A formula for the specialization of skew Schur functions

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Abstract

We give a formula for $s_{\lambda/\mu}(1, q, q^2, \dots)/s_\lambda(1, q, q^2, \dots)$, which generalizes a result of Okounkov and Olshanski about $f^{\lambda/\mu}/f^\lambda$.

Keywords: skew Schur function, q-analogue, *jeu de taquin*

1 Introduction

For the notation and terminology below on symmetric functions, see Stanley [6] or Macdonald [4]. Let μ be a partition of some nonnegative integer. A *reverse tableau* of shape μ is an array of positive integers of shape μ which is weakly decreasing in rows and strictly decreasing in columns. Let $\mathrm{RT}(\mu, n)$ be the set of all reverse tableaux of shape μ whose entries belong to $\{1, 2, \dots, n\}$.

Recall that f^λ and $f^{\lambda/\mu}$ denote the number of SYT (standard Young tableaux) of shape λ and λ/μ respectively. Okounkov and Olshanski [5] give the following surprising formula.

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Lemma 1. *Let* $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \vdash m$ *and* $\mu = \{\mu_1, \mu_2, \ldots, \mu_n\} \vdash k$ *with* $\mu \subseteq \lambda$. *Then*

$$\frac{(m)_k\, f^{\lambda/\mu}}{f^\lambda} = \sum_{T \in \mathrm{RT}(\mu,n)} \prod_{u \in \mu} (\lambda_{T(u)} - c(u)) \tag{1}$$

where $c(u)$ *and* $T(u)$ *are the content and entry of the square* u *respectively, and* $(m)_k = m(m-1)\cdots(m-k+1)$.

In this paper, we generalize the above result to a q-analogue. Our main result is the following.

Theorem 2. *We have*

$$\frac{s_{\lambda/\mu}(1, q, q^2, \ldots)}{s_\lambda(1, q, q^2, \ldots)} = \sum_{T \in \mathrm{RT}(\mu,n)} \prod_{u \in \mu} \left(q^{1-T(u)} (1 - q^{\lambda_{T(u)} - c(u)}) \right). \tag{2}$$

2 Proof of the main result

Denote by the symbol $(x \downharpoonleft k)$ the k-th falling q-factorial power of a variable x,

$$(x \downharpoonleft k) = \begin{cases} (1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1}), & \text{if } k = 1, 2, \ldots, \\ 1, & \text{if } k = 0. \end{cases}$$

In particular, for nonnegative integers n and k, we use $[k]!$ to denote $(k \downharpoonleft k)$, and $\left[{n \atop k}\right] = \frac{[n]!}{[k]![n-k]!}$ for $n \geq k$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and λ/μ be a skew shape. We define

$$t_{\lambda/\mu,n}(q) = s_{\lambda/\mu}(1, q, q^2, \ldots) \prod_{u \in \lambda/\mu} (1 - q^{n+c(u)}). \tag{3}$$

Lemma 3 ([6])**.** *Let* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *and* $\nu_i = \lambda_i + n - i$. *Then:*

(1) $t_{\lambda/\mu,n}(q) = \det \left[\left[{n + \lambda_i - i \atop \lambda_i - \mu_j - i + j} \right] \right]_{i,j=1}^n$

(2) $\displaystyle\prod_{u \in \lambda} (1 - q^{n+c(u)}) = \prod_{i=1}^n \frac{[\nu_i]!}{[n-i]!}$.

The *shifted q-Schur function* is defined as follows:

$$ {}_q s^*_\mu(x_1,\dots,x_n) = \frac{\det[(x_i+n-i\downharpoonright\mu_j+n-j)]}{\det[(x_i+n-i\downharpoonright n-j)]}, \tag{4} $$

where $1\le i,j\le n$.

Lemma 4. *Let $\lambda=(\lambda_1,\dots,\lambda_n)$. Then we have*

$$ \frac{s_{\lambda/\mu}(1,q,q^2,\dots)}{s_\lambda(1,q,q^2,\dots)} = {}_q s^*_\mu(\lambda_1,\dots,\lambda_n). $$

Proof. By Lemma 3 we have

$$ \begin{aligned} \frac{s_{\lambda/\mu}(1,q,q^2,\dots)}{s_\lambda(1,q,q^2,\dots)} &= \frac{t_{\lambda/\mu,n}(q)}{t_{\lambda,n}(q)}\prod_{u\in\mu}(1-q^{n+c(u)}) \\ &= \frac{\det\left[\begin{bmatrix} n+\lambda_i-i \\ \lambda_i-\mu_j-i+j\end{bmatrix}\right]}{\det\left[\begin{bmatrix} n+\lambda_i-i \\ \lambda_i-i+j\end{bmatrix}\right]}\prod_{j=1}^{n}\frac{[\mu_j+n-j]!}{[n-j]!} \\ &= {}_q s^*_\mu(\lambda_1,\dots,\lambda_n) \end{aligned} $$

□

We first consider the denominator of (4).

