

# SOME ASPECTS OF $(r, k)$ -PARKING FUNCTIONS

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ABSTRACT. An  $(r, k)$ -parking function of length  $n$  may be defined as a sequence  $(a_1, \dots, a_n)$  of positive integers whose increasing rearrangement  $b_1 \leq \dots \leq b_n$  satisfies  $b_i \leq k + (i - 1)r$ . The case  $r = k = 1$  corresponds to ordinary parking functions. We develop numerous properties of  $(r, k)$ -parking functions. In particular, if  $F_n^{(r, k)}$  denotes the Frobenius characteristic of the action of the symmetric group  $\mathfrak{S}_n$  on the set of all  $(r, k)$ -parking functions of length  $n$ , then we find a combinatorial interpretation of the coefficients of the power series  $\left(\sum_{n \geq 0} F_n^{(r, 1)} t^n\right)^k$  for any  $k \in \mathbb{Z}$ . For instance, when  $k > 0$  this power series is just  $\sum_{n \geq 0} F_n^{(r, k)} t^n$ . We also give a  $q$ -analogue of this result. For fixed  $r$ , we can use the symmetric functions  $F_n^{(r, 1)}$  to define a multiplicative basis for the ring  $\Lambda$  of symmetric functions. We investigate some of the properties of this basis.

## 1. INTRODUCTION

Parking functions were first defined by Konheim and Weiss as follows. We have  $n$  cars  $C_1, \dots, C_n$  and  $n$  parking spaces  $1, 2, \dots, n$ . Each car  $C_i$  has a preferred space  $a_i$ . The cars go one at a time in order to their preferred space. If it is empty they park there; otherwise they park at the next available space (in increasing order). If all the cars are able to park, then the sequence  $\alpha = (a_1, \dots, a_n)$  is called a *parking function* of length  $\ell(\alpha) = n$ . For instance,  $(3, 1, 4, 3)$  is not a parking function since the last car will go to space 3, but spaces 3 and 4 are already occupied. It is easy to see that  $(a_1, \dots, a_n) \in [n]^n$  (where  $[n] = \{1, 2, \dots, n\}$ ) is a parking function if and only if its increasing rearrangement  $b_1 \leq b_2 \leq \dots \leq b_n$  satisfies  $b_i \leq i$ .

Let  $\text{PF}_n$  denote the set of all parking functions of length  $n$ . A fundamental result of Konheim and Weiss [2] (earlier proved in an equivalent form by Steck [7]—see Yan [8, §1.4] for a discussion) states that  $\#\text{PF}_n = (n + 1)^{n-1}$ . An elegant proof of this result was given by Pollak (reported in [3]), which we now sketch since it will be generalized later. Suppose that we have the same  $n$  cars, but now there are  $n + 1$  spaces  $1, 2, \dots, n + 1$ . The spaces are arranged on a circle. The cars follow the same algorithm as before, but once a car reaches space  $n + 1$  and is unable to park, it can continue around the circle to spaces  $1, 2, \dots$  until it can finally park. Of course all the cars can park this way, so at the end there will be one empty space. Note that their preferences  $(a_1, \dots, a_n) \in [n + 1]^n$  will be a parking function if and only if the empty space is  $n + 1$ . If the empty space is  $e$  and the preferences are changed to  $(a_1 + i, \dots, a_n + i)$  for some  $i$ , where  $a_j + i$  is taken modulo  $n + 1$  so that  $a_j + i \in [n + 1]$ , then the empty space becomes  $e + i$ . Hence given  $(a_1, \dots, a_n) \in [n + 1]^n$ , exactly one of the vectors  $(a_1 + i, \dots, a_n + i)$  will be a parking function. It follows that  $\#\text{PF}_n = \frac{1}{n+1}(n + 1)^n = (n + 1)^{n-1}$ .

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We will use notation and terminology on symmetric functions from [6, Chap. 7]. The symmetric group  $\mathfrak{S}_n$  acts on  $\text{PF}_n$  by permuting coordinates. Let  $F_n := \text{ch PF}_n$  denote the Frobenius characteristic of this action of  $\mathfrak{S}_n$ , as defined in [6, §7.18]. Hence  $F_n$  is a homogeneous symmetric function of degree  $n$ , called the *parking function symmetric function*. If  $\alpha = (a_1, \dots, a_n)$  is a sequence of positive integers with  $m_i$   $i$ 's (so  $\sum m_i = n$ ), then the Frobenius characteristic of the action of  $\mathfrak{S}_n$  on the set of permutations of the terms of  $\alpha$  is the complete symmetric function  $h_{m_1} h_{m_2} \cdots$  (with  $h_0 = 1$ ). Hence to compute  $F_n$ , take all vectors  $(b_1, \dots, b_n) \in \text{PF}_n$  with  $b_1 \leq b_2 \leq \cdots \leq b_n$  (the number of such vectors is the Catalan number  $C_n$ ) and add the corresponding  $h_\lambda$  for each. For instance, when  $n = 3$  the weakly increasing parking functions are 111, 112, 113, 122, 123, so  $F_3 = h_3 + 3h_2h_1 + h_1^3$ .

The symmetric function  $F_n$  has many remarkable properties, summarized (in a dual form, and with equation (1.2) below not included) in [6, Exer. 7.48(f)].

**Proposition 1.1.** *We have*

$$\begin{aligned}
(1.1) \quad F_n &= \sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)-1} z_\lambda^{-1} p_\lambda \\
&= \frac{1}{n+1} \sum_{\lambda \vdash n} s_\lambda (1^{n+1}) s_\lambda \\
&= \frac{1}{n+1} \sum_{\lambda \vdash n} \left[ \prod_i \binom{\lambda_i + n}{\lambda_i} \right] m_\lambda \\
&= \sum_{\lambda \vdash n} \frac{n(n-1) \cdots (n - \ell(\lambda) + 2)}{d_1(\lambda)! \cdots d_n(\lambda)!} h_\lambda \\
(1.2) \quad &= \sum_{\lambda \vdash n} \varepsilon_\lambda \frac{(n+2)(n+3) \cdots (n + \ell(\lambda))}{d_1(\lambda)! \cdots d_n(\lambda)!} e_\lambda \\
\omega F_n &= \frac{1}{n+1} \left[ \prod_i \binom{n+1}{\lambda_i} \right] m_\lambda,
\end{aligned}$$

where  $d_i(\lambda)$  denotes the number of parts of  $\lambda$  equal to  $i$  and  $\varepsilon_\lambda = (-1)^{n-\ell(\lambda)}$ . Moreover,

$$(1.3) \quad F_n = \frac{1}{n+1} [t^n] H(t)^{n+1},$$

where  $[t^n]f(t)$  denotes the coefficient of  $t^n$  in the power series  $f(t)$ , and

$$H(t) = \sum_{n \geq 0} h_n t^n = \frac{1}{(1-x_1 t)(1-x_2 t) \cdots}.$$

Note in particular that the coefficient of  $h_\lambda$  in equation (1.3) is the number of weakly increasing parking functions of length  $n$  whose entries occur with multiplicities  $\lambda_1, \lambda_2, \dots$ .

A further important property of  $F_n$ , an immediate consequence of equation (1.3) and the Lagrange inversion formula, is the following. Let

$$(1.4) \quad E(t) = \sum_{n \geq 0} e_n t^n = \prod_i (1 + x_i t),$$

and let  $G(t)^{\langle -1 \rangle}$  denote the compositional inverse of the power series  $G(t)$  (which will exist as a formal power series if  $G(t) = a_1 t + a_2 t^2 + \dots$ , where  $a_1 \neq 0$ ). Then

$$(1.5) \quad \sum_{n \geq 1} F_n t^n = (tE(-t))^{\langle -1 \rangle}.$$

There are several known generalizations of parking functions. In particular, if  $\mathbf{u} = (u_1, \dots, u_n)$  is a weakly increasing sequence of positive integers, then a  $\mathbf{u}$ -parking function is a sequence  $(a_1, \dots, a_n) \in \mathbb{P}^n$  (where  $\mathbb{P} = \{1, 2, \dots\}$ ) such that its increasing rearrangement  $b_1 \leq b_2 \leq \dots \leq b_n$  satisfies  $b_i \leq u_i$ . Thus an ordinary parking function corresponds to  $\mathbf{u} = (1, 2, \dots, n)$ . For the general theory of  $\mathbf{u}$ -parking functions, see the survey [8, §13.4]. We will be interested here in the special case  $\mathbf{u} = (k, r+k, 2r+k, \dots, (n-1)r+k)$ , where  $r, k \geq 1$ . We call such a  $\mathbf{u}$ -parking function an  $(r, k)$ -parking function. With this terminology, an ordinary parking function is a  $(1, 1)$ -parking function. We call an  $(r, 1)$ -parking function simply an  $r$ -parking function.

