mathbf{x})$ respectively the *power symmetric function* and *monomial symmetric function* (see e.g. [10]). That is, $p_\lambda(\mathbf{x}) = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(\mathbf{x})$ where $p_k(\mathbf{x}) = \sum_{i \geq 1} x_i^k$, and $m_\lambda(\mathbf{x}) = \sum_{\alpha} \mathbf{x}^\alpha$ where the sum is over all the distinct sequences α whose positive parts are $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ (in any order). Recall that the power symmetric functions form a basis of the symmetric functions. For a symmetric function $f(\mathbf{x})$ we denote by $[p_\lambda(\mathbf{x})]f(\mathbf{x})$ the coefficient of $p_\lambda(\mathbf{x})$ of the decomposition of $f(\mathbf{x})$ in this basis.

2 Strategy

In this section, we first translate the problem of determining the separation probabilities σ_λ^α into the problem of enumerating certain sets $\mathcal{S}_\lambda^\alpha$. Then, we introduce a symmetric function $G_n^\alpha(\mathbf{x}, t)$ whose coefficients in one basis are the cardinalities $\#\mathcal{S}_\lambda^\alpha$, while the coefficient in another basis count certain “colored” separated factorizations of the permutation $(1, \dots, n)$. Lastly, we give an exact counting formulas for these colored separated factorizations. Our main results will follow as

corollaries in Section 3.

For a composition $\alpha = (\alpha_1, \dots, \alpha_k)$ of size $m \leq n$, we denote by \mathcal{A}_n^α the set of tuples $A = (A_1, \dots, A_k)$ of pairwise disjoint subsets of $[n]$ with $\#A_i = \alpha_i$ for all i in $[k]$. Now, recall from the introduction that σ_λ^α is the probability for the product of a uniformly random permutation of cycle type λ with a *uniformly random* n -cycle to be A -separated for a *fixed* tuple A in \mathcal{A}_n^α . Alternatively, it can be defined as the probability for the product of a uniformly random permutation of cycle type λ with a *fixed* n -cycle to be A -separated for a *uniformly random* tuple A in \mathcal{A}_n^α (since the only property that matters is that the elements in A are randomly distributed in the n -cycle).

Definition 1. For an integer partition λ of n , and a composition α of $m \leq n$, we denote by $\mathcal{S}_\lambda^\alpha$ the set of pairs (π, A) , where A is a tuple in \mathcal{A}_n^α and π is a permutation in \mathcal{C}_λ such that the product $\pi \circ (1, 2, \dots, n)$ is A -separated.

From the above discussion we obtain for any composition $\alpha = (\alpha_1, \dots, \alpha_k)$ of size m ,

$$\sigma_\lambda^\alpha = \frac{\#\mathcal{S}_\lambda^\alpha}{\binom{n}{\alpha_1, \alpha_2, \dots, \alpha_k, n-m} \#\mathcal{C}_\lambda}. \quad (3)$$

Enumerating the sets $\mathcal{S}_\lambda^\alpha$ directly seems rather challenging. However, we will show below how to enumerate a related class of “colored” separated permutations denoted by $\mathcal{T}_\gamma^\alpha(r)$. We call *cycle-coloring* of a permutation $\pi \in \mathfrak{S}_n$ in $[q]$, a mapping c from $[n]$ to $[q]$ such that if $i, j \in [n]$ belong to the same cycle of π then $c(i) = c(j)$. We think of $[q]$ as the set of *colors*, and $c^{-1}(i)$ as set of *elements colored* i .

Definition 2. Let $\gamma = (\gamma_1, \dots, \gamma_\ell)$ be a composition of size n , and let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a composition of size $m \leq n$ and length k . For a non-negative integer r we define $\mathcal{T}_\gamma^\alpha(r)$ as the set of quadruples (π, A, c_1, c_2) , where π is a permutation of $[n]$, $A = (A_1, \dots, A_k)$ is in \mathcal{A}_n^α , and

- (i) c_1 is a cycle-coloring of π in $[\ell(\gamma)]$ such that there are γ_i element colored i for all i in $[\ell(\gamma)]$,
- (ii) c_2 is a cycle-coloring of the product $\pi \circ (1, 2, \dots, n)$ in $[k+r]$ such that every color in $[k+r]$ is used and for all i in $[k]$ the elements in the subset A_i are colored i .

Note that condition (ii) in Definition 2 and the definition of cycle-coloring implies in particular that the product $\pi \circ (1, 2, \dots, n)$ is A -separated.

In order to relate the cardinalities of the sets $\mathcal{S}_\lambda^\alpha$ and $\mathcal{T}_\gamma^\alpha(r)$, it is convenient to use symmetric functions (in the variables $\mathbf{x} = \{x_1, x_2, x_3, \dots\}$). Namely, given a composition α of $m \leq n$, we define

$$G_n^\alpha(\mathbf{x}, t) := \sum_{\lambda \vdash n} p_\lambda(\mathbf{x}) \sum_{(\pi, A) \in \mathcal{S}_\lambda^\alpha} t^{\text{excess}(\pi, A)},$$

where the outer sum runs over all the integer partitions of n , and $\text{excess}(\pi, A)$ is the number of cycles of the product $\pi \circ (1, 2, \dots, n)$ containing none of the elements in A . Recall that the power symmetric functions $p_\lambda(\mathbf{x})$ form a basis of the symmetric functions, so that the contribution of a partition λ to $G_n^\alpha(\mathbf{x}, t)$ can be recovered by extracting the coefficient in this basis. In particular,

$$\#\mathcal{S}_\lambda^\alpha = [p_\lambda(\mathbf{x})] G_n^\alpha(\mathbf{x}, 1) \quad (4)$$

where $[\cdot]$ extracts the corresponding coefficient in the basis of power symmetric functions. As we prove now, the sets $\mathcal{T}_\gamma^\alpha(r)$ are related to the coefficients of $G_n^\alpha(\mathbf{x}, t)$ in the basis of monomial symmetric functions.

Proposition 3. If α is a composition of length k , then

$$G_n^\alpha(\mathbf{x}, t+k) = \sum_{r \geq 0} \sum_{\gamma \vdash n} m_\gamma(\mathbf{x}) \binom{t}{r} \#\mathcal{T}_\gamma^\alpha(r), \quad (5)$$

where the inner sum is over all integer partitions of n .

Proof. Let λ be a partition, and π be a permutation of cycle type λ . Then clearly, the symmetric function $p_\lambda(\mathbf{x})$ can be interpreted as the generating function of the cycle-colorings of π , that is, for any sequence $\gamma = (\gamma_1, \dots, \gamma_\ell)$ of non-negative integer, the coefficient $[\mathbf{x}^\gamma]p_\lambda(\mathbf{x})$ is the number of cycle-colorings of π such that γ_i elements are colored i , for all $i > 0$. Moreover, if π is A -separated for a certain $A = (A_1, \dots, A_k) \in \mathcal{A}_n^\alpha$, then $(t+k)^{\text{excess}(S, \pi)}$ represents the number of cycle-colorings of the permutation $\pi \circ (1, 2, \dots, n)$ in $[k+t]$ (not necessarily using every color) such that for all $i \in [k]$ the elements in the subset A_i are colored i . Therefore, for a partition γ and an integer t , the coefficient $[\mathbf{x}^\gamma]G_n^\alpha(\mathbf{x}, t+k)$ counts the quadruple (π, A, c_1, c_2) , where π, A, c_1, c_2 are as in the definition of $\mathcal{T}_\gamma^\alpha(t+k)$ except that c_2 might actually use only a subset of the colors $[k+t]$. Note however that all the colors in $[k]$ will necessarily be used by c_2 , and that we can partition the quadruples according to the subset of colors of size $k+r$ they use. This gives

$$[\mathbf{x}^\gamma]G_n^\alpha(\mathbf{x}, t+k) = \sum_{r \geq 0} \binom{t}{r} \#\mathcal{T}_\gamma^\alpha(r).$$

Now extracting the coefficient of \mathbf{x}^γ in the right-hand side of (5) gives the same result. This proves (5) when t is an integer (since both sides are symmetric functions in \mathbf{x}), hence for all t (since both sides are polynomial in t). \square

In order to obtain an explicit expression for the series $G_n^\alpha(\mathbf{x}, t)$ it remains to enumerate the sets $\mathcal{T}_\gamma^\alpha(r)$ which is done below.

Proposition 4. Let r be a non-negative integer, let α be a composition of size m and length k , and let γ be a partition of size $n \geq m$ and length ℓ . Then, the set $\mathcal{T}_\gamma^\alpha(r)$ specified by Definition 2 has cardinality

$$\#\mathcal{T}_\gamma^\alpha(r) = \frac{n(n-\ell)!(n-k-r)!}{(n-k-\ell-r+1)!} \binom{n+k-1}{n-m-r}, \quad (6)$$

if $n-k-\ell-r+1 \geq 0$, and 0 otherwise.

