

# New examples of period collapse

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## Abstract

“Period collapse” refers to any situation where the period of the Ehrhart function of a polytope is less than the denominator of that polytope. We study several interesting situations where this occurs, primarily involving triangles. For example: 1) we determine exactly when the Ehrhart function of a right triangle with legs on the axes and slant edge with irrational slope is a polynomial; 2) we find triangles with periods given by any even-index  $k$ -Fibonacci number, and larger denominators; 3) we construct several higher dimensional examples. Several related issues are also discussed, including connections with symplectic geometry.

## 1 Introduction

### 1.1 Background

Let  $\mathcal{P} \subset \mathbb{R}^d$  be a convex polytope. The counting function

$$I_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d)$$

for a positive integer  $t$  is called the *Ehrhart function* of  $\mathcal{P}$ . A classical result of Ehrhart [7] asserts that when  $\mathcal{P}$  is rational,  $I_{\mathcal{P}}(t)$  is a “quasipolynomial” in  $t$ . This means that the function  $I_{\mathcal{P}}(t)$  is a polynomial in  $t$ , with periodic coefficients of integral period. The minimum common period of these coefficients is called the *period* of  $\mathcal{P}$ . While it is known that the period of  $\mathcal{P}$  is bounded from above by the minimum integer  $\mathcal{D}$  such that the vertices of  $\mathcal{D} \cdot \mathcal{P}$  are integral, called the *denominator* of  $\mathcal{P}$ , the precise relationship between  $\mathcal{P}$  and its period can be quite subtle.

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For example, in their study of vertices of Gelfand-Tsetlin polytopes, De Leora and McAllister [6] constructed an infinite family of non-integral polytopes for which the Ehrhart function is still a polynomial. Later, McAllister and Woods [10] extended this result to any dimension  $d \geq 2$ . They showed that, given  $\mathcal{D}$  and  $s$  such that  $s|\mathcal{D}$ , there exists a  $d$ -dimensional polytope with denominator  $\mathcal{D}$  whose Ehrhart quasi-polynomial has period  $s$ . Other interesting related work appears in (for example) [2, 8, 4, 14].

Any situation where the period of  $\mathcal{P}$  is smaller than its denominator is called *period collapse*. In this paper, we further study this phenomenon through several interesting examples, which we now explain.

## 1.2 Irrational triangles

As alluded to above, traditionally the objects of study in Ehrhart theory are *rational* polytopes. The first question we are concerned with here is how frequently an irrational polytope has an Ehrhart function that is a quasi-polynomial or a polynomial. We should think of the denominator of an irrational polytope as being infinite, so our question is about a particularly extreme form of period collapse.

An interesting class of examples comes from fixing positive numbers  $u$  and  $v$  with  $u/v$  irrational, and studying the Ehrhart function of the triangle  $\mathcal{T}_{u,v} \subset \mathbb{R}^2$  with vertices  $(0,0)$ ,  $(1/u, 0)$ , and  $(0, 1/v)$ . It turns out that one can completely determine when the Ehrhart function of such a polytope is a quasipolynomial or a polynomial. To state our result, first recall that any polytope whose Ehrhart function is a polynomial is called *pseudo-integral*. In analogy with this, we will call an (irrational) polytope *pseudo-rational* if its Ehrhart function is a quasipolynomial. Of course, if  $\mathcal{T}$  is pseudo-rational, then any scaling of  $\mathcal{T}$  by a positive integer is as well; we will therefore call a pseudo-rational triangle *primitive* if no scaling  $1/t \cdot \mathcal{T}$  for an integer  $t > 1$  is pseudo-rational.

We can now state precisely which triangles in the family  $\mathcal{T}_{u,v}$  are pseudo-rational and pseudo-integral. In fact,  $u$  and  $v$  must be certain special conjugate quadratic irrationalities:

**Theorem 1.1.** *Let  $u$  and  $v$  be positive numbers with  $u/v$  irrational.*

(i) *The triangle  $\mathcal{T}_{u,v}$  is primitive and pseudo-rational if and only if*

$$\begin{aligned} u + v &= \alpha \\ 1/u + 1/v &= \beta, \end{aligned} \tag{1.1}$$

*for positive integers  $\alpha, \beta$ . The period of this quasipolynomial divides  $\alpha$ .*

(ii) The triangle  $\mathcal{T}_{u,v}$  is primitive and pseudo-integral if and only if (1.1) is satisfied, and in addition, either  $\alpha = 1$  or

$$(\alpha, \beta) \in \{(3, 3), (2, 4)\}.$$

To simplify the notation, we call a pair of positive numbers  $(u, v)$  with  $u/v$  irrational and  $(u, v)$  satisfying (1.1) *admissible*, and we also call a triangle  $\mathcal{T}_{u,v}$  admissible when  $(u, v)$  is an admissible pair. To get a feel for Theorem 1.1, the following example, which we prove in §2.1, is illustrative.