NOTE. Our terminology is not universally used. For instance, if  $(a_1, \dots, a_n)$  is what we call an  $(r, r)$ -parking function, then Bergeron [1] would call  $(a_1 - 1, \dots, a_n - 1)$  an  $r$ -parking function.

Pollak's proof that  $\#\text{PF}_n = (n+1)^{n-1}$  extends easily to  $(r, k)$ -parking functions. Namely, we now have  $rn$  cars  $C_1, \dots, C_{rn}$  and  $rn+k-1$  spaces  $1, 2, \dots, rn+k-1$ . We consider preferences  $\alpha = (a_1, \dots, a_n)$ ,  $1 \leq a_i \leq rn+k-1$ , where cars  $C_{r(i-1)+1}, \dots, C_{ri}$  all prefer  $a_i$ . The cars use the same parking algorithm as before. It is not hard to check that all the cars can park if and only if  $\alpha$  is an  $(r, k)$ -parking function. Now arrange  $rn+k$  spaces on a circle, allow the preferences  $1 \leq a_i \leq rn+k$ , and park as in Pollak's proof. Then  $\alpha$  is an  $(r, k)$ -parking function if and only if the space  $rn+k$  is empty. Reasoning as in Pollak's proof gives the following result, which in an equivalent form is due to Steck [7].

**Theorem 1.2.** *Let  $\text{PF}_n^{(r,k)}$  denote the set of  $(r, k)$ -parking functions of length  $n$ . Then*

$$\#\text{PF}_n^{(r,k)} = k(rn+k)^{n-1}.$$

The results in Proposition 1.1 can be extended to  $(r, k)$ -parking functions (Theorem 2.1). Most of them appear in Bergeron [1, Prop. 1] for the case  $k = r$ . (Bergeron and his collaborators have gone on to generalize their results in a series of papers on rectangular parking functions.) One of our key results (Theorem 3.1) connects  $r$ -parking functions to  $(r, k)$ -parking functions as follows.

Let  $\text{PF}_n^{(r,k)}$  denote the set of all  $(r, k)$ -parking functions of length  $n$ , and let  $F_n^{(r,k)}$  denote the Frobenius characteristic  $\text{ch PF}_n^{(r)}$  of the action of  $\mathfrak{S}_n$  on  $\text{PF}_n^{(r,k)}$  by permuting coordinates. Define

$$\begin{aligned} \mathcal{P}^{(r,k)}(t) &= \sum_{n \geq 0} F_n^{(r,k)} t^n \\ \mathcal{P}^{(r)}(t) &= \mathcal{P}^{(r,1)}(t), \end{aligned}$$

Then (Theorem 3.1)

$$(1.6) \quad \mathcal{P}^{(r)}(t)^k = \mathcal{P}^{(r,k)}(t).$$

Equation (1.6) suggests looking at  $\mathcal{P}^{(r)}(t)^k$  for negative integers  $k$ . We obtain parking function interpretations of the coefficients of such power series in Section 4. As some motivation

for what to expect, consider two power series  $A(t), B(t)$ , with  $B(0) = 0$ , that are related by

$$A(t) = \frac{1}{1 - B(t)} = 1 + B(t) + B(t)^2 + \dots.$$

Thus

$$(1.7) \quad B(t) = 1 - \frac{1}{A(t)},$$

and often  $B(t)$  will be a generating function for certain “prime” objects, while  $A(t)$  will be a generating function for all objects, i.e., products of primes. See for instance [5, Prop. 4.7.11]. We will see examples of this relationship with our generating functions for parking functions.

For instance, if we set

$$(1.8) \quad \mathcal{P}^{(r,k)}(t)^{-1} = 1 - \sum_{n \geq 1} G_n^{(r,k)} t^n,$$

then  $G_n^{(1,1)}$  is the Frobenius characteristic of the action of  $\mathfrak{S}_n$  on *prime* parking functions of length  $n$ , i.e., parking functions that remain parking functions when some term equal to 1 is deleted (a concept due to Gessel [6, Exer. 5.49(f)]). An increasing parking function  $b_1 b_2 \cdots b_n$  can be uniquely factored  $\beta_1 \cdots \beta_k$ , such that (1) if  $b_j$  is the first term of  $\beta_i$  then  $b_j = j$ , and (2) if we subtract from each term of  $\beta_i$  one less than its first element (so it now begins with a 1), then we obtain a prime parking function.

As a direct generalization of the previous example,  $G_n^{(r,1)}$  is the Frobenius characteristic of the action of  $\mathfrak{S}_n$  on sequences  $a_1 a_2 \cdots a_n$  such that some  $a_i = 1$ , and if remove this term then we obtain an  $(r, r)$ -parking function. More generally, if  $1 \leq k \leq r$  then  $G_n^{(r,k)}$  is the Frobenius characteristic of the action of  $\mathfrak{S}_n$  on sequences  $a_1 a_2 \cdots a_n$  such that we can remove some term less than  $k+1$  and obtain an  $(r, r)$  parking function (Theorem 4.3). For instance, when  $r = 2$  and  $n = 3$  the increasing sequences with this property are 111, 112, 113, 114, 122, 123, 124, 222, 223, 224. Hence  $G_3^{(2,2)} = 2h_1^3 + 6h_2 h_1 + 2h_3$ . The situation for  $\mathcal{P}^{(r,k)}(t)^{-j}$  when  $j > r$  is more complicated (Theorem 4.1).

## 2. EXPANSIONS OF $F_n^{(r,k)}$

In this section we consider the expansion of  $F_n^{(r,k)}$  into the six classical bases for symmetric functions. These expressions are defined even when  $k$  is an indeterminate, so we can use any of them to define  $F_n^{(r,k)}$  in this situation. For later combinatorial applications we will only consider the case when  $k$  is an integer. We use notation from [6, Ch. 7] regarding symmetric functions. We also use multinomial coefficient notation such as

$$\binom{k}{d_1, \dots, d_n, k - \sum d_i} = \frac{k(k-1) \cdots (k - \sum d_i + 1)}{d_1! \cdots d_n!},$$

where  $d_1, \dots, d_n$  are nonnegative integers and  $k$  may be an indeterminate. As usual we abbreviate  $\binom{k}{d, k-d}$  as  $\binom{k}{d}$ .

**Theorem 2.1.** Recall that  $d_i(\lambda)$  denotes the number of parts of  $\lambda$  equal to  $i$ . Then  $F_0^{(r,k)} = 1$ , and for  $n \geq 1$  we have

$$(2.1) \quad F_n^{(r,k)} = \frac{k}{rn+k} \sum_{\lambda \vdash n} \binom{rn+k}{d_1(\lambda), \dots, d_n(\lambda), rn+k-\ell(\lambda)} h_\lambda$$

$$(2.2) \quad = \frac{k}{rn+k} \sum_{\lambda \vdash n} \varepsilon_\lambda \binom{rn+k+\ell(\lambda)-1}{d_1(\lambda), \dots, d_n(\lambda), rn+k-1} e_\lambda$$

$$= \frac{k}{rn+k} \sum_{\lambda \vdash n} \left[ \prod_i \binom{\lambda_i + rn + k - 1}{\lambda_i} \right] m_\lambda$$

$$= \frac{k}{rn+k} \sum_{\lambda \vdash n} s_\lambda(1^{rn+k}) s_\lambda$$

$$(2.3) \quad = k \sum_{\lambda \vdash n} z_\lambda^{-1} (rn+k)^{\ell(\lambda)-1} p_\lambda$$

$$\omega F_n^{(r,k)} = \frac{k}{rn+k} \sum_{\lambda \vdash n} \left[ \prod_i \binom{rn+k}{\lambda_i} \right] m_\lambda,$$

Moreover,

$$(2.4) \quad F_n^{(r,k)} = \frac{k}{rn+k} [t^n] H(t)^{rn+k}.$$

*Proof.* Define two elements  $\alpha$  and  $\beta$  of  $[rn+k]^n$  to be *equivalent* if their difference is a multiple of  $(1, 1, \dots, 1) \bmod rn+k$ . This defines an equivalence relation on  $[rn+k]^n$ , and each equivalence class contains  $rn+k$  elements. It follows from the proof of Theorem 1.2 that each equivalence class contains exactly  $k$   $(r, k)$ -parking functions. Moreover, all the elements  $\alpha$  in each equivalence class have the same multiset of part multiplicities, i.e., the multiset  $\{d_1, \dots, d_{rn+k}\}$ , where  $d_i$  is the number of  $i$ 's in  $\alpha$ .