The rest of this section is devoted to the proof of (6). In order to count the quadruples (π, A, c_1, c_2) satisfying Definition 2, we shall start by choosing π, c_1, c_2 before choosing the tuple A . For compositions $\gamma = (\gamma_1, \dots, \gamma_\ell)$, $\delta = (\delta_1, \dots, \delta_{\ell'})$ of n we denote by $\mathcal{B}_{\gamma, \delta}$ the set of triples (π, c_1, c_2) , where π is a permutation of $[n]$, c_1 is a cycle-coloring of π such that γ_i elements are colored i for all $i \in [\ell]$, and c_2 is a cycle-coloring of the permutation $\pi \circ (1, 2, \dots, n)$ such that δ_i elements are colored i for all $i \in [\ell']$. The problem of counting such sets was first considered by Jackson [5] who actually enumerated the union $\bigcup_{\gamma, \delta \vdash n, \ell(\gamma)=i, \ell(\delta)=j} \mathcal{B}_{\gamma, \delta}$ using representation theory.

It was later proved in [8] that

$$\#\mathcal{B}_{\gamma, \delta} = \frac{n(n-\ell)!(n-\ell')!}{(n-\ell-\ell'+1)!}, \quad (7)$$

if $n-\ell-\ell'+1 \geq 0$, and 0 otherwise. The proof of (7) in [8] uses a refinement of a bijection designed in [9] in order to prove Jackson's formula (a proof using representation theory is in fact possible

but we have not found it in the literature). Another bijective proof of (7) is given in [2] and we shall discuss it further in Section 5.

One of the striking feature of the counting formula (7) is that it depends on the compositions γ, δ only through their lengths ℓ, ℓ' . This “symmetry” feature will prove particularly handy for enumerating $\mathcal{T}_\gamma^\alpha(r)$. Let r, α, γ be as in Proposition 4, and let $\delta = (\delta_1, \dots, \delta_{k+r})$ be a composition of n of length $k+r$. We denote by $\mathcal{T}_{\gamma, \delta}^\alpha$ the set of quadruples (π, A, c_1, c_2) in $\mathcal{T}_\gamma^\alpha(r)$ such that the cycle-coloring c_2 has δ_i elements colored i for all i in $[k+r]$ (equivalently, $(\pi, c_1, c_2) \in \mathcal{B}_{\gamma, \delta}$). We also denote $d_\delta^\alpha = \prod_{i=1}^k \binom{\delta_i}{\alpha_i}$. It is easily seen that for any triple $(\pi, c_1, c_2) \in \mathcal{B}_{\gamma, \delta}$, the number d_δ^α counts the tuples $A \in \mathcal{A}_n^\alpha$ such that $(\pi, A, c_1, c_2) \in \mathcal{T}_{\gamma, \delta}^\alpha$. Therefore,

$$\#\mathcal{T}_\gamma^\alpha(r) = \sum_{\delta \models n, \ell(\delta)=k+r} \#\mathcal{T}_{\gamma, \delta}^\alpha = \sum_{\delta \models n, \ell(\delta)=k+r} d_\delta^\alpha \#\mathcal{B}_{\gamma, \delta},$$

where the sum is over all the compositions of n of length $k+r$. Using (7) then gives

$$\#\mathcal{T}_\gamma^\alpha(r) = \frac{n(n-\ell)!(n-k-r)!}{(n-k-\ell-r+1)!} \sum_{\delta \models n, \ell(\delta)=k+r} d_\delta^\alpha$$

if $n-k-\ell-r+1 \geq 0$, and 0 otherwise. In order to complete the proof of Proposition 4, it only remains to prove the following lemma.

Lemma 5. If α has size m and length k , then

$$\sum_{\delta \models n, \ell(\delta)=k+r} d_\delta^\alpha = \binom{n+k-1}{n-m-r}.$$

Proof. We give a bijective proof illustrated in Figure 1. One can represent a composition $\delta = (\delta_1, \dots, \delta_{k+r})$ as a sequence of rows of boxes (the i th row has δ_i boxes). With this representation, $d_\delta^\alpha := \prod_{i=1}^k \binom{\delta_i}{\alpha_i}$ is the number of ways of choosing α_i boxes in the i th row for $i = 1, \dots, k$. Hence $\sum_{\delta \models n, \ell(\delta)=k+r} d_\delta^\alpha$ counts α -marked compositions of size n and length $k+r$, that is, sequences of $k+r$ non-empty rows of boxes with some marked boxes in the first k rows, with a total of n boxes, and α_i marks in the i th row for $i = 1, \dots, k$; see Figure 1. Now α -marked compositions of size n and length $k+r$ are clearly in bijection (by adding a marked box to each of the rows $1, \dots, k$, and marking the last box of each of the rows $k+1, \dots, k+r$) with α' -marked compositions of size $n+k$ and length $k+r$ such that the last box of each row is marked, where $\alpha' = (\alpha_1+1, \alpha_2+1, \dots, \alpha_k+1, 1, 1, \dots, 1)$ is a composition of length $k+r$. Lastly, these objects are clearly in bijection (by concatenating all the rows) with sequences of $n+k$ boxes with $m+k+r$ marks, one of which is on the last box. There are $\binom{n+k-1}{n-m-r}$ such sequences, which concludes the proof of Lemma 5 and Proposition 4. \square



Figure 1: A $(2, 1, 2)$ -marked composition of size $n = 12$ and length 5 and its bijective transformation into a sequence $n+k = 15$ boxes with $m+k+r = 10$ marks, one of which is on the last box.

3 Main results

In this section, we exploit the consequences of Propositions 3 and 4 in order to derive our main results. All the results will be consequences of the following theorem.

Theorem 6. For any composition α of $m \leq n$ of length k , the generating function $G_n^\alpha(\mathbf{x}, t + k)$ in the variables t and $\mathbf{x} = \{x_1, x_2, \dots\}$ has the following explicit expression in the bases $m_\lambda(\mathbf{x})$ and $\binom{t}{r}$:

$$G_n^\alpha(\mathbf{x}, t + k) = \sum_{r=0}^{n-m} \sum_{\lambda \vdash n, \ell(\lambda) \leq n-k-r+1} m_\lambda(\mathbf{x}) \binom{t}{r} \frac{n(n-\ell(\lambda))!(n-k-r)!}{(n-k-r-\ell(\lambda)+1)!} \binom{n+k-1}{n-m-r}. \quad (8)$$

Moreover, for any partition λ of n , one has $\#\mathcal{S}_\lambda^\alpha = [p_\lambda(\mathbf{x})]G_n^\alpha(\mathbf{x}, 1)$ and $\sigma_\lambda^\alpha = \frac{\#\mathcal{S}_\lambda^\alpha}{\binom{n}{\alpha_1, \alpha_2, \dots, \alpha_k, n-m} \#\mathcal{C}_\lambda}$.

Theorem 6 is the direct consequence of Propositions 3 and 4. One of the striking feature of (8) is that the expression of $G_n^\alpha(\mathbf{x}, t + k)$ depends on α only through its size and length. This ‘‘symmetry property’’ then obviously also holds for $\#\mathcal{S}_\lambda^\alpha = [p_\lambda(\mathbf{x})]G_n^\alpha(\mathbf{x}, 1)$, and translates into the formula (1) for separation of probabilities as stated below.

Corollary 7. Let λ be a partition of n , and let $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_k)$ be composition of the same size m and length k . Then,

$$\#\mathcal{S}_\lambda^\alpha = \#\mathcal{S}_\lambda^\beta. \quad (9)$$

or, equivalently, in terms of separation probabilities, $\frac{\sigma_\lambda^\alpha}{\prod_{i=1}^k \alpha_i!} = \frac{\sigma_\lambda^\beta}{\prod_{i=1}^k \beta_i!}$.

We now derive explicit formulas for the separation probabilities for the product of a uniformly random permutation π , with particular constraints on its cycle type, with a uniformly random n -cycle. We focus on two constraints: when π has p cycles and for n even and π is a fixed-point-free involution.

3.1 Case when π has p cycles

Let $\mathcal{C}(n, p)$ denote the set of permutations of $[n]$ having p cycles. Recall that the numbers $c(n, p) = \#\mathcal{C}(n, p) = [x^p]x(x+1)(x+2)\cdots(x+n-1)$ are called the *signless Stirling numbers of the first kind*. We denote by $\sigma^\alpha(n, p)$ the probability that the product of a uniformly random permutation in $\mathcal{C}(n, p)$ with a uniformly random n -cycle is A -separated for a given set A in \mathcal{A}_n^α . By a reasoning similar to the one used in the proof of Equation (3), one gets

$$\sigma^\alpha(n, p) = \frac{1}{\binom{n}{\alpha_1, \alpha_2, \dots, \alpha_k, n-m} c(n, p)} \sum_{\lambda \vdash n, \ell(\lambda)=p} \#\mathcal{S}_\lambda^\alpha. \quad (10)$$

We now compute the probabilities $\sigma^\alpha(n, p)$ explicitly.