**Example 1.2.** Let  $u/v$  be irrational. The pseudo-integral triangle in the family  $\mathcal{T}_{u,v}$  with smallest area corresponds to  $(u, v) = (\tau^2, 1/\tau^2)$ , where  $\tau = \frac{1+\sqrt{5}}{2}$  is the Golden Ratio.

One of the key steps in the proof of the “only if” direction of the first bullet point involves a slightly stronger statement than what is required, which is of potentially independent interest. Recall that a sequence  $f(n)$  is  $\mathcal{P}$ -recursive, of order  $k$ , if there are polynomials  $p_0, \dots, p_k$ , not all 0, such that the recurrence relation

$$p_k(n+k)f(n+k) + \dots + p_0(n)f(n) = 0$$

holds for all nonnegative integer  $n$ . In general, it can be difficult to show that a sequence is not  $\mathcal{P}$ -recursive. However, natural examples of sequences which are not  $\mathcal{P}$ -recursive are given by the following.

**Theorem 1.3.** *Let  $u$  and  $v$  be positive numbers with  $u/v$  irrational, and assume that  $1/u + 1/v$  and  $u + v$  are rational, but  $\mathcal{T}_{u,v}$  is not a positive integer scaling of a primitive pseudo-rational triangle. Then the sequence  $f(n) := I_{\mathcal{T}_{u,v}}(n)$  is not  $\mathcal{P}$ -recursive.*

To introduce our final set of results, note that since the Ehrhart functions for pseudo-rational  $\mathcal{T}_{u,v}$  are quasipolynomials, one can ask to what degree some of the basic results from Ehrhart theory in the rational case apply. In fact, versions of Ehrhart-Macdonald reciprocity, as well as the nonnegativity theorem and monotonicity theorem of the third author, hold for these triangles; see Proposition 2.4.

Although our primary interest here is for triangles, we can also give examples of irrational polytopes with quasipolynomial Ehrhart functions in any dimension; see Example 2.5, Example 2.6, and Example 2.7.

### 1.3 Criteria for period collapse for rational triangles

When  $(u, v)$  are rational, the period collapse question for  $\mathcal{T}_{u,v}$  is less well understood than in the irrational case. Nevertheless, we find many new examples of rational triangles of this form exhibiting significant period collapse. The key is the following criterion.

**Theorem 1.4.** *Let  $u = q/p$  and  $v = s/r$  in lowest terms. Then  $q$  is a period of the Ehrhart quasipolynomial for  $\mathcal{T}_{u,v}$  if*

$$s|p, \quad p|(rq+1), \quad \text{and} \quad \gcd\left(\frac{rq+1}{p}, s\right) = 1. \quad (1.2)$$

For example, if  $q = 1$ , then one obtains the McAllister and Woods example of period collapse mentioned above as a corollary of Theorem 1.4. Indeed, the theorem implies that the triangle with vertices  $(0, 0)$ ,  $(p, 0)$  and  $(0, \frac{p-1}{p})$  is a pseudo-integral triangle with denominator  $p$ . This triangle is unimodularly equivalent to the pseudo-integral triangle found by McAllister and Woods [10, Theorem 2.2], which has vertices  $(0, 0)$ ,  $(p, 0)$  and  $(1, \frac{p-1}{p})$ , via the map

$$\varphi(x) = x \begin{pmatrix} -1 & 0 \\ -p & 1 \end{pmatrix} + (p, 0).$$

Theorem 1.4 can be used to construct other pseudo-integral triangles, via the following result.

**Corollary 1.5.** *Let  $u = q/p, v = s/r$  in lowest terms. The triangle  $\mathcal{T}_{u,v}$  is pseudo-integral if*

$$s|p, \quad p|(rq+1), \quad \gcd\left(\frac{rq+1}{p}, s\right) = 1 \quad (1.3)$$

and

$$q|r, \quad r|(sp+1), \quad \gcd\left(\frac{sp+1}{r}, q\right) = 1. \quad (1.4)$$

The criteria of (1.2) also have a nice relationship with the  $k$ -Fibonacci numbers. Specifically, we can use Theorem 1.4 to construct triangles with period dividing any even-index  $k$ -Fibonacci number, