Rodica Simion and Shuffle Posets

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version of 9 July 2001

I wrote one joint paper [4] with Rodica Simion. Rodica had read my paper [5] and knew from there that certain classes of graded partially ordered sets had interesting connections with the theory of symmetric functions and the representation theory of the symmetric group. The basic property of such posets is *local rank-symmetry*, i.e, every interval is graded and for all i has as many elements of rank i as of corank i. For any finite graded poset P with $\hat{0}$ and $\hat{1}$ Ehrenborg [2][7, Exer. 7.48] defined a certain generating function $F_P(\boldsymbol{x})$ for the flag f-vector of P (which encodes the number of chains of P whose elements have specified ranks). When P is locally rank-symmetric the generating function F_P is actually a symmetric function of the variables $\boldsymbol{x} = (x_1, x_2, \ldots)$, so the machinery of symmetric functions can be brought to bear. The symmetric function F_P is associated in a natural way with a virtual representation of the symmetric group $\mathfrak{S}_n$ (where $n = \mathrm{rank}(P)$) whose dimension is the number of maximal chains of P, and one can ask when this virtual representation is actually a "nice" permutation representation of $\mathfrak{S}_n$ acting on the maximal chains. Such an action can be found if P has a CL-labeling λ (in the sense of Björner-Wachs) with a special property called an *S-labeling*. One glaring defect of this general theory was the dearth of interesting examples.

In the spring of 1997 both Rodica and I were participants in the Combinatorial Program at MSRI in Berkeley. One day she walked into my office and asked "Did you realize that posets of shuffles are locally rank-symmetric?" Posets of shuffles (or shuffle posets) are an intriguing generalization of finite boolean algebras discovered by Curtis Greene [3]. I had heard Curtis Greene lecture on them but had given them no thought since. In particular it never

[1]Partially supported by NSF grant #DMS-9743966.

occurred to me that shuffle posets might be locally rank-symmetric. Rodica's question immediately seized my attention and relegated all other mathematical projects to the back burner. Here was a totally unexpected example to which we could try to apply the elaborate theory of locally rank-symmetric posets. One interesting aspect of shuffle posets is that they are not self-dual. All other known "natural" examples of locally rank-symmetric posets are in fact locally self-dual, i.e., every interval is isomorphic to its dual.

Collaborating on mathematics with Rodica was an exhilirating experience. She had an enthusiasm for mathematical research which infected anyone she worked with. Each day she would come to my office with her latest ideas, which we would attempt to iron out together into a coherent and elegant theory. Occasionally we would carry on our discussions at a cafe rather than at MSRI. Rodica made three fundamental contributions to the paper, in addition to numerous small improvements, examples, etc. (1) The idea of looking at shuffle posets in the first place was Rodica's. (2) She discovered the correct labeling rule for the CL property. Unlike previous examples of locally rank-symmetric posets [5][6], a stronger and more manageable labeling condition known as EL was not sufficient. When Rodica discovered the CL labeling she was looking for EL labelings and didn't realize at first that her "faulty" EL-labeling was precisely what was needed for a CL labeling! (3) A special property of shuffle posets not directly relevant to local rank-symmetry is that every interval of a shuffle poset is a product of shuffle posets. This allows one to define a monoid of "multiplicative functions" on shuffle posets and suggests the problems of "determining" this monoid, i.e., describing it in a concrete way without reference to shuffle posets. Exactly such a program appears in [1, §5.2] for the lattice of partitions of a set, where the monoid consists of power series with constant term 0 under the operation of composition. The first step needed to determine the shuffle poset monoid is a combinatorial description of multiplication in the monoid. This is equivalent to determining the number of elements t in a shuffle poset such that the intervals $[\hat{0}, t]$ and $[t, \hat{1}]$ are isomorphic to a prescribed product of shuffle posets. This intricate combinatorial problem was solved by Rodica. I supplied the argument for stating this result as an operation on generating functions [5, Thm. 5.2].

It is clear from the above discussion that Rodica made several fundamental contributions to our joint paper. We had some additional plans for collaboration which her premature death tragically cut short. It is a great loss for all of combinatorics, as well as for me personally, that Rodica is no longer here to discover beautiful new mathematics and to inspire other researchers to do the same.

References

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