Theorem 8. Let α be a composition of m with k parts. Then,

$$\sigma^\alpha(n, p) = \frac{(n-m)! \prod_{i=1}^k \alpha_i!}{c(n, p)} \sum_{r=0}^{n-m} \frac{c(n-k-r+1, p)}{(n-k-r+1)!} \binom{1-k}{r} \binom{n+k-1}{n-m-r}, \quad (11)$$

where $c(n, p)$ are signless Stirling numbers of the first kind.

For instance, when $m = n$ in the preceding theorem, one gets

$$\sigma^\alpha(n, p) = \frac{\prod_{i=1}^k \alpha_i!}{c(n, p)} \frac{c(n - k + 1, p)}{(n - k + 1)!}.$$

This is the probability that the cycles of the product of a uniformly random permutation in $\mathcal{C}(n, p)$ with a uniformly random n -cycle refine a given set partition of $[n]$ having blocks of sizes $\alpha_1, \alpha_2, \dots, \alpha_k$.

Via (10), proving Theorem 8 amounts to enumerate $\mathcal{S}^\alpha(n, p) := \bigcup_{\lambda \vdash n, \ell(\lambda)=p} \mathcal{S}_\lambda^\alpha$, and by (4) and Theorem 6 one gets

$$\begin{aligned} \#\mathcal{S}^\alpha(n, p) &= \sum_{\lambda \vdash n, \ell(\lambda)=p} [p_\lambda(\mathbf{x})] G_n^\alpha(\mathbf{x}, 1) = \\ &= \sum_{r=0}^{n-m} \sum_{\ell=1}^{n-k-r+1} A(n, p, \ell) \binom{1-k}{r} \frac{n(n-\ell)!(n-k-r)!}{(n-k-r-\ell+1)!} \binom{n+k-1}{n-m-r}, \end{aligned} \quad (12)$$

where $A(n, p, \ell) := \sum_{\mu \vdash n, \ell(\mu)=p} [p_\mu(\mathbf{x})] \sum_{\lambda \vdash n, \ell(\lambda)=\ell} m_\lambda(\mathbf{x})$.

Lemma 9. For any positive integers $p, \ell \leq n$

$$\sum_{\mu \vdash n, \ell(\mu)=p} [p_\mu(\mathbf{x})] \sum_{\lambda \vdash n, \ell(\lambda)=\ell} m_\lambda(\mathbf{x}) = \frac{(-1)^{\ell-p}}{\ell!} \binom{n-1}{\ell-1} c(\ell, p), \quad (13)$$

where $c(a, b)$ are the signless Stirling numbers of the first kind.

Proof. For this proof we use the principal specialization of symmetric functions, that is, their evaluation at $\mathbf{x} = 1^a := \{1, 1, \dots, 1, 0, 0, \dots\}$ (a ones). Since $p_\gamma(1^a) = a^{\ell(\gamma)}$ for any positive integer a , one gets

$$\sum_{\lambda \vdash n, \ell(\lambda)=\ell} m_\lambda(1^a) = \sum_{p=1}^n a^p \sum_{\mu \vdash n, \ell(\mu)=p} [p_\mu(\mathbf{x})] \sum_{\lambda \vdash n, \ell(\lambda)=\ell} m_\lambda(\mathbf{x}). \quad (14)$$

Now, since $m_\lambda(1^a)$ counts the a -tuples of non-negative integers such that the positive ones are the same as the parts of λ (in some order), the left-hand side of (14) counts the a -tuples of non-negative integers with ℓ positive ones summing to n . Thus,

$$\sum_{\lambda \vdash n, \ell(\lambda)=\ell} m_\lambda(1^a) = \binom{n-1}{\ell-1} \binom{a}{\ell}. \quad (15)$$

The right-hand sides of (14) and (15) are polynomials in a , so they are equal as polynomials and one can identify their coefficients. Extracting the coefficient of a^p gives (13) since $[a^p] \binom{a}{\ell} = \frac{(-1)^{\ell-p}}{\ell!} c(\ell, p)$. \square

Using Lemma 9 in (12) gives

$$\#\mathcal{S}^\alpha(n, p) = n! \sum_{r=0}^{n-m} \binom{1-k}{r} \binom{n+k-1}{n-m-r} \sum_{\ell=1}^{n-k-r+1} \frac{(-1)^{\ell-p} c(\ell, p)}{\ell!} \binom{n-k-r}{\ell-1}, \quad (16)$$

which we simplify using the following lemma.

Lemma 10. For any nonnegative integer a ,
$$\sum_{q=0}^a \frac{(-1)^{q+1-p} c(q+1, p)}{(q+1)!} \binom{a}{q} = \frac{c(a+1, p)}{(a+1)!}.$$

Proof. The left-hand side equals $[x^p] \sum_{q=0}^a \binom{x}{q+1} \binom{a}{q}$. Using the Chu-Vandermonde identity this equals $[x^p] \binom{x+a}{a+1}$ which is precisely the right-hand side. \square

Using Lemma 10 in (16) gives

$$\#\mathcal{S}^\alpha(n, p) = n! \sum_{r=0}^{n-m} \frac{c(n-k-r+1, p)}{(n-k-r+1)!} \binom{1-k}{r} \binom{n+k-1}{n-m-r}, \quad (17)$$

which is equivalent to (11) via (3). This completes the proof of Theorem 8. \square

In the case $p = 1$, the expression (11) for the probability $\sigma^\alpha(1) = \sigma_{(n)}^\alpha$ can be written as a sum of $m - k$ terms instead. We state this below.

Corollary 11. Let α be a composition of m with k parts. Then the separation probabilities $\sigma_{(n)}^\alpha$ (separation for the product of two uniformly random n -cycles) is

$$\sigma_{(n)}^\alpha = \frac{(n-m)! \prod_{i=1}^k \alpha_i!}{(n+k)(n-1)!} \left(\frac{(-1)^{n-m} \binom{n-1}{k-2}}{\binom{n+m}{m-k}} + \sum_{r=0}^{m-k} \frac{(-1)^r \binom{m-k}{r} \binom{n+r+1}{m}}{\binom{n+k+r}{r}} \right).$$

The equation in Corollary 11 already stated in the introduction, is particularly simple when $m - k$ is small. For $\alpha = 1^k$ (i.e. $m = k$) one gets the result stated at the beginning of this paper:

$$\sigma_{(n)}^{1^k} = \begin{cases} \frac{1}{k!} & \text{if } n - k \text{ odd,} \\ \frac{1}{k!} + \frac{2}{(k-2)!(n-k+1)(n+k)} & \text{if } n - k \text{ even.} \end{cases} \quad (18)$$

In order to prove Corollary 11 we start with the expression obtained by setting $p = 1$ in (11):

$$\begin{aligned} \sigma_{(n)}^\alpha &= \frac{(n-m)! \prod_{i=1}^k \alpha_i!}{(n-1)!} \sum_{r=0}^{n-m} \binom{1-k}{r} \frac{1}{n-k-r+1} \binom{n+k-1}{n-m-r} \\ &= \frac{(n-m)! \prod_{i=1}^k \alpha_i!}{(n-1)!} [x^{n-m}] (1+x)^{1-k} \sum_{r=0}^{n+k-1} \frac{x^r}{r+m-k+1} \binom{n+k-1}{r}. \end{aligned} \quad (19)$$

We now use the following polynomial identity.