For  $n \geq 1$  let  $D_n^{(r,k)}$  denote the Frobenius characteristic of the action of  $\mathfrak{S}_n$  on  $[rn+k]^n$  by permuting coordinates. It follows that

$$F_n^{(r,k)} = \frac{k}{rn+k} D_n^{(r,k)}.$$

Hence if we set  $q = 1, k = n$ , and  $n = rn+k$  in Exercise 7.75(a) of [6] then we get

$$D_n^{(r,k)} = \sum_{\lambda \vdash n} s_\lambda(1^{rn+k}) s_\lambda.$$

(Exercise 7.75 deals with  $\mathfrak{S}_k$  acting on submultisets  $M$  of  $\{1^n, \dots, k^n\}$ . Replace  $M$  with the vector  $(d_1, \dots, d_k)$ , where  $d_i$  is the multiplicity of  $i$  in  $M$ , to get our formulation.) Therefore

$$F_n^{(r,k)} = \frac{k}{rn+k} \sum_{\lambda \vdash n} s_\lambda(1^{rn+k}) s_\lambda.$$

The remainder of the proof is routine symmetric function manipulation.  $\square$

A further important property of  $F_n^{(r,k)}$  in the case  $k = r$ , an immediate consequence of equation (2.4) and the Lagrange inversion formula [6, Thm. 5.4.2], is the following.

Let  $E(t)$  be given by equation (1.4). Then

$$(2.5) \quad \sum_{n \geq 0} F_n^{(r,r)} t^{n+1} = (tE(-t)^r)^{\langle -1 \rangle}$$

### 3. A RELATION BETWEEN $r$ -PARKING FUNCTIONS AND $(r, k)$ -PARKING FUNCTIONS

In this section we give a combinatorial proof of the following result.

**Theorem 3.1.** *Let  $k, r \in \mathbb{P}$ . Then  $\mathcal{P}^{(r)}(t)^k = \mathcal{P}^{(r,k)}(t)$ .*

*Proof.* We need to give a bijection  $\psi: (\text{PF}_n^{(r,1)})^k \rightarrow \text{PF}_n^{(r,k)}$  such that if  $\psi(\alpha_1, \dots, \alpha_k) = \beta$ , then  $\ell(\alpha_1) + \dots + \ell(\alpha_k) = \ell(\beta)$ . Note that we consider the empty sequence  $\emptyset$  to be an  $(r, j)$ -parking function for any  $r$  and  $j$ .

Given  $(\alpha_1, \dots, \alpha_k) \in (\text{PF}_n^{(r,1)})^k$ , define  $\alpha'_i$  to be the sequence obtained by adding  $r(\ell(\alpha_1) + \dots + \ell(\alpha_{i-1})) + i - 1$  to every term of  $\alpha_i$ . For instance, if  $r = 2$  and

$$(\alpha_1, \dots, \alpha_5) = ((1, 2), \emptyset, \emptyset, (1), (1, 3, 4)),$$

then  $\alpha'_1 = (1, 2)$ ,  $\alpha'_2 = \alpha'_3 = \emptyset$ ,  $\alpha'_4 = (8)$ , and  $\alpha'_5 = (11, 13, 14)$ .

It is easily seen that  $\psi$  is the desired bijection. In particular, the inverse  $\psi^{-1}$  has the following description. Given  $\beta = (b_1, \dots, b_n) \in \text{PF}_n^{(r,k)}$ , let  $c_i = b_i - ri + r - 1$ . (The term  $r - 1$  could be replaced by any constant independent from  $i$ ; we made the choice so  $c_1 = 0$ .) Let  $c_{j_1} < \dots < c_{j_r}$  be the left-to-right maxima of the sequence  $c_1, \dots, c_n$ , so  $j_1 = 1$ . Factor  $\beta$  (regarded as a word  $b_1 \dots b_n$ ) as  $\beta_1 \dots \beta_r$ , where  $\beta_i$  begins with  $b_{j_i}$ . Subtract a constant  $t_i$  from each term of  $\beta_i$  so that we obtain a sequence (or word)  $\beta'_i$  beginning with a 1. Insert  $c_{j_{i+1}} - c_{j_i} - 1$  empty words  $\emptyset$  between  $\beta'_i$  and  $\beta'_{i+1}$ , and place empty words at the end so that there are  $k$  words in all. These words  $\alpha_1, \dots, \alpha_k$  then satisfy  $\psi^{-1}(\beta) = (\alpha_1, \dots, \alpha_k)$ .  $\square$

**Example 3.2.** Suppose that  $r = 2, k = 7$ , and

$$\beta = (1, 2, 2, 10, 12, 14, 15, 19, 22).$$

Then  $(c_1, \dots, c_9) = (0, -1, -3, 3, 3, 3, 3, 4, 5)$ . The left-to-right maxima are  $c_1 = 0, c_4 = 3, c_8 = 4, c_9 = 5$ . Thus  $\beta_1 = (1, 2, 2), \beta_2 = (10, 12, 14, 15), \beta_3 = (19)$ , and  $\beta_4 = (22)$ . Hence  $\beta'_1 = (1, 2, 2), \beta'_2 = (1, 3, 5, 6), \beta'_3 = \beta'_4 = (1)$ . Between  $\beta'_1$  and  $\beta'_2$  insert  $c_4 - c_1 - 1 = 2$  copies of  $\emptyset$ . Similarly since  $c_8 - c_4 - 1 = c_9 - c_8 - 1 = 0$  we insert no further copies of  $\emptyset$  between remaining  $\beta'_i$ 's. We now have the six words  $\beta'_1, \emptyset, \emptyset, \beta'_2, \beta'_3, \beta'_4$ . Since  $k = 7$  we insert one  $\emptyset$  at the end, finally obtaining

$$\psi^{-1}(\beta) = ((1, 2, 2), \emptyset, \emptyset, (1, 3, 5, 6), (1), (1), \emptyset).$$

Theorem 3.1 has a natural  $q$ -analogue. We simply state the relevant result since the bijection in the proof of Theorem 3.1 is compatible with our  $q$ -analogue, so the proof carries over. More specifically, using the notation of equation (3.1) below it is easy to check that if  $\beta \in \text{PF}_n^{(r,k)}$  and  $\psi^{-1}(\beta) = (\alpha_1, \dots, \alpha_k)$ , then

$$s^{(r,k)}(\beta) = \sum_{j=1}^k (s^{(r,1)}(\alpha_j) + (k-j)\ell(\alpha_j)).$$

Given an  $(r, k)$ -parking function  $\alpha = (a_1, \dots, a_n)$  of length  $n$ , note that the largest possible value of  $\sum a_i$  is  $k + (k + r) + \dots + (k + (n - 1)r) = kn + \binom{n}{2}r$ . Define

$$(3.1) \quad s^{(r,k)}(\alpha) = kn + \binom{n}{2}r - \sum_{i=1}^n a_i.$$

When  $k = r$  this is a well-known statistic on parking functions, sometimes used in the variant form  $\sum a_i$ . See for instance [4][8, §§1.2.2, 1.3.3]. Note that the action of  $\mathfrak{S}_n$  on  $(r, k)$ -parking functions  $\alpha$  of length  $n$  is compatible with this statistic, i.e., if  $w \in \mathfrak{S}_n$  then  $s^{(r,k)}(w \cdot \alpha) = w \cdot s^{(r,k)}(\alpha)$ .