Lemma 5.

$$ \det\left[(x_i+n-i\downharpoonright n-j)\right] = \prod_{i=2}^{n}\prod_{j=0}^{i-2} q^{x_i+n-i-j}\cdot\prod_{i<j}(1-q^{x_i-x_j-i+j}) \tag{5} $$

Proof. For $j=1,\cdots,n-1$, we subtract from the j-th column of the determinant in the left hand the $(j+1)$-th column, multiplied by $(1-q^{x_1+j})$. The determinant becomes

$$ \prod_{i=2}^{n}(1-q^{x_1-x_i-1+i})\prod_{i=2}^{n}q^{x_i+n+2-2i}\cdot\det[(x_{i+1}+n-i-1\downharpoonright n-j-1)]_{i,j=1}^{n-1}, $$

and then the result follows by induction. □

The following lemma is almost the same as Lemma 2.1 in [5], just lifted to the q-analogue.

Lemma 6. *We have*

$$\frac{(x+1\downharpoonleft k+1)-(y\downharpoonleft k+1)}{q^y-q^{x+1}}=\sum_{l=0}^{k} q^{-l}(y\downharpoonleft l)(x-l\downharpoonleft k-l).$$

Proof. We have

$$\begin{aligned}(q^y-q^{x+1})&\sum_{l=0}^{k} q^{-l}(y\downharpoonleft l)(x-l\downharpoonleft k-l)\\ &=\sum_{l=0}^{k}(q^{y-l}-q^{x+1-l})(y\downharpoonleft l)(x-l\downharpoonleft k-l)\\ &=\sum_{l=0}^{k}(y\downharpoonleft l)(x-l\downharpoonleft k-l)(1-q^{x+1-l})-\sum_{l=0}^{k}(y\downharpoonleft l)(x-l\downharpoonleft k-l)(1-q^{y-l})\\ &=\sum_{l=0}^{k}(y\downharpoonleft l)(x-l+1\downharpoonleft k-l+1)-\sum_{l=0}^{k}(y\downharpoonleft l+1)(x-l\downharpoonleft k-l).\end{aligned}$$

Since all summands cancel each other except $(x+1\downharpoonleft k+1)-(y\downharpoonleft k+1)$, the result follows. □

For two partitions μ and ν, we write $\mu\succ\nu$ if $\mu_i\geq\nu_i\geq\mu_{i+1}$, $i=1,2,\ldots$. Thus given a reverse tableau $T\in\mathrm{RT}(\mu,n)$, we can regard it as a sequence

$$\mu=\mu^{(1)}\succ\mu^{(2)}\succ\cdots\succ\mu^{(n+1)}=\emptyset,$$

where $\mu^{(i)}$ is the shape of the reverse tableau consisting of entries of T no less than i.

Now we can give the proof of Theorem 1.

Proof. By Lemma 4, it is equivalent to prove that

$${}_qs^*_\mu(\lambda_1,\ldots,\lambda_n)=\sum_{\nu\prec\mu} q^{-|\nu|}(\lambda_1\downharpoonleft\mu/\nu)\ {}_qs^*_\nu(\lambda_2,\ldots,\lambda_n),\ n\geq l(\mu). \tag{6}$$

Recall that the numerator of ${}_qs^*_\mu(\lambda_1,\ldots,\lambda_n)$ is

$$\det[(\lambda_i+n-i\downharpoonleft\mu_j+n-j)]. \tag{7}$$

For all $j = 1, 2, \ldots, n-1$, we subtract from the j-th column of (7) the $(j+1)$-th column, multiplied by $(\lambda_1 - \mu_{j+1} + j \downharpoonleft \mu_j - \mu_{j+1} + 1)$. Then for all $j < n$, the (i, j)-th entry of (7) becomes

$$\begin{aligned}(\lambda_i + n - i \downharpoonleft \mu_{j+1} + n - j - 1)((\lambda_i - \mu_{j+1} + j + 1 - i \downharpoonleft \mu_j - \mu_{j+1} + 1)\\ - (\lambda_1 - \mu_{j+1} + j \downharpoonleft \mu_j - \mu_{j+1} + 1)).\end{aligned} \tag{8}$$

We can now apply Lemma 6, where we set

$$\begin{array}{ll} x = \lambda_1 - \mu_{j+1} + j - 1, & k = \mu_j - \mu_{j+1}, \\ y = \lambda_i - \mu_{j+1} + j + 1 - i, & l = \nu_j - \mu_{j+1}. \end{array}.$$