Lemma 12. For non-negative integers a, b , one has the following identity between polynomials in x :

$$\sum_{i=0}^a \frac{x^i}{i+b+1} \binom{a}{i} = \frac{1}{(a+1)} \left(\frac{1}{\binom{a+b+1}{b} (-x)^{b+1}} - \sum_{i=0}^b \frac{\binom{b}{i} (x+1)^{a+i+1}}{\binom{a+i+1}{i} (-x)^{i+1}} \right). \quad (20)$$

Proof. It is easy to see that the left-hand side of (20) is equal to $\frac{1}{x^{b+1}} \int_0^x (1+t)^a t^b dt$. Now this integral can be computed via integration by parts. By a simple induction on b , this gives the right-hand side of (20). \square

Now using (20) in (19), with $a = n + k - 1$ and $b = m - k$, gives

$$\begin{aligned}\sigma_{(n)}^\alpha &= \frac{(n-m)! \prod_{i=1}^k \alpha_i!}{(n+k)(n-1)!} [x^{n-m}] \left(\frac{(1+x)^{1-k}}{\binom{n+m}{m-k} (-x)^{m-k+1}} - \sum_{r=0}^{m-k} \frac{\binom{m-k}{r} (1+x)^{n+r+1}}{\binom{n+k+r}{r} (-x)^{r+1}} \right) \\ &= \frac{(n-m)! \prod_{i=1}^k \alpha_i!}{(n+k)(n-1)!} \left(\frac{(-1)^{n-m} \binom{n-1}{k-2}}{\binom{n+m}{m-k}} + \sum_{r=0}^{m-k} \frac{(-1)^r \binom{m-k}{r} \binom{n+r+1}{m}}{\binom{n+k+r}{r}} \right).\end{aligned}$$

This completes the proof of Corollary 11. \square

3.2 Case when π is a fixed-point-free involution

Given a composition α of $m \leq 2N$ with k parts, we define

$$H_N^\alpha(t) := \sum_{(\pi, A) \in \mathcal{S}_{2N}^\alpha} t^{\text{excess}(\pi, A)},$$

where $\text{excess}(\pi, A)$ is the number of cycles of the product $\pi \circ (1, 2, \dots, 2N)$ containing none of the elements of A and where π is a fixed-point-free involution of $[2N]$. Note that $H_N^\alpha(t) = [p_{2N}(\mathbf{x})] G_{2N}^\alpha(\mathbf{x}, t)$. We now give an explicit expression for this series.

Theorem 13. For any composition α of $m \leq 2N$ of length k , the generating series $H_N^\alpha(t+k)$ is given by

$$H_N^\alpha(t+k) = N \sum_{r=0}^{\min(2N-m, N-k+1)} \binom{t}{r} 2^{k+r-N} \frac{(2N-k-r)!}{(N-k-r+1)!} \binom{2N+k-1}{2N-m-r}. \quad (21)$$

Consequently the separation probabilities for the product of a fixed-point-free involution with a $2N$ -cycle are given by

$$\sigma_{2N}^\alpha = \frac{\prod_{i=1}^k \alpha_i!}{(2N-1)!(2N-1)!!} \sum_{r=0}^{\min(2N-m, N-k+1)} \binom{1-k}{r} 2^{k+r-N-1} \frac{(2N-k-r)!}{(N-k-r+1)!} \binom{2N+k-1}{2N-m-r}. \quad (22)$$

Remark 14. It is possible to prove Theorem 13 directly using ideas similar to the one used to prove Theorem 6 in Section 2. This will be explained in more details in Section 5. In the proof given below, we instead obtain Theorem 13 as a consequence of Theorem 6.

The rest of this section is devoted to the proof of Theorem 13. Since $H_N^\alpha(t) = [p_{2N}(\mathbf{x})] G_{2N}^\alpha(\mathbf{x}, t)$, Theorem 6 gives

$$\begin{aligned}H_N^\alpha(t+k) &= \\ &\sum_{r=0}^{2N-m} \binom{t}{r} \binom{2N+k-1}{2N-m-r} \sum_{s=0}^{N-k-r+1} \frac{2N(N-s)!(2N-k-r)!}{(N-k-r-s+1)!} [p_{2N}(\mathbf{x})] \sum_{\lambda \vdash 2N, \ell(\lambda)=N+s} m_\lambda(\mathbf{x}).\end{aligned} \quad (23)$$

We then use the following result.

Lemma 15. For any nonnegative integer $s \leq N$,

$$[p_{2N}(\mathbf{x})] \sum_{\lambda \vdash 2N, \ell(\lambda)=N+s} m_\lambda(\mathbf{x}) = \frac{(-1)^s}{2^s s! (N-s)!}.$$

Proof. For partitions λ, μ of n , we denote $S_{\lambda, \mu} = [p_\lambda(\mathbf{x})]m_\mu(\mathbf{x})$ and $R_{\lambda, \mu} = [m_\lambda(\mathbf{x})]p_\mu(\mathbf{x})$. The matrices $S = (S_{\lambda, \mu})_{\lambda, \mu \vdash n}$ and $R = (R_{\lambda, \mu})_{\lambda, \mu \vdash n}$ are the transition matrices between the bases $\{p_\lambda\}_{\lambda \vdash n}$ and $\{m_\lambda\}_{\lambda \vdash n}$ of symmetric functions of degree n , hence $S = R^{-1}$. Moreover the matrix R is easily seen to be lower triangular in the *dominance order*, that is, $R_{\lambda, \mu} = 0$ unless $\lambda_1 + \lambda_2 + \dots + \lambda_i \leq \mu_1 + \mu_2 + \dots + \mu_i$ for all $i \geq 1$ ([10, Prop. 7.5.3]). Thus the matrix $S = R^{-1}$ is also lower triangular in the dominance order. Since the only partition of $2N$ of length $N + s$ that is not larger than the partition 2^N in the dominance order is $1^{2s}2^{N-s}$, one gets

$$[p_{2^N}(\mathbf{x})] \sum_{\lambda \vdash 2N, \ell(\lambda)=N+s} m_\lambda(\mathbf{x}) = [p_{2^N}(\mathbf{x})] m_{1^{2s}2^{N-s}}(\mathbf{x}). \quad (24)$$

To compute this coefficient we use the standard scalar product $\langle \cdot, \cdot \rangle$ on symmetric functions (see e.g. [10, Sec. 7]) defined by $\langle p_\lambda, p_\mu \rangle = z_\lambda$ if $\lambda = \mu$ and 0 otherwise, where z_λ was defined at the end of Section 1. From this definition one immediately gets

$$[p_{2^N}] m_{1^{2s}2^{N-s}} = \frac{1}{z_{2^N}} \langle p_{2^N}, m_{1^{2s}2^{N-s}} \rangle = \frac{1}{N!2^N} \langle p_{2^N}, m_{1^{2s}2^{N-s}} \rangle. \quad (25)$$

Let $\{h_\lambda\}$ denote the complete symmetric functions. It is known that $\langle h_\lambda, m_\mu \rangle = 1$ if $\lambda = \mu$ and 0 otherwise, therefore $\langle p_{2^N}, m_{1^{2s}2^{N-s}} \rangle = [h_{1^{2s}2^{N-s}}] p_{2^N}$. Lastly, since $p_{2^N} = (p_2)^N$ and $p_2 = 2h_2 - h_1^2$ one gets

$$\langle p_{2^N}, m_{1^{2s}2^{N-s}} \rangle = [h_{1^{2s}2^{N-s}}] p_{2^N} = [h_1^{2s} h_2^{N-s}] (2h_2 - h_1^2)^N = 2^{N-s} (-1)^s \binom{N}{s}. \quad (26)$$

Putting together (24), (25) and (26) completes the proof. \square

By Lemma 15, Equation (23) becomes

$$\begin{aligned} H_N^\alpha(t+k) &= \sum_{r=0}^{2N-m} \binom{t}{r} \binom{2N+k-1}{2N-m-r} \sum_{s=0}^{N-k-r+1} \frac{2N(N-s)!(2N-k-r)!}{(N-k-r-s+1)!} \frac{(-1)^s}{2^s s! (N-s)!} \\ &= 2N \sum_{r=0}^{2N-m} \binom{t}{r} \frac{(2N-k-r)!}{(N-k-r+1)!} \binom{2N+k-1}{2N-m-r} \sum_{s=0}^{N-k-r+1} \binom{N-k-r+1}{s} \frac{(-1)^s}{2^s} \\ &= 2N \sum_{r=0}^{\min(2N-m, N-k+1)} \binom{t}{r} \frac{(2N-k-r)!}{(N-k-r+1)!} \binom{2N+k-1}{2N-m-r} (1/2)^{N-k-r+1}, \end{aligned}$$

where the last equality uses the binomial theorem. This completes the proof of Equation (21). Equation (22) then immediately follows from the case $t = 1 - k$ of (21) via (3). This completes the proof of Theorem 13. \square

4 Adding fixed points to the permutation π

In this section we obtain a relation between the separation probabilities σ_λ^α and $\sigma_{\lambda'}^\alpha$, when the partition λ' is obtained from λ by adding some parts of size 1. Our main result is given below.