Given a sequence  $\beta = (b_1, \dots, b_n) \in \mathbb{P}^n$ , let  $U_\beta$  denote the Frobenius characteristic of the action by permuting coordinates of  $\mathfrak{S}_n$  on all permutations of the terms of  $\beta$ . Hence if  $m_i$  is the number of  $i$ 's in  $\beta$  then  $U_\beta = h_{m_1} h_{m_2} \dots$ . Given  $r, k, n \geq 1$ , define

$$F_n^{(r,k)}(q) = \sum_{\beta} q^{s^{(r,k)}(\beta)} U_\beta,$$

where  $\beta$  runs over all increasing  $(r, k)$ -parking functions of length  $n$ . Write

$$\begin{aligned} \mathcal{P}^{(r,k)}(q, t) &= \sum_{n \geq 0} F_n^{(r,k)}(q) t^n \\ \mathcal{P}^{(r)}(q, t) &= \mathcal{P}^{(r,1)}(q, t). \end{aligned}$$

Thus  $\mathcal{P}^{(r,k)}(1, t) = \mathcal{P}^{(r,k)}(t)$ .

**Theorem 3.3.** *We have*

$$\mathcal{P}^{(r,k)}(q, t) = \prod_{i=0}^{k-1} \mathcal{P}^{(r)}(q, q^i t).$$

Equation (1.7) gives a relationship between a generating function  $A(t)$  for all objects and  $B(t)$  for prime objects. There is another basic relationship of this nature between exponential generating functions  $A(t)$  for all objects and  $B(t)$  for “connected” objects, namely, the *exponential formula*  $A(t) = \exp B(t)$  or  $B(t) = \log A(t)$ . See [6, §5.1]. Thus we can ask whether there is a combinatorial interpretation of the coefficients of  $\log \mathcal{P}^{(r,k)}(t)$ . Recall that  $D_n^{(r,k)}$  denotes the Frobenius characteristic of the action of  $\mathfrak{S}_n$  on  $[rn + k]^n$  by permuting coordinates, as in the proof of Theorem 2.1. The case  $k = r$  is handled by the following result.

**Proposition 3.4.** *We have*

$$\log \mathcal{P}^{(r,r)}(t) = \sum_{n \geq 1} D_n^{(r,r)} \frac{t^n}{n}.$$

*Proof.* The proof is a simple consequence of the following variant of the Lagrange inversion formula appearing in [6, Exer. 5.56]: for any power series  $F(t) = a_1 t + a_2 t^2 + \dots \in \mathbb{C}[[t]]$  with  $a_1 \neq 0$  we have

$$(3.2) \quad n[t^n] \log \frac{F^{(-1)}(t)}{t} = [t^n] \left( \frac{t}{F(t)} \right)^n.$$

Choose  $F(t) = tE(-t)^r$ , where  $E(t)$  is given by equation (1.4). Now

$$\frac{1}{E(-t)} = H(t) = \sum_{n \geq 0} h_n t^n.$$

Hence by equation (2.5), we see that equation (3.2) becomes

$$n[t^n] \log \mathcal{P}^{(r,r)}(t) = [t^n] H(t)^{nr}.$$

It is clear that  $[t^n] H(t)^{nr} = D_n^{(r,r)}$ , so the proof follows.  $\square$

#### 4. A DUAL TO $(r, k)$ -PARKING FUNCTIONS

Equation (1.6) suggests looking at  $\mathcal{P}^{(r)}(t)^k$  for negative integers  $k$ . We obtain an object “dual” (in the sense of combinatorial reciprocity) to  $(r, k)$ -parking functions.

We define  $F_n^{(r,k)}$  for  $k \leq 0$  by (2.1) (therefore all the equations in Theorem 2.1 hold for  $k \leq 0$ ). It follows from the definition of  $\mathcal{P}^{(r,k)}(t)$  and equation (1.6) that

$$\mathcal{P}^{(r)}(t)^k = \mathcal{P}^{(r,k)}(t) = \sum_{n \geq 0} F_n^{(r,k)} t^n$$

holds for all  $k > 0$ . Thus it also holds for all  $k \leq 0$ . Comparing the coefficients of  $t^n$  with those in equation (1.8), namely,

$$\mathcal{P}^{(r)}(t)^{-k} = 1 - \sum_{n \geq 1} G_n^{(r,k)} t^n, \quad \text{for all } k \geq 0,$$

and combining with (2.1), we see that

$$(4.1) \quad G_n^{(r,k)} = -F_n^{(r,-k)} = \frac{k}{rn - k} \sum_{\lambda \vdash n} \binom{rn - k}{d_1(\lambda), \dots, d_n(\lambda)} h_\lambda, \quad \text{for all } k \geq 0, n \geq 1.$$

We then have the following combinatorial interpretation of  $G_n^{(r,k)}$ .

**Theorem 4.1.** *If  $rn - k > 0$ , then  $G_n^{(r,k)}$  is the Frobenius characteristic of the action of  $\mathfrak{S}_n$  on the set  $S$  of  $n$ -tuples whose increasing rearrangements have the following form:*

$$(4.2) \quad (\underbrace{w, \dots, w}_{q(w) \text{ } w\text{'s}}, b_{q(w)+1}, b_{q(w)+2}, \dots, b_n),$$

where  $w \in [k]$  and  $q(w)$  is the smallest integer such that  $w \leq q(w)r$ , and

$$(4.3) \quad b_j \leq \min\{(j-1)r, w-1+rn-k\} \quad \text{for } j = q(w)+1, q(w)+2, \dots, n.$$

Note that  $w \leq \min\{(j-1)r, w-1+rn-k\}$  for all  $j \geq q(w)+1$ ; therefore (4.3) is equivalent to

$$(4.4) \quad b_j \leq \min\{(j-1)r, w-1+rn-k\} \quad \text{whenever } b_j > w.$$

In other words, a weakly increasing integer sequence  $b$  is in  $S$  if and only if it satisfies the following properties.

- I.  $b_1 = w$  for some  $w \in [k]$ , and  $b_n - b_1 < rn - k$ .
- II.  $b_{q(w)} = w$ .
- III.  $b_j \leq (j-1)r$  for all  $j \in [n]$  whenever  $b_j > w$ .



**Example 4.2.** Let  $r = 1$ ,  $k = 2$ , and  $n = 5$ . The coefficient of  $t^5$  in  $-\mathcal{P}^{(1)}(t)^{-2}$  is

$$2h_3h_1^2 + 2h_2^2h_1 + 4h_3h_2 + 4h_4h_1 + 2h_5.$$

This symmetric function is the Frobenius characteristic of the action of  $\mathfrak{S}_5$  on all sequences  $(a_1, \dots, a_5) \in \mathbb{P}^5$  whose increasing rearrangement  $b_1 \geq \dots \geq b_5$  satisfies either of the conditions (1)  $b_1 = 1$ ,  $b_2 \leq 1$  (so in fact  $b_2 = 1$ ),  $b_3 \leq 2$ ,  $b_4 \leq 3$ ,  $b_5 \leq 3$ , or (2)  $b_1 = b_2 = 2$ ,  $b_3 \leq 2$  (so in fact  $b_3 = 2$ ),  $b_4 \leq 3$ ,  $b_5 \leq 4$ . We get the fourteen increasing sequences (orbit representatives) 11111, 11112, 11113, 11122, 11123, 11133, 11222, 11233, 11223, 22222, 22223, 22224, 22233, 22234.

**A special case.** When  $k \in \{1, \dots, r\}$ , for all  $w \in [k]$  we have  $q(w) = 1$  and  $(n-1)r \leq rn - k \leq w - 1 + rn - k$ . Therefore (4.4) becomes  $b_j \leq (j-1)r$  for all  $j > 1$ , so  $b$  having the form (4.2) is equivalent to  $b_1 \in [k]$  and  $(b_2, b_3, \dots, b_n)$  is a weakly increasing  $(r, r)$ -parking functions of length  $n-1$ . Thus Theorem 4.1 becomes the following result.

**Theorem 4.3.** *If  $k \in \{1, \dots, r\}$ , then  $G_n^{(r,k)}$  is the Frobenius characteristic of the action of  $\mathfrak{S}_n$  on the distinct  $n$ -tuples we get by adjoining  $1, 2, \dots$ , or  $k$  to  $(r, r)$ -parking functions of length  $n-1$ ; or equivalently, the  $n$ -tuples whose increasing rearrangements start with  $1, 2, \dots$ , or  $k$  and followed by weakly increasing  $(r, r)$ -parking functions of length  $n-1$ .*

Theorem 4.1 is a consequence of the following result, which will be proved right below Proposition 4.6. fix

**Proposition 4.4.** *Suppose that  $rn - k > 0$ . Given  $a = (a_1, \dots, a_n) \in [rn - k]^n$ , let  $p \in [rn - k]$  be the smallest positive integer  $i$  such that the increasing rearrangements of  $a$  and  $(a + p) \bmod rn - k$  coincide, where  $a + i := (a_1 + i, \dots, a_n + i)$  and  $a_j + i \bmod rn - k$  is the  $a_j + i$  taken modulo  $rn - k$  so that  $a_j + i \in [rn - k]$ ; equivalently,  $p = \#R_a$ , where  $R_a$  is the set of increasing rearrangements of vectors  $a + i \bmod rn - k$  ( $i \in \mathbb{Z}$ ).*

*Then the number of increasing vectors  $b \in S$  such that the increasing rearrangement of  $(b \bmod rn - k)$  is in  $R_a$  is  $\frac{pk}{rn-k}$ .*

Theorem 4.1 follows as each  $b \in S$  corresponds to a unique set  $R_a$  (the vector  $a$  may not be unique).

*Remark 4.5.* The reason why we need the vector  $b \bmod rn - k$  is that we may have  $b \in S \setminus [rn - k]^n$  and  $b \bmod rn - k \in S$ . For instance, when  $r = 2, n = 4, k = 3, rn - k = 5$ , we have  $(6, 2, 2, 4) \in S \setminus [rn - k]^n$  and  $(1, 2, 2, 4) \in S$ .

**A special case.** When  $k \in \{1, \dots, r\}$ , it follows from (4.3) that  $b_n \leq (n-1)r \leq rn - k$  for all  $b \in S$ ; therefore  $b \bmod rn - k = b$ . In other words, we only need to consider  $b$  instead of  $b \bmod rn - k$ . Thus, combined with Theorem 4.1, Proposition 4.4 becomes as follows.

**Proposition 4.6.** *If  $k \in \{1, \dots, r\}$ , then for any given  $(a_1, \dots, a_n) \in [rn - k]^n$ , there are exactly  $k$   $i$ 's  $(\bmod rn - k)$  such that the vector  $(a_1 + i, \dots, a_n + i) \bmod rn - k$  is an  $(r, r)$ -parking function of length  $n-1$  adjoined by  $1, 2, \dots$ , or  $k$ , where  $a_j + i \bmod rn - k$  is the  $a_j + i$  taken modulo  $rn - k$  so that  $a_j + i \in [rn - k]$ .*

*Proof of Proposition 4.4.* The case  $k = 0$  is trivial. Assume that  $k \geq 1$ . It suffices to prove the proposition for a weakly increasing sequence  $a = (a_1, \dots, a_n)$  with  $a_1 = 1$ . For convenience, let  $N := rn - k > 0$  and denote the increasing rearrangement of a sequence  $x$  by  $x_\uparrow$ .

We have two cases:  $p < N$  and  $p = N$ .

**Case 1.**  $p < N$ .

Then  $a$  has the form:

$$(4.5) \quad \left( \underbrace{1, \dots, 1}_{d \text{ 1's}}, \underbrace{1+p, \dots, 1+p}_{d \text{ (1+p)'s}}, \underbrace{1+2p, \dots, 1+2p}_{d \text{ (1+2p)'s}}, \dots, \underbrace{1+(\ell-1)p, \dots, 1+(\ell-1)p}_{d \text{ (1+(\ell-1)p)'s}} \right),$$

where  $d, \ell \in \mathbb{P}$  with  $\ell > 1$  such that  $\ell d = n$  and  $\ell p = N$ . Thus  $k = rn - N = (rd - p)\ell$ .

The following fact can be verified immediately from the definition of  $S$  and  $R_a$ .

**Lemma 4.7.** *If  $b \in S$ , then  $b+i \in S$  for all  $i \in \{0, -1, \dots, -b_1+1\}$ . Further, if  $(b \bmod N)_\uparrow \in R_a$ , then  $(b+i \bmod N)_\uparrow \in R_a$ .*

*In particular, when  $i = -b_1 + 1$ , the smallest coordinate of  $b+i$  is 1. According to (4.4), we have  $b+i \in [N]^n$ , and therefore  $b+i \bmod N = b+i$ . If  $(b \bmod N)_\uparrow \in R_a$ , then  $(b+i)_\uparrow = (b+i \bmod N)_\uparrow \in R_a$ .*

We also need the following lemma.

**Lemma 4.8.** *We have  $a+i \in S$  if and only if  $i \in \{0, 1, \dots, rd-p-1\}$ .*

On the strength of Lemmas 4.7 and 4.8 and the fact that  $R_a = \{a+i : 0 \leq i \leq p-1\}$ , the number of vectors  $b \in S$  such that  $(b \bmod rn-k)_\uparrow \in R_a$  is  $rd-p = \frac{pk}{N}$ , as desired.

*Proof of Lemma 4.8.* If  $a+i \in S$  with the form (4.2), then applying (4.3) to  $a+i$  and  $d+1$  yields  $1+p+i \leq rd$ , and therefore  $i \leq rd-p-1$ .

On the other hand, for any  $i \in \{0, 1, \dots, rd-p-1\}$ , we have  $a+i \in S$ . In fact, the vector  $a+i = \left( \underbrace{w, \dots, w}_{d \text{ } w\text{'s}}, \underbrace{w+p, \dots, w+p}_{d \text{ (w+p)'s}}, \underbrace{w+2p, \dots, w+2p}_{d \text{ (w+2p)'s}}, \dots, \underbrace{w+(\ell-1)p, \dots, w+(\ell-1)p}_{d \text{ (w+(\ell-1)p)'s}} \right)$ ,

where  $w = 1+i \leq rd-p \leq (rd-p)\ell = k$ . Property I then follows from  $(a+i)_n - (a+i)_1 = (\ell-1)p < \ell p = N$ . Property II holds since  $q(\cdot)$  is weakly increasing and  $q(w) \leq q(rd) = d$ . Finally, Property III is satisfied because  $(a+i)_{jd+1} = \dots = (a+i)_{(j+1)d} = w+jp \leq j(w+p) \leq j(rd-p+p) = (jd)r$  for all  $j \in [\ell-1]$ .  $\square$

**Case 2.**  $p = N$ .

Namely, the vectors  $(a+i \bmod N)_\uparrow$ ,  $i \in [N]$  are distinct. We will determine explicitly the  $\frac{pk}{rn-k} = k$  vectors in  $S$  desired in Proposition 4.4.

For convenience, we denote  $x_j = a_{j+1} (\leq N)$ ,  $j = 0, \dots, n-1$ , and consider the weakly increasing sequence  $x = (x_0, \dots, x_{n-1})$  with  $x_0 = 1$ . Then  $x \in S$  if and only if  $x_j \leq rj$  for all  $j \in [n-1]$ . In general, a weakly increasing integer sequence  $y$  is in  $S$  if and only if

- I'**.  $y_0 = w$  for some  $w \in [k]$ , and  $y_{n-1} - y_0 < N$ .
- II'**.  $y_{q(w)-1} = w$ .
- III'**.  $y_j \leq jr$  for all  $j \in [n-1]$  whenever  $y_j > w$ .

In the rest of the proof, all variables are integers, and for a vector  $y$ , we denote by  $y_j$  its  $(j+1)$ -th coordinate.

Let  $\Delta_j := rj - x_j$ ,  $j = 0, 1, \dots, n-1$ . Then  $\Delta_0 = -1$ , and  $x \in S$  if and only if  $\Delta_j \geq 0$  for all  $j \in [n-1]$ .

**Lemma 4.9.** *There exists  $i \in \mathbb{Z}$  such that the vector  $(x + i \bmod N)_\uparrow \in S$ , with the smallest coordinate equal to 1. More precisely, if  $x \in S$ , then we can take  $i = 0$ ; otherwise, take  $i = 1 - x_j$ , where  $j$  is the largest number in  $[n - 1]$  such that  $\Delta_j = \min_{j' \in [n-1]} \Delta_{j'}$ .*

*Proof.* Assume that  $x \notin S$ , then  $\Delta_j \leq -1$  and  $j \in [n - 1]$  for the  $j$  taken in the lemma. Taking  $i = 1 - x_j$ , we get

$$\begin{aligned} & x + i \bmod N \\ = & (2 - x_j + N, x_1 - x_j + 1 + N, \dots, x_{j-1} - x_j + 1 + N, 1, x_{j+1} - x_j + 1, \dots, x_{n-1} - x_j + 1), \end{aligned}$$

and thus

$$\begin{aligned} \alpha & := (x + i \bmod N)_\uparrow \\ = & (1, \underbrace{x_{j+1} - x_j + 1}_{\alpha_1}, \dots, \underbrace{x_{n-1} - x_j + 1}_{\alpha_{n-1-j}}, \underbrace{2 - x_j + N}_{\alpha_{n-j}}, \underbrace{x_1 - x_j + 1 + N}_{\alpha_{n-j+1}}, \dots, \underbrace{x_{j-1} - x_j + 1 + N}_{\alpha_{n-1}}). \end{aligned}$$

It follows from the definition of  $j$  that  $x_j \geq rj + 1$ , and for  $j' > j$  we have  $\Delta_{j'} \geq \Delta_j + 1$ , and therefore  $x_{j'} - x_j \leq r(j' - j) - 1$ ; for  $j' < j$  we have  $\Delta_{j'} \geq \Delta_j$ , and therefore  $x_{j'} - x_j \leq r(j' - j)$ . Thus

$$\begin{aligned} \alpha_u & = x_{j+u} - x_j + 1 \leq r(j + u - j) - 1 + 1 = ru, \quad u \in [n - 1 - j], \\ \alpha_{n-j} & = 2 - x_j + rn - k \leq 2 - rj - 1 + rn - 1 = r(n - j), \\ \alpha_{n-j+u} & = x_u + 1 - x_j + rn - k \leq r(u - j + n), \quad u \in [j - 1]. \end{aligned}$$

Hence  $\alpha \in S$ . □

On the strength of Lemma 4.9, we can assume that  $x \in S$  with  $x_0 = 1$ . The following result determines the  $k$  vectors in  $S$  desired in Proposition 4.4.

**Lemma 4.10.** *Let  $0 = j_0 < j_1 < j_2 < \dots$  be the elements of the subset*

$$J^* := \{j \in J : \Delta_{j'} > \Delta_j, \text{ for all } n - 1 \geq j' > j\} \subseteq J := \{0\} \cup \{j \in [n - 1] : x_j > x_{j-1}\}$$

and  $m$  be the nonnegative integer determined by

$$-1 = \Delta_{j_0} < \Delta_{j_1} < \dots < \Delta_{j_m} \leq k - 2 < \Delta_{j_{m+1}} < \dots$$

(if  $j_{m+1}$  does not exist, then set  $j_{m+1}$  and  $\Delta_{j_{m+1}}$  to be infinity). In particular,  $j_1$  is the largest number in  $[n - 1]$  such that  $\Delta_{j_1} = \min_{j \in [n-1]} \Delta_j \geq 0$ .

Then  $y$  is a weakly increasing sequence in  $S$  such that  $(y \bmod N)_\uparrow \in R_x$  if and only if

- (1)  $y = x + i$  with  $0 \leq i \leq \Delta_{j_1} \wedge (k - 1)$ , where  $\wedge$  represents the minimum function; or
- (2)  $y = (x + i_1 \bmod N)_\uparrow + i_2$  with
  - (i)  $i_1 = 1 - x_{j_v}$  for some  $v \in [m]$ , and
  - (ii)  $0 \leq i_2 = y_0 - 1 \leq \Delta_{j_{v+1}} \wedge (k - 1) - \Delta_{j_v} - 1 < k - 1$ .

Further, the  $k$  vectors given in (1) and (2) are distinct.

*Remark 4.11.* Note that (1) is the special case of (2) with  $i_1 = 0 = v$  and  $i_2 = i$ .

*Proof.* As a consequence of  $p = N$ , the vectors  $(x + i_1 \bmod N)_\uparrow$  with  $i_1$  given in (1) ( $i_1 = i$ ) and (2), whose smallest coordinates are all 1, are distinct. Thus the  $k$  vectors given in (1) and (2) are distinct.

(1) If  $y = x + i \in S$ , then by definition we have  $1 \leq (x + i)_0 \leq k$  and  $(x + i)_{j_1} \leq rj_1$ . Thus  $0 \leq i \leq \Delta_{j_1} \wedge (k - 1)$ .

Conversely, for any  $y = x + i$  with  $0 \leq i \leq \Delta_{j_1} \wedge (k - 1)$ , we have  $(y \bmod N)_\uparrow \in R_x$ ,  $y_{n-1} - y_0 = x_{n-1} - x_0 < N$ , and  $1 \leq w := y_0 = (x + i)_0 \leq (1 + \Delta_{j_1}) \wedge k \leq k$ , and Property I' follows.

For Property II', notice that for any  $j \in [n - 1]$  such that  $x_j \geq 2$ , since  $rj - x_j = \Delta_j \geq \Delta_{j_1}$ , we have  $j \geq (2 + \Delta_{j_1})/r > w/r$ , and therefore  $j \geq q(w)$ . Hence  $y_{q(w)-1} = w$ .

Finally for Property III', for all  $j \in [n - 1]$ , since  $\Delta_{j_1} \leq \Delta_j$ , we get  $x_j - x_{j_1} \leq r(j - j_1)$ , and therefore  $y_j = (x + i)_j = x_j + i \leq r(j - j_1) + \Delta_{j_1} = rj$ .

(2) If  $y$  is a weakly increasing sequence in  $S$  such that  $(y \bmod N)_\uparrow \in R_x$  but  $y$  does not have the form described in (1), then by Lemma 4.7 we get  $\alpha := y - i_2 \in S$ ,  $\alpha_0 = 1$  and  $\alpha \in R_x$ , where  $i_2 = y_0 - 1 \geq 0$ .

Since  $\alpha \neq x$ , we have  $\alpha = (x + i_1 \bmod N)_\uparrow$  for some  $i_1 \in \{-1, -2, \dots, 1 - N\}$ . Recall that  $\alpha_0 = 1$ , and thus  $i_1 = 1 - x_j$  for some  $j \in [n - 1]$ . If there is more than one  $j$  such that  $i_1 = 1 - x_j$ , we choose the smallest one, i.e., the  $j \in J$ . Then

$$\begin{aligned} & x + i_1 \bmod N \\ &= (2 - x_j + N, x_1 - x_j + 1 + N, \dots, x_{j-1} - x_j + 1 + N, 1, x_{j+1} - x_j + 1, \dots, x_{n-1} - x_j + 1), \end{aligned}$$

and

$$\begin{aligned} & \alpha := (x + i_1 \bmod N)_\uparrow \\ &= (1, \underbrace{x_{j+1} - x_j + 1}_{\alpha_1}, \dots, \underbrace{x_{n-1} - x_j + 1}_{\alpha_{n-1-j}}, \underbrace{2 - x_j + N}_{\alpha_{n-j}}, \underbrace{x_1 - x_j + 1 + N}_{\alpha_{n-j+1}}, \dots, \underbrace{x_{j-1} - x_j + 1 + N}_{\alpha_{n-1}}). \end{aligned}$$

Recall that  $\alpha \in S$  if and only if

$$(4.6) \quad \alpha_u \leq ru, \quad \text{for all } u \in [n - 1].$$

Applying to  $u = 1, \dots, n - 1 - j$  leads to

$$x_{j'} - x_j + 1 \leq r(j' - j), \text{ i.e., } \Delta_j < \Delta_{j'}, \quad \text{for all } j' < j \leq n - 1;$$

applying to  $u = n - j$  leads to

$$2 - x_j + rn - k \leq r(n - j), \text{ i.e., } \Delta_j \leq k - 2.$$

Therefore  $j = j_v$  for some  $v \in [m]$ .

Conversely, from the above argument we see that if  $i_1 = 1 - x_j$  with  $j = j_v$  for some  $v \in [m]$ , then we have  $\alpha_u \leq ru$  for all  $u \in [n - j]$ . Further, we have

$$\alpha_{n-j+u} = x_u - x_j + 1 + N \leq ru + \Delta_j - rj + 1 + rn - k < r(n - j + u)$$

for all  $u \in [j - 1]$ . Hence  $\alpha \in S$ .

It remains to show that  $\alpha + i_2 \in S$  if only if  $i_2$  satisfies the inequality in (ii).

If  $\alpha + i_2 \in S$ , then applying (4.6) to  $\alpha' := \alpha + i_2$  and  $u = j_{v+1} - j_v$  (if exists) leads to

$$x_{j_{v+1}} - x_{j_v} + 1 + i_2 \leq r(j_{v+1} - j_v), \text{ i.e., } i_2 \leq \Delta_{j_{v+1}} - \Delta_{j_v} - 1;$$

applying (4.6) to  $\alpha' := \alpha + i_2$  and  $u = n - j_v$  leads to

$$2 - x_{j_v} + rn - k + i_2 \leq r(n - j_v), \text{ i.e., } i_2 \leq k - 2 - \Delta_{j_v}.$$

Recall that  $i_2 = y_0 - 1 \geq 0$ , and thus  $i_2$  satisfies the inequality in (ii).

Conversely, if  $i_2$  satisfies the inequality in (ii), then  $\alpha' \in S$ . In fact, we have  $1 \leq w := 1 + i_2 \leq k$  and  $\alpha'_{n-1} - \alpha'_0 = \alpha_{n-1} - \alpha_0 < N$ , and Property I' then follows.

For Property II', by the definition of  $j_{v+1}$ , we have  $\Delta_u \geq \Delta_{j_{v+1}}$  for any  $j_v < u \in J$ , and hence for any  $j_v < u \leq n-1$  such that  $x_u > x_{j_v}$ . Thus  $ru - x_u \geq \Delta_{j_{v+1}}$ . It follows that

$$ru \geq \Delta_{j_{v+1}} + x_u > \Delta_{j_{v+1}} + x_{j_v} = \Delta_{j_{v+1}} + rj_v - \Delta_{j_v}$$

and

$$u - j_v > (\Delta_{j_{v+1}} - \Delta_{j_v})/r \geq w/r, \text{ i.e., } u \geq q(w) + j_v.$$

Hence  $\alpha'_{q(w)-1} = x_{q(w)-1+j_v} - x_{j_v} + w = w$ .

Finally for Property III', from the above argument we see that  $\alpha'_u \leq ru$  for  $u = j_{v+1} - j_v, n - j_v$ . Further, we have

$$\alpha'_{n-j_v+u} = x_u - x_{j_v} + 1 + N + i_2 \leq ru - x_{j_v} + 1 + rn - k + k - 2 - \Delta_{j_v} < r(n - j_v + u)$$

for all  $u \in [j_v - 1]$ . For  $j_v + 1 \leq u \leq n-1$  such that  $x_u > x_{j_v}$  and  $u \in J$ , we have  $\Delta_u \geq \Delta_{j_{v+1}}$  by the definition of  $j_{v+1}$ , and therefore

$$\alpha'_{u-j_v} = x_u - x_{j_v} + w \leq (ru - \Delta_{j_{v+1}}) - x_{j_v} + (\Delta_{j_{v+1}} - \Delta_{j_v}) = r(u - j_v).$$

Hence  $\alpha' \in S$ , as desired. □

□

NOTE. We have been unable to find a satisfactory  $q$ -analogue of Theorem 4.1, generalizing Theorem 3.3.

## 5. THE $r$ -PARKING FUNCTION BASIS

Equation (1.6) and other considerations suggest looking at products of the symmetric functions  $F_n^{(r)}$  for various values of  $n$ . Thus for any partition  $\lambda$  define

$$F_\lambda^{(r)} = F_{\lambda_1}^{(r)} F_{\lambda_2}^{(r)} \cdots,$$

where  $F_0 = 1$ .

Recall that  $\Lambda$  denotes the ring of all symmetric functions that can be written as an integer linear combination of the monomial symmetric functions  $m_\lambda$  (or equivalently,  $s_\lambda$ ,  $h_\lambda$ , or  $e_\lambda$ ).

**Proposition 5.1.** *Fix  $r \geq 1$ . Then the symmetric functions  $F_\lambda^{(r)}$ , where  $\lambda$  ranges over all partitions of all  $n \geq 0$ , form an integral basis for the ring  $\Lambda$ .*

*Proof.* We need to show that for each  $n$ , the set  $\{F_\lambda^{(r)} : \lambda \vdash n\}$  is an integral basis for the (additive) group  $\Lambda^n$  of all homogeneous symmetric functions of degree  $n$  contained in  $\Lambda$ . Let  $\lambda^1, \lambda^2, \dots$  be any ordering of the partitions of  $n$  that is compatible with refinement, that is, if  $\lambda^i$  is a refinement of  $\lambda^j$  then  $i \leq j$ . Now  $F_n^{(r)} = h_n + \dots \in \Lambda^n$ . Hence  $F_\lambda^{(r)} = h_\lambda +$  terms involving  $h_\mu$  where  $\mu$  refines  $\lambda$ . Hence the transition matrix for expressing the  $F_\lambda^{(r)}$ 's in terms of the  $h_\lambda$ 's is lower triangular with 1's on the main diagonal. Since the  $h_\lambda$ 's form an integral basis, the same is true of the  $F_\lambda^{(r)}$ 's. □

Now that for each  $r \geq 1$  we have this ‘‘parking function basis’’  $\{F_\lambda^{(r)}\}$ , we can ask about its expansion in terms of other bases and vice versa. If we restrict ourselves to the six ‘‘standard’’ bases (where the power sums  $p_\lambda$  are a basis over  $\mathbb{Q}$  but not  $\mathbb{Z}$ ), we thus have

twelve transition matrices to consider. We can also ask about various scalar products such as  $\langle F_\lambda^{(r)}, F_\mu^{(r)} \rangle$ . Moreover, we could also consider the basis  $\{\tilde{F}_\lambda^{(r)}\}$  dual to  $\{F_\lambda^{(r)}\}$ , i.e.,

$$\langle F_\lambda^{(r)}, \tilde{F}_\mu^{(r)} \rangle = \delta_{\lambda\mu}.$$

However, these dual bases will not yield any new coefficients since the dual basis to a standard basis is also a standard basis (up to a normalizing factor in the case of  $p_\lambda$ ). We have not systematically investigated these problems. Some miscellaneous results are below.

We first consider scalar products  $\langle F_\mu^{(r,k)}, F_\lambda^{(r,k)} \rangle$ . We can give an explicit formula when  $\mu = (n)$ . In fact, we can give a more general result where  $F_\lambda^{(r,k)}$  is replaced with a ‘‘mixed’’ product.

**Theorem 5.2.** *Let  $\lambda \vdash n$ , and let  $r, r_1, r_2, \dots$  be positive integers. Let  $k, k_1, k_2, \dots$  be integers or even indeterminates. Then*

$$\left\langle F_n^{(r,k)}, \prod_i F_{\lambda_i}^{r_i, k_i} \right\rangle = \frac{k}{rn+k} \prod_{i \geq 1} \frac{k_i}{r_i \lambda_i + k_i} \binom{(rn+k)(r_i \lambda_i + k_i) + \lambda_i - 1}{\lambda_i}.$$

*First proof.* If  $\lambda = (\lambda_1, \lambda_2, \dots)$  then write  $[t^\lambda]$  for the operator that takes the coefficient of  $t_1^{\lambda_1} t_2^{\lambda_2} \dots$ . By equation (2.4) we have

$$\left\langle F_n^{(r,k)}, \prod_i F_{\lambda_i}^{r_i, k_i} \right\rangle = \frac{k}{rn+k} \prod_i \frac{k_i}{\lambda_i + k_i} [t^\lambda] \langle H(1)^{rn+k}, H(t_1)^{r_1 \lambda_1 + k_1} H(t_2)^{r_2 \lambda_2 + k_2} \dots \rangle.$$

Writing  $H(u)^b = \prod (1 - x_i u)^b$ , taking logarithms, expanding in terms of the power sums  $p_k$ , and then exponentiating, we get the well-known result

$$H(u)^b = \sum_\mu z_\mu^{-1} b^{\ell(\mu)} p_\mu u^{|\mu|},$$

where  $\mu$  ranges over all partitions of all integers  $j \geq 0$ . (For the case  $b = 1$ , see [6, (7.22)].)

Since  $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ , we get

$$\begin{aligned} \left\langle F_n^{(r,k)}, \prod_i F_{\lambda_i}^{(r_i, k_i)} \right\rangle &= \frac{k}{rn+k} \prod_i \left( \frac{k_i}{r_i \lambda_i + k_i} \right. \\ &\quad \left. [t^\lambda] \left\langle \sum_{u+n} z_\mu^{-1} (rn+k)^{\ell(\mu)} p_\mu, \prod_{i \geq 1} \left( \sum_{\nu+\lambda_i} z_\nu^{-1} (r_i \lambda_i + k_i)^{\ell(\nu)} t_i^{|\nu|} p_\nu \right) \right\rangle \right) \\ (5.1) \quad &= \frac{k}{rn+k} \prod_i \frac{k_i}{r_i \lambda_i + k_i} \cdot \prod_{i \geq 1} \left( \sum_{\nu+\lambda_i} z_\nu^{-1} (rn+x)^{\ell(\nu)} (r_i \lambda_i + k_i)^{\ell(\nu)} \right). \end{aligned}$$

Now in general (equivalent for instance to [5, Prop. 1.3.7]),

$$\sum_{\nu+m} z_\nu^{-1} u^{\ell(\nu)} = \binom{u+m-1}{m}.$$

Hence the proof follows immediately from equation (5.1).

*Second proof.* From equation (2.4) we see that

$$\left\langle F_n^{(r,k)}, \prod_i F_{\lambda_i}^{r_i, k_i} \right\rangle = \frac{k}{rn+k} \left( \prod_{i \geq 1} \frac{k_i}{r_i \lambda_i + k_i} \right)$$

$$\cdot \left\langle \sum_{a_1 + \dots + a_{rn+k} = n} h_{a_1} \cdots h_{a_{rn+k}}, \prod_i \left( \sum_{b_{i,1} + \dots + b_{i,r_i n + k_i} = \lambda_i} h_{b_{i,1}} \cdots h_{b_{i,r_i n + k_i}} \right) \right\rangle,$$

where  $a_i, b_{i,j} \geq 0$ . Let

$$Z = \frac{k}{rn+k} \prod_{i \geq 1} \frac{k_i}{r_i \lambda_i + k_i}.$$

Now  $\langle h_\lambda, h_\mu \rangle$  is equal to the number of matrices  $(a_{ij})_{i,j \geq 1}$  of nonnegative integers with row sum vector  $\lambda$  and column sum vector  $\mu$  [6, (7.31)]. Hence  $\frac{1}{Z} \langle F_n^{(r,k)}, \prod_i F_{\lambda_i}^{r_i, k_i} \rangle$  is equal to the total number of  $(rn+k) \times (\sum_i (r_i \lambda_i + k_i))$  matrices of nonnegative integers whose entries sum to  $n$ , such that the first  $r_1 \lambda_1 + k_1$  columns sum to  $\lambda_1$ , the next  $r_2 \lambda_2 + k_2$  columns sum to  $\lambda_2$ , etc. Since  $\sum \lambda_i = n$ , if the conditions on the columns is satisfied then the entries will automatically sum to  $n$ . By elementary and well-known reasoning, the number of ways to write  $\lambda_i$  as an ordered sum of  $(rn+k)(r_i \lambda_i + k_i)$  nonnegative integers is  $\binom{(rn+k)(r_i \lambda_i + k_i) + \lambda_i - 1}{\lambda_i}$ , and the proof follows.  $\square$

We now consider the expansion of the symmetric functions  $p_\lambda$ ,  $h_\lambda$ , and  $e_\lambda$  in terms of the basis  $F_n^{(r)}$  (for fixed  $r$ , which we may even regard as an indeterminate).

**Proposition 5.3.** *For  $n \geq 1$  we have*

$$\begin{aligned} F_n^{(r, -rn-1)} &= (-1)^n (rn+1) e_n \\ F_n^{(r, -rn)} &= -r p_n \\ F_n^{(r, -rn+1)} &= (1-rn) h_n. \end{aligned}$$

*Proof.* Putting  $k = -rn - 1$  in equation (2.3) gives  $(-1)^n (rn+1) \sum_{\lambda \vdash n} z_\lambda^{-1} (-1)^{n-\ell(\lambda)} p_\lambda$ . It is well-known that this sum is just  $e_n$ , and the proof of the first equation follows. (We could also substitute  $k = -rn - 1$  in equation (2.2) and simplify.) The other two equations are similar.  $\square$

Now by Proposition 5.3 we have (writing  $d_i = d_i(\lambda)$ )

$$\begin{aligned}
(-1)^n (rn + 1) e_n &= F_n^{(r, -rn-1)} \\
&= [t^n] \left( \sum_{i \geq 0} F_i^{(r)} t^i \right)^{-rn-1} \\
&= [t^n] \sum_{j \geq 0} (-1)^j \binom{rn+j}{j} \left( \sum_{i \geq 1} F_i^{(r)} t^i \right)^j \\
&= \sum_{a_1 + \dots + a_j = n} (-1)^j \binom{rn+j}{j} F_{a_1}^{(r)} \dots F_{a_j}^{(r)} \\
&= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \binom{rn + \ell(\lambda)}{d_1, d_2, \dots, rn} F_\lambda^{(r)},
\end{aligned}$$

where the penultimate sum is over all  $2^{n-1}$  compositions of  $n$ . We have therefore expressed  $e_n$  as a linear combination of  $F_\lambda^{(r)}$ 's. In exactly the same way we obtain

$$\begin{aligned}
-rp_n &= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \binom{rn + \ell(\lambda) - 1}{d_1, d_2, \dots, rn-1} F_\lambda^{(r)} \\
-(rn-1)h_n &= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \binom{rn + \ell(\lambda) - 2}{d_1, d_2, \dots, rn-2} F_\lambda^{(r)}.
\end{aligned}$$

(For  $r = n = 1$ , the last equation becomes  $0 = 0$ , but it is clear that  $h_1 = F_1^{(r)}$ .) Since  $\{e_\mu\}$ ,  $\{p_\mu\}$ ,  $\{h_\mu\}$  and  $\{F_\lambda^{(r)}\}$  are multiplicative bases, we have in principle expressed each  $e_\mu$ ,  $p_\mu$ , and  $h_\mu$  as a linear combination of  $F_\lambda^{(r)}$ 's. We leave open, however, whether there is some more elegant form of these expansions, e.g., a simple combinatorial interpretation of the coefficients.

Similarly, since Theorem 2.1 in the case  $k = 1$  gives the expansion of  $F_n^{(r)}$  in terms of the multiplicative bases  $p_\mu$ ,  $h_\mu$ , and  $e_\mu$ , we in principle also have an expansion of  $F_\lambda^{(r)}$  in terms of these bases, but perhaps a better description is available. We cannot expect a simple product formula for the coefficients in general since for instance the coefficient of  $p_3 p_6$  in the power sum expansion of  $F_{(3,2,1,1,1,1)}^{(1)}$  is equal to  $2 \cdot 7 \cdot 157/3$ .

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