Then (8) equals

$$\begin{aligned}-(1 - q^{\lambda_1 - \lambda_i + i - 1}) q^{\lambda_i - \mu_{j+1} + j + 1 - i} \sum_{\nu_j = \mu_{j+1}}^{\mu_j} \{ q^{\mu_{j+1} - \nu_j} (\lambda_1 - \nu_j + j - 1 \downharpoonleft \mu_j - \nu_j) \\ \cdot (\lambda_i + n - i \downharpoonleft \nu_j + n - j - 1)\},\end{aligned}$$

and thus the determinant (7) equals

$$\begin{aligned}\prod_{i=2}^{n} ((1 - q^{\lambda_1 - \lambda_i + i - 1}) q^{\lambda_i - i}) \prod_{j=1}^{n-1} q^{j+1-\nu_j} \sum_{\nu \prec \mu} ((\lambda_1 \downharpoonleft \mu/\nu) \\ \det[(\lambda_{i+1} + n - i - 1 \downharpoonleft \nu_j + n - j - 1)]_{i,j=1}^{n-1}).\end{aligned}$$

On the other hand, by Lemma 5 we have

$$\frac{\det[(\lambda_i + n - i \downharpoonleft n - j)]_{i,j=1}^{n}}{\det[(\lambda_{i+1} + n - i - 1 \downharpoonleft n - j - 1)]_{i,j=1}^{n-1}} = \prod_{i=2}^{n} \left((1 - q^{\lambda_1 - \lambda_i + i - 1}) q^{\lambda_i - i}\right) \prod_{j=1}^{n-1} q^{j+1}.$$

Combining the above two identities together, we then obtain (6). □

Corollary 7. *The rational function*

$$\frac{s_{\lambda/\mu}(1, q, q^2, \ldots)}{(1-q)^{|\mu|} s_\lambda(1, q, q^2, \ldots)}$$

is a Laurent polynomial in q with nonnegative integer coefficients.

For the special case when $\mu = 1$, we give a simple formula for $s_{\lambda/1}/(1-q)s_\lambda$ in Corollary 8 below. Before giving a combinatorial proof of this result, we first introduce some notation.

The acronym SSYT stands for a semistandard Young tableau where 0 is allowed as a part. *Jeu de taquin* (jdt) is a kind of transformation between skew tableaux obtained by moving entries around, such that the property of being a tableau is preserved. For example, given a tableau T of shape λ, we first delete the entry $T(i, j)$ for some box (i, j). If $T(i, j-1) > T(i-1, j)$, we then move $T(i, j-1)$ to box (i, j); otherwise, we move $T(i-1, j)$ to (i, j). Continuing this moving process, we eventually obtain a tableau of shape $\lambda/1$. On the other hand, given a tableau of shape $\lambda/1$, we can regard $(0, 0)$ as an empty box. By moving entries in a reverse way, we then get a tableau of shape λ with a empty box after every step. For more information about jdt, readers can refer to [6, Ch. 7, App. I].

The following result was first obtained by Kerov [2, Thm. 1 and (2.2)] (after sending $q \mapsto q^{-1}$) and by Garsia and Haiman [1, (I.15), Thm. 2.3] (setting $t = q^{-1}$) by algebraic reasoning. For further information see [3, p. 9].

Corollary 8. *We have*

$$\frac{s_{\lambda/1}(1, q, q^2, \dots)}{(1-q)s_\lambda(1, q, q^2, \dots)} = \sum_{u\in\lambda} q^{c(u)}. \tag{9}$$

Proof. We define two sets in the following way:

$$T_{\lambda/1} = \{(T, k) \mid T \text{ is a SSYT of shape } \lambda/1, \text{ and } k \in \mathbb{N}\},$$
$$T_\lambda = \{(T, u) \mid T \text{ is a SSYT of shape } \lambda, \text{ and } u \in \lambda\}.$$

It suffices to prove that there is bijection $\varphi : T_\lambda \to T_{\lambda/1}$, say $\varphi(T, u) = (T_\varphi, k)$, such that $|T| + c(u) = |T_\varphi| + k$.

We define φ in the following way. Given $(T, u) \in T_\lambda$, let $k = T(u) + c(u)$. To obtain T_φ, we first delete the entry $T(u)$ from T, and then carry out the jdt operation. Since T is a SSYT we have $k \geq 0$, and thus the definition is reasonable.

On the other hand, given $(T_\varphi, k) \in T_{\lambda/1}$, we carry out the jdt operation to T_φ step-by-step in the reverse way. After t steps, if we get a SSYT by filling the empty box u_t with $k - c(u_t)$, then we call u_t a *nice* box. It's obvious that a nice box exists. Let $u = (i, j)$ be the first nice box and T be the corresponding SSYT. We just need to prove that u is also the only nice box. Otherwise, we assume that there exists another nice box $u' = (i', j')$, and T' is the corresponding SSYT. Then we have $i' \geq i$ and $j' \geq j$. Let $a_{i,j}$ and $a_{i',j'}$ be the entries of (i, j) and (i', j') in T' respectively. Since T' is a SSYT, we must have $a_{i',j'} \geq a_{i,j} + i' - i$. Since $u = (i, j)$ is a nice box and T is a SSYT, we have $a_{i,j} > k + i - j$ when $j' = j$, and $a_{i,j} \geq k + i - j$ when $j' > j$. In either case we get a contradiction, since $a_{i',j'} = k + i' - j'$ by the definition of T'.

□

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