Theorem 16. Let λ be a partition of n with parts of size at least 2 and let λ' be the partition obtained from λ by adding r parts of size 1. Then for any composition $\alpha = (\alpha_1, \dots, \alpha_k)$ of $m \leq n+r$

of length k ,

$$\#\mathcal{S}_{\lambda'}^{\alpha} = \sum_{p=0}^{m-k} \left(\frac{n+p}{n} \binom{n+m+r-p}{n+m} + \frac{m-p}{n} \binom{n+m+r-p-1}{n+m} \right) \binom{m-k}{p} \#\mathcal{S}_{\lambda}^{(m-k-p+1, 1^{k-1})}. \quad (27)$$

Equivalently, in terms of separation probabilities,

$$\sigma_{\lambda'}^{\alpha} = \frac{n!}{\binom{n+r}{\alpha_1, \dots, \alpha_k, n+r-m}} \binom{n+r}{r} \sum_{p=0}^{m-k} \frac{\left(\frac{n+p}{n} \binom{n+m+r-p}{n+m} + \frac{m-p}{n} \binom{n+m+r-p-1}{n+m} \right) \binom{m-k}{p}}{(n+m+p)!(m-k-p+1)!} \sigma_{\lambda}^{(m-k-p+1, 1^{k-1})}. \quad (28)$$

For instance, when $\alpha = 1^k$ Theorem 16 gives

$$\sigma_{\lambda'}^{1^k} = \frac{\binom{n+r-k}{r}}{\binom{n+r}{r}^2} \left(\binom{n+r+k}{n+k} + \frac{k}{n} \binom{n+r+k-1}{n+k} \right) \sigma_{\lambda}^{1^k}.$$

The rest of the section is devoted to proving Theorem 16. Observe first that (28) is a simple restatement of (27) via (3) (using the fact that $\#\mathcal{C}_{\lambda'} = \binom{n+r}{n} \#\mathcal{C}_{\lambda}$). Thus it only remains to prove (27), which amounts to enumerating $\mathcal{S}_{\lambda'}^{\alpha}$. For this purpose, we will first define a mapping Ψ from $\mathcal{S}_{\lambda'}^{\alpha}$ to $\hat{\mathcal{S}}_{\lambda}^{\alpha}$, where $\hat{\mathcal{S}}_{\lambda}^{\alpha}$ is a set closely related to $\mathcal{S}_{\lambda}^{\alpha}$. We shall then count the number of preimages of each element in $\hat{\mathcal{S}}_{\lambda}^{\alpha}$ under the mapping Ψ . Roughly speaking, if (π', A) is in $\mathcal{S}_{\lambda'}^{\alpha}$ and the tuple $A = (A_1, \dots, A_k)$ is thought as “marking” some elements in the cycles of the permutation $\omega = \pi' \circ (1, 2, \dots, n+r)$, then the mapping Ψ simply consists in removing all the fixed points of π' from the cycle structure of ω and transferring their “marks” to the element preceding them in the cycle structure of ω .

We introduce some notation. A *multisubset* of $[n]$ is a function M which associates to each integer $i \in [n]$ its *multiplicity* $M(i)$ which is a non-negative integer. The integer i is said to be *in* the multisubset M if $M(i) > 0$. The *size* of M is the sum of multiplicities $\sum_{i=1}^n M(i)$. For a composition $\alpha = (\alpha_1, \dots, \alpha_k)$, we denote by $\hat{\mathcal{A}}_n^{\alpha}$ the set of tuples (M_1, \dots, M_k) of *disjoint* multisubsets of $[n]$ (i.e., no element $i \in [n]$ is in more than one multisubset) such that the multisubset M_j has size α_j for all $j \in [k]$. For $M = (M_1, \dots, M_k)$ in $\hat{\mathcal{A}}_n^{\alpha}$ we say that a permutation π is M -separated if no cycle of π contains elements of more than one of the multisubsets M_j . Lastly, for a partition λ of n we denote by $\hat{\mathcal{S}}_{\lambda}^{\alpha}$ the set of pairs (π, M) where π is a permutation in \mathcal{C}_{λ} , and M is a tuple in $\hat{\mathcal{A}}_n^{\alpha}$ such that the product $\pi \circ (1, 2, \dots, n)$ is M -separated.

We now set $\lambda, \lambda', \alpha, k, m, n, r$ to be as in Theorem 16, and define a mapping Ψ from $\mathcal{S}_{\lambda'}^{\alpha}$ to $\hat{\mathcal{S}}_{\lambda}^{\alpha}$. Let π' be a permutation of $[n+r]$ of cycle type λ' , and let $e_1 < e_2 < \dots < e_n \in [n+r]$ be the elements not fixed by π' . We denote $\varphi(\pi')$ the permutation π defined by setting $\pi(i) = \pi'(j)$ if $\pi'(e_i) = \pi'(e_j)$.

Remark 17. If $e_1 < e_2 < \dots < e_n \in [n+r]$ are the elements not fixed by π' and $\pi = \varphi(\pi')$, then the cycle structure of the permutation $\pi' \circ (1, 2, \dots, n+r)$ is obtained from the cycle structure of $\pi \circ (1, 2, \dots, n)$ by replacing each element $i \in [n-1]$ by the sequence of elements $F_i = e_i, e_i + 1, e_i + 2, \dots, e_{i+1} - 1$, and replacing the element n by the sequence of elements $F_n = e_n, e_n + 1, e_n + 2, \dots, n+r, 1, 2, \dots, e_1 - 1$. In particular, the permutations $\pi \circ (1, 2, \dots, n)$ and $\pi' \circ (1, 2, \dots, n+r)$ have the same number of cycles.

Now given a pair (π', A) in $\mathcal{S}_{\lambda'}^{\alpha}$, where $A = (A_1, \dots, A_k)$, we consider the pair $\Psi(\pi', A) = (\pi, M)$, where $\pi = \varphi(\pi')$ and $M = (M_1, \dots, M_k)$ is a tuple of multisubsets of $[n]$ defined as follows: for all

$j \in [k]$ and all $i \in [n]$ the multiplicity $M_j(i)$ is the number of elements in the sequence F_i belonging to the subset A_j (where the sequence F_i is defined as in Remark 17). It is easy to see that Ψ is a mapping from $\mathcal{S}_\lambda^\alpha$ to $\hat{\mathcal{S}}_\lambda^\alpha$.

We are now going to evaluate $\#\mathcal{S}_\lambda^\alpha$ by counting the number of preimages of each element in $\hat{\mathcal{S}}_\lambda^\alpha$ under the mapping Ψ . As we will see now, the number of preimages of a pair (π, M) in $\hat{\mathcal{S}}_\lambda^\alpha$ only depends on M .

Lemma 18. Let $(\pi, M) \in \hat{\mathcal{S}}_\lambda^\alpha$, where $M = (M_1, \dots, M_k)$. Let s be the number of elements appearing in the multisets M_1, \dots, M_k , and let $x = \sum_{j=1}^k M_j(n)$ be the multiplicity of the integer n . Then the number of preimages of the pair (π, M) under the mapping Ψ is

$$\#\Psi^{-1}(\pi, M) = \begin{cases} \binom{n+r+s}{n+m} & \text{if } x = 0, \\ x \binom{n+r+s}{n+m} + \binom{n+r+s-1}{n+m} & \text{otherwise.} \end{cases} \quad (29)$$

Proof. We adopt the notation of Remark 17. In order to construct a preimage (π', A) of (π, M) , where $A = (A_1, \dots, A_k)$, one has to

- (i) choose for all $i \in [n]$ the length $f_i > 0$ of the sequence F_i (with $\sum_{i=1}^n f_i = n+r$),
- (ii) choose the position $b \in [f_n]$ corresponding to the integer $n+r$ in the sequence F_n ,
- (iii) if $M_j(i) > 0$ for some $i \in [n]$ and $j \in [k]$, then choose which $M_j(i)$ elements in the sequence F_i are in the subset A_j .

Indeed, by Remark 17 the choices (i), (ii) determines the permutation $\pi' \in \mathcal{C}_\lambda$ (the non-initial elements of the sequences F_i will be the fixed-points of π'), while the choice (iii) determines the tuple of subsets $A = (A_1, \dots, A_k)$.

We will now count the number ways of making the choices (i), (ii), (iii) by encoding such choices as rows of (marked and unmarked) boxes as illustrated in Figure 2. We treat separately the cases $x = 0$ and $x \neq 0$. Suppose first $x = 0$. To each $i \in [n]$ we associate a row of boxes R_i encoding the choices (i), (ii), (iii) as follows:

- (1) if $i \neq n$ and $M_j(i) = 0$ for all $j \in [k]$, then the row R_i is made of f_i boxes, the first of which is marked,
- (2) if $i \neq n$ and $M_j(i) > 0$ for some $j \in [k]$, then the row R_i is made of $f_i + 1$ boxes, with the first box being marked and $M_j(i)$ other boxes being marked (the marks represent the choice (iii)),
- (3) the row R_n is made of $f_n + 1$ boxes, with the first box being marked and an additional box being marked and called *special marked box* (the special marked box represents the choice (ii)).

There is no loss of information in concatenating the rows R_1, R_2, \dots, R_n given that M is known (indeed the row R_i starts at the $(i + N_i)$ th marked box, where $N_i = \sum_{h < i} \sum_{j=1}^k M_j(h)$). This concatenation results in a row of $n+r+s+1$ boxes with $n+m+1$ marks with the first box being marked (and the last mark being “special”); see Figure 2. Moreover there are $\binom{n+r+s}{n+m}$ such rows of boxes and any of them can be obtained for some choices of (i), (ii), (iii). This proves the case $x = 0$ of Lemma 18.

We now suppose $x > 0$. We reason similarly as above but there are now two possibilities for the row R_n , depending on whether or not the integer $n+r$ belongs to one of the subsets A_1, \dots, A_k . In order to encode a preimage such that $n+r$ belong to one of the subsets A_1, \dots, A_k the condition (3) above must be changed to

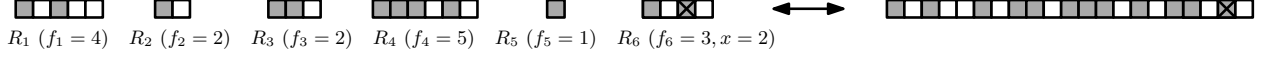


Figure 2: Example of choices (1),(2),(3) encoded by a sequence of boxes, some of which being marked (indicated in grey) and one of which being special (indicated with a cross). Here $n = 6$, $k = 2$, $r = 11$, $x = 0$ and the multisubsets M_1, M_2 are defined by $M_1(1) = 1$, $M_2(3) = 1$, $M_1(4) = 3$, and $M_j(i) = 0$ for the other values of i, j .

(3') the row R_n is made of $f_n + 1$ boxes, with the first box being marked and x other boxes being marked, one of which called *special marked box*.

In this case, concatenating the rows R_1, R_2, \dots, R_n gives a row of $n + r + s$ boxes with $n + m$ marks, with the first box being marked and one of the x last marked boxes being special. There are $x \binom{n+r+s-1}{n+m-1}$ such rows and each of them comes from a unique choice of (i), (ii) and (iii).

Lastly, in order to encode a preimage such that $n + r$ does not belong to one of the subsets A_1, \dots, A_k the condition (3) above must be changed to

(3'') the row R_n is made of $f_n + 1$ boxes, with the first box being marked and $x + 1$ other boxes being marked, one of which called *special marked box*.

In this case, concatenating the rows R_1, R_2, \dots, R_n gives a row of $n + r + s$ boxes with $n + m + 1$ marks, with the first box being marked and one of the $x + 1$ last marked box being special. There are $(x + 1) \binom{n+r+s-1}{n+m}$ such rows and each of them comes from a unique choice of (i), (ii) and (iii).

Thus, in the case $x > 0$ one has

$$\#\Psi^{-1}(\pi, M) = x \binom{n+r+s-1}{n+m-1} + (x+1) \binom{n+r+s-1}{n+m} = x \binom{n+r+s}{n+m} + \binom{n+r+s-1}{n+m}.$$

This completes the proof. \square

We now complete the proof of Theorem 16. For any composition $\gamma = (\gamma_1, \dots, \gamma_k)$, we denote by $\hat{\mathcal{S}}_\lambda^{\alpha, \gamma}$ the set of pairs (π, M) in $\hat{\mathcal{S}}_\lambda^\alpha$, where $M = (M_1, \dots, M_k)$, such that for all $j \in [k]$ the multisubset M_j contains exactly γ_j distinct elements. Summing Equation (29) gives

$$\sum_{(\pi, M) \in \hat{\mathcal{S}}_\lambda^{\alpha, \gamma}} \#\Psi^{-1}(\pi, M) = \left((\mathbb{E}(X) + \mathbb{P}(X = 0)) \binom{n+r+|\gamma|}{n+m} + \mathbb{P}(X > 0) \binom{n+r+|\gamma|-1}{n+m} \right) \#\hat{\mathcal{S}}_\lambda^{\alpha, \gamma}, \quad (30)$$

where X is the random variable defined as $X = \sum_{j=1}^k M_j(n)$ for a pair (π, M) chosen uniformly randomly in $\hat{\mathcal{S}}_\lambda^{\alpha, \gamma}$, $\mathbb{E}(X)$ is the expectation of this random variable, and $\mathbb{P}(X > 0) = 1 - \mathbb{P}(X = 0)$ is the probability that X is positive.

Lemma 19. With the above notation, $\mathbb{E}(X) = \frac{m}{n}$, and $\mathbb{P}(X > 0) = \frac{|\gamma|}{n}$.

Proof. The proof is simply based on a cyclic symmetry. For $i \in [n]$ we consider the random variable $X_i = \sum_{j=1}^k M_j(i)$ for a pair (π, M) chosen uniformly randomly in $\hat{\mathcal{S}}_\lambda^{\alpha, \gamma}$. It is easy to see that all the variables $X_1, \dots, X_n = X$ are identically distributed since the set $\hat{\mathcal{S}}_\lambda^{\alpha, \gamma}$ is unchanged by cyclically shifting the value of the integers $1, 2, \dots, n$ in pairs $(\pi, M) \in \hat{\mathcal{S}}_\lambda^{\alpha, \gamma}$. Therefore,

$$n \mathbb{E}(X) = \sum_{i=1}^n \mathbb{E}(X_i) = \mathbb{E} \left(\sum_{i=1}^n X_i \right) = \mathbb{E}(m) = m,$$

and

$$n \mathbb{P}(X > 0) = \sum_{i=1}^n \mathbb{P}(X_i > 0) = \mathbb{E} \left(\sum_{i=1}^n 1_{X_i > 0} \right) = \mathbb{E}(|\gamma|) = |\gamma|.$$

□

We now enumerate the set $\hat{\mathcal{S}}_\lambda^{\alpha, \gamma}$. Observe that any pair (π, M) in $\hat{\mathcal{S}}_\lambda^{\alpha, \gamma}$ can be obtained (in a unique way) from a pair (π, A) in $\mathcal{S}_\lambda^\gamma$ by transforming $A = (A_1, \dots, A_k)$ into $M = (M_1, \dots, M_k)$ as follows: for each $j \in [k]$ one has to assign a positive multiplicity $M_j(i)$ for all $i \in A_j$ so as to get a multisubset M_j of size α_j . There are $\binom{\alpha_j - 1}{\gamma_j - 1}$ ways of performing the later task, hence

$$\#\hat{\mathcal{S}}_\lambda^{\alpha, \gamma} = \prod_{i=1}^k \binom{\alpha_i - 1}{\gamma_i - 1} \#\mathcal{S}_\lambda^\gamma.$$

Using this result and Lemma 19 in (30) gives

$$\sum_{(\pi, M) \in \hat{\mathcal{S}}_\lambda^{\alpha, \gamma}} \#\Psi^{-1}(\pi, M) = \left(\frac{m+n-|\gamma|}{n} \binom{n+r+|\gamma|}{n+m} + \frac{|\gamma|}{n} \binom{n+r+|\gamma|-1}{n+m} \right) \prod_{i=1}^k \binom{\alpha_i - 1}{\gamma_i - 1} \#\mathcal{S}_\lambda^\gamma.$$

Observe that the above expression is 0 unless γ is less than α componentwise. Finally, one gets

$$\#\mathcal{S}_\lambda^\alpha = \sum_{\gamma \leq \alpha, \ell(\gamma)=k} \left(\frac{m+n-|\gamma|}{n} \binom{n+r+|\gamma|}{n+m} + \frac{|\gamma|}{n} \binom{n+r+|\gamma|-1}{n+m} \right) \prod_{i=1}^k \binom{\alpha_i - 1}{\gamma_i - 1} \#\mathcal{S}_\lambda^\gamma, \quad (31)$$

where the sum is over compositions γ with k parts that are less than α componentwise. Lastly, by Corollary 7, the cardinality $\#\mathcal{S}_\lambda^\alpha$ only depends on the composition α through the length and size of α so one can use equation (31) with $\alpha = (m - k + 1, 1^{k-1})$ in order to obtain (27). This completes the proof of Theorem 16. □

5 Bijective proofs and interpretation in terms of maps

In this section we explain how certain results of this paper can be interpreted in terms of *maps*, and can be proved bijectively. In particular, we shall interpret the sets $\mathcal{T}_{\gamma, \delta}^\alpha$ of “separated colored factorizations” (defined in Section 2) in terms of maps. We can then extend a bijection from [1] in order to prove bijectively the symmetry property stated in Corollary 7.

5.1 Interpretations of (separated) colored factorizations in terms of maps

We first recall some definitions about maps. Our *graphs* are undirected, and they can have multiple edges and loops. Our *surfaces* are two-dimensional, compact, boundaryless, orientable, and considered up to homeomorphism; such a surface is characterized by its genus. A connected graph is *cellularly embedded* in a surface if its edges are not crossing and its *faces* (connected components of the complementary of the graph) are simply connected. A *map* is a cellular embedding of a connected graph in an orientable surface considered up to homeomorphism. A map is represented in Figure 3. By cutting an edge in its midpoint one gets two *half-edges*. A map is *rooted* if one of its half-edges is distinguished as the *root*. In the following we shall consider rooted bipartite maps, and consider the unique proper coloring of the vertices in black and white such that the root half-edge is incident to a black vertex.

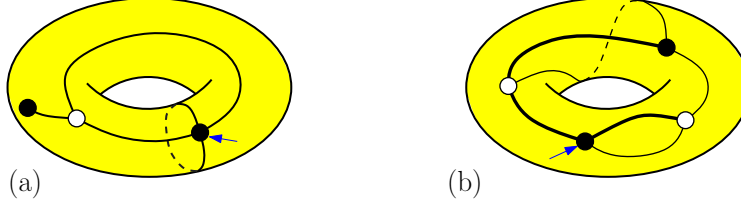


Figure 3: (a) A rooted bipartite one-face map. (b) A rooted bipartite tree-rooted map (the spanning tree is indicated by thick lines). The rooting is indicated by an arrow preceding the root half-edge in clockwise order around the incident vertex.

By a classical encoding (see e.g. [6]), for any partitions λ, μ of n , the solutions $(\pi_1, \pi_2) \in \mathcal{C}_\lambda \times \mathcal{C}_\mu$ of the equation $\pi_1 \circ \pi_2 = (1, 2, \dots, n)$ are in bijection with the rooted one-face bipartite maps such that black and white vertices have degrees given by the permutations λ and μ respectively (that is, the number of vertices of degree i is equal to the number of parts equal to i). Let $\gamma = (\gamma_1, \dots, \gamma_\ell)$, $\delta = (\delta_1, \dots, \delta_{\ell'})$ be compositions of n and let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a composition of $m \leq n$. A rooted bipartite map is (γ, δ) -colored if its black vertices are colored in $[\ell]$ (that is, every vertex is assigned a “color” in $[\ell]$) in such a way that γ_i edges are incident to black vertices of color i , and its white vertices are colored in $[\ell']$ in such a way that δ_i edges are incident to white vertices of color i . Through the above mentioned encoding, the set $\mathcal{B}_{\gamma, \delta}$ of colored factorizations of the n -cycles defined in Section 2 corresponds to the set of (γ, δ) -colored rooted bipartite one-face maps. Similarly, the sets $\mathcal{T}_{\gamma, \delta}^\alpha$ of “separated colored factorizations” corresponds to the set of (γ, δ) -colored rooted bipartite one-face maps with some marked edges, such that for all $i \in [k]$ exactly α_i marked edges are incident to white vertices colored i .

The results in this paper can then be interpreted in terms of maps. For instance, one can interpret (8) in the case $m = k = 0$ as follows:

$$\sum_{\lambda \vdash n} \sum_{M \in \mathcal{B}_\lambda} p_\lambda(\mathbf{x}) t^{\#\text{white vertices}} = G_n^\emptyset(\mathbf{x}, t) = \sum_{r=1}^n \sum_{\lambda \vdash n, \ell(\lambda) \leq n-r+1} m_\lambda(\mathbf{x}) \binom{t}{r} \frac{n(n-\ell(\lambda))!(n-r)!}{(n-r-\ell(\lambda)+1)!} \binom{n-1}{n-r},$$

where \mathcal{B}_λ is the set of rooted bipartite one-face maps such that black vertices have degrees given by the permutation λ . The results in Subsection 3.2 can also be interpreted in terms of *general* (i.e., non-necessarily bipartite) maps. Indeed, the set $\mathcal{M}_N = \mathcal{B}_{2N}$ can be interpreted as the set of general rooted one-face maps with N edges (because a bipartite map in which every black vertex has degree two can be interpreted as a general map upon contracting the black vertices). Therefore one can interpret (21) in the case $m = k = 0$ as follows:

$$\sum_{M \in \mathcal{M}_N} t^{\#\text{vertices}} = H_N^\emptyset(t) = N \sum_{r=1}^{N+1} \binom{t}{r} 2^{r-N} \frac{(2N-r)!}{(N-r+1)!} \binom{2N-1}{2N-r}. \quad (32)$$

This equation is exactly the celebrated Harer-Zagier formula [4].

5.2 Bijection for separated colored factorizations, and symmetry

In this section, we explain how some of our proofs could be made bijective. In particular we will use bijective results obtained in [1] in order to prove the symmetry result stated in Corollary 7.

We first recall the bijection obtained in [1] about the sets $\mathcal{B}_{\gamma,\delta}$. We call *tree-rooted map* is a rooted map with a marked spanning tree; see Figure 3(b). We say that a bipartite tree-rooted map is (ℓ, ℓ') -labelled if it has ℓ black vertices labelled with distinct labels in $[\ell]$, and ℓ' white vertices labelled with distinct labels in $[\ell']$. It was shown in [1] that for any compositions $\gamma = (\gamma_1, \dots, \gamma_\ell)$, $\delta = (\delta_1, \dots, \delta_{\ell'})$ of n , the set $\mathcal{B}_{\gamma,\delta}$ is in bijection with the set of (ℓ, ℓ') -labelled bipartite tree-rooted maps such that the black (resp. white) vertex labelled i has degree γ_i (resp. δ_i).

From this bijection, it is not too hard to derive the enumerative formula (7) (see Remark 21). We now adapt the bijection established in [1] to the sets $\mathcal{T}_{\gamma,\delta}^\alpha$ of “separated colored factorizations”. For a composition $\alpha = (\alpha_1, \dots, \alpha_k)$, a (ℓ, ℓ') -labelled bipartite maps is said to be α -marked if α_i edges incident to the white vertex labelled i are marked for all i in $[k]$; see Figure 4.

Theorem 20. The bijection in [1] extends into a bijection between the set $\mathcal{T}_{\gamma,\delta}^\alpha$ and the set of α -marked (ℓ, ℓ') -labelled bipartite tree-rooted maps with n edges such that the black (resp. white) vertex labelled i has degree γ_i (resp. δ_i).

By Theorem 20, the set $\mathcal{T}_\gamma^\alpha(r)$ specified by Definition 2 is in bijection with the set $\tilde{\mathcal{T}}_\gamma^\alpha(r)$ of α -marked $(\ell, k+r)$ -labelled bipartite tree-rooted maps with n edges such that the black vertex labelled i has degree γ_i . We will now explain bijectively why

$$\#\mathcal{T}_\gamma^\alpha(r) = \#\mathcal{T}_\gamma^\beta(r), \quad (33)$$

whenever the compositions α and β have the same length and size. Observe that, in turn, Equation (33) readily implies Corollary 7.

In order to prove (33) bijectively, it is convenient to interpret maps as graphs endowed with a rotation-system. A *rotation-system* of a graph G is an assignment for each vertex v of G of a cyclic ordering of the half-edges incident to v . Any map M defines a rotation-system $\rho(M)$ of the underlying graph: the cyclic orderings are given by the clockwise order of the half-edges around the vertices. This correspondence is in fact bijective (see e.g. [7]): for any connected graph G the mapping ρ gives a bijection between maps having underlying graph G and the rotation-systems of G . Using the “rotation-system” interpretation, any map can be represented in the plane (with edges allowed to cross each other) by choosing the clockwise order of the half-edges around each vertex to represent the rotation-system; this is the convention used in Figures 4 and 5.

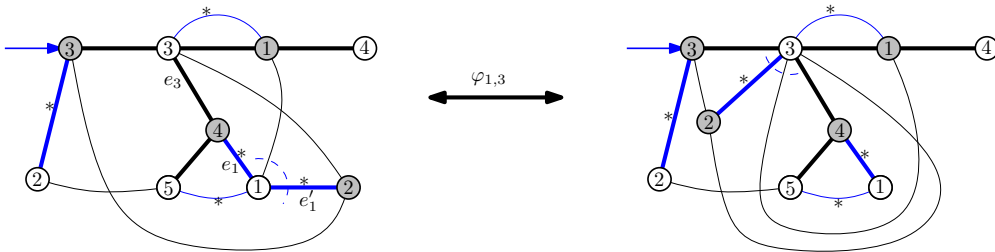


Figure 4: A $(3, 1, 1)$ -marked $(4, 5)$ -labelled bipartite tree-rooted maps (left) and the $(2, 1, 2)$ -marked $(4, 5)$ -labelled bipartite tree-rooted maps obtained by applying the mapping $\varphi_{1,3}$. In this figure, the maps are represented using the “rotation system interpretation”, so that the edge-crossings are irrelevant. The spanning trees are drawn in thick lines, the marked edges are indicated by stars, and the root half-edge is indicated by an arrow preceding it clockwise around the incident vertex.

We now prove (33). It is sufficient to consider the case $\alpha = (\alpha_1, \dots, \alpha_k)$, $\beta = (\beta_1, \dots, \beta_k)$ with $\beta_i = \alpha_i - 1$, $\beta_j = \alpha_j + 1$ and $\alpha_s = \beta_s$ for $s \neq i, j$. Let M be an α -marked (ℓ, ℓ') -labelled bipartite

tree-rooted map. We consider the path joining the white vertices i and j in the spanning tree of M . Let e_i and e_j be the edges of this path incident to the white vertices i and j respectively; see Figure 4. We consider the first marked edge e'_i following e_i in clockwise order around the vertex i (note that $e_i \neq e'_i$ since $\alpha_i = \beta_i + 1 > 1$). We then define $\varphi_{i,j}(M)$ as the map obtained by ungluing from the vertex i the half-edge of e'_i as well as all the half-edges appearing strictly between e_i and e'_i , and gluing them (in the same clockwise order) in the corner following e_j clockwise around the vertex j . Figure 4 illustrates the mapping $\varphi_{1,3}$. It is easy to see that $\varphi_{i,j}(M)$ is a tree-rooted map, and that $\varphi_{i,j}$ and $\varphi_{j,i}$ are reverse mappings. Therefore $\varphi_{i,j}(M)$ is a bijection between $\tilde{\mathcal{T}}_\gamma^\alpha(r)$ and $\tilde{\mathcal{T}}_\gamma^\beta(r)$. This proves (33).

Remark 21. By an argument similar to the one used above to prove (33), one can prove that if $\gamma, \gamma', \delta, \delta'$ are compositions of n such that $\ell(\gamma) = \ell(\gamma')$ and $\ell(\delta) = \ell(\delta')$ then $\mathcal{B}_{\gamma,\delta} = \mathcal{B}_{\gamma',\delta'}$ (this is actually done in a more general setting in [2]). From this property one can compute the cardinality of $\mathcal{B}_{\gamma,\delta}$ by choosing the most convenient compositions γ, δ of length ℓ and ℓ' . We take $\gamma = (n - \ell + 1, 1, 1, \dots, 1)$ and $\delta = (n - \ell + 1, 1, 1, \dots, 1)$, so that $\#\mathcal{B}_{\gamma,\delta}$ is the number of (ℓ, ℓ') -labelled bipartite tree-rooted maps with the black and white vertices labelled 1 of degree $n - \ell + 1$ and $n - \ell' + 1$ respectively, and all the other vertices of degree 1. In order to construct such an object (see Figure 5), one must choose the unrooted plane tree (1 choice), the labelling of the vertices $((\ell - 1)!(\ell' - 1)!$ choices), the $n - \ell - \ell' + 1$ edges not in the tree $\binom{n - \ell}{n - \ell - \ell' + 1} \binom{n - \ell'}{n - \ell - \ell' + 1} (n - \ell' - \ell' + 1)!$ choices), and lastly the root (n choices). This gives (7).

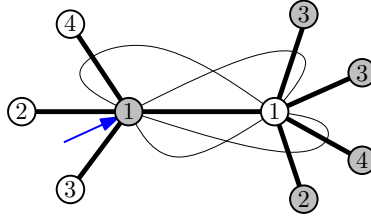


Figure 5: A tree-rooted map in $\mathcal{B}_{\gamma,\delta}$, where $\gamma = (8, 1, 1, 1, 1)$, $\delta = (9, 1, 1, 1)$. Here the map is represented using the “rotation system interpretation”, so that the edge-crossings are irrelevant.

5.3 A direct proof of Theorem 13

In Section 3 we obtained Theorem 13 as a consequence of Theorem (6). Here we explain how to obtain it directly.

First of all, by a reasoning identical to the one used to derive (5) one gets

$$H_N^\alpha(t+k) = \sum_{r=0}^{2N-m} \binom{t}{r} \#\mathcal{U}^\alpha(r), \quad (34)$$

where $\mathcal{U}^\alpha(r)$ is the set of triples (π, A, c_2) where π is a fixed-point free involution of $[2N]$, A is in \mathcal{A}_n^α and c_2 is a cycle-coloring of the product $\pi \circ (1, 2, \dots, 2N)$ in $[k+r]$ such that every color in $[k+r]$ is used and for all i in $[k]$ the elements in the subset A_i are colored i .

In order to enumerate $\mathcal{U}^\alpha(r)$ one considers for each composition $\gamma = (\gamma_1, \dots, \gamma_\ell)$ the set \mathcal{M}_γ of pairs (π, c_2) , where π is a fixed-point-free involution of $[2N]$ and c_2 is a cycle-coloring of the

permutation $\pi \circ (1, 2, \dots, 2N)$ such that γ_i elements are colored i for all $i \in [\ell]$. One then uses the following analogue of (7):

$$\#\mathcal{M}_\gamma = \frac{N(2N - \ell)!}{(N - \ell + 1)!} 2^{\ell - N}. \quad (35)$$

Using this result in conjunction with Lemma 5, one then obtains the following analogue of (6)

$$\#\mathcal{U}^\alpha(r) = \frac{N(2N - k - r)!}{(N - k - r + 1)!} \binom{2N + k - 1}{2N - m - r}.$$

Plugging this result in (34) completes the proof of Theorem 13.

Similarly as (7), Equation (35) can be obtained bijectively. Indeed by a classical encoding, the set \mathcal{M}_γ is in bijection with the set of rooted one-face maps with vertices colored in $[\ell]$ in such a way that for all $i \in [\ell]$, there are exactly γ_i half-edges incident to vertices of color i . Using this interpretation, it was proved in [1] that the set \mathcal{M}_γ is in bijection with the set of tree-rooted maps with ℓ vertices labelled with distinct labels in $[\ell]$. The later set is easy to enumerate (using symmetry as in Remark 21) and one gets (35).

6 Concluding remarks: strong separation and connection coefficients

Given a tuple $A = (A_1, \dots, A_k)$ of disjoint subsets of $[n]$, a permutation π is said to be *strongly A-separated* if each of the subset A_i , $i \in [k]$ is included in a distinct cycle of π . Given a partition λ of n and a composition α of $m \leq n$, we denote by π_λ^α the probability that the product $\omega \circ \rho$ is *strongly A-separated*, where π (resp. ρ) is a uniformly random permutation of type λ (resp. (n)) and A is a fixed tuple in \mathcal{A}_n^α . In particular, for a composition α of size $m = n$, one gets

$$\pi_\lambda^\alpha = \frac{K_{\lambda, (n)}^\alpha}{(n - 1)! \#\mathcal{C}_\lambda},$$

where $K_{\lambda, (n)}^\alpha$ is the *connection coefficient of the symmetric group* counting the number of solutions $(\omega, \rho) \in \mathcal{C}_\lambda \times \mathcal{C}_{(n)}$, of the equation $\omega \circ \rho = \pi$ where π is a fixed permutation of cycle type α .

We now argue that the separation probabilities $\{\sigma_\lambda^\alpha\}_{\alpha \models m}$ computed in this paper are enough to determine the probabilities $\{\pi_\lambda^\alpha\}_{\alpha \models m}$. Indeed, it is easy to prove that

$$\sigma_\lambda^\alpha = \sum_{\beta \preceq \alpha} R_{\alpha, \beta} \pi_\lambda^\beta, \quad (36)$$

where the sum is over the compositions $\beta = (\beta_1, \dots, \beta_\ell)$ of size $m = |\alpha|$ such that there exists $0 = j_0 < j_1 < j_2 < \dots < j_k = \ell$ such that $(\beta_{j_{i-1}+1}, \beta_{j_{i-1}+1}, \dots, \beta_{j_i})$ is a composition of α_i for all $i \in [k]$, and $R_{\alpha, \beta} = \prod_{i=1}^k R_i$ where R_i is the number of ways of partitioning a set of size α_i into blocks of respective sizes $\beta_{j_{i-1}+1}, \beta_{j_{i-1}+1}, \dots, \beta_{j_i}$. Moreover, the matrix $(R_{\alpha, \beta})_{\alpha, \beta \models m}$ is invertible (since upper triangular for the lexicographic ordering of compositions). Thus, from the separation probabilities $\{\sigma_\lambda^\alpha\}_{\alpha \models m}$ one can deduce the strong separation probabilities $\{\pi_\lambda^\alpha\}_{\alpha \models m}$ and in particular, for $m = n$, the connection coefficients $K_{\lambda, (n)}^\alpha$ of the symmetric group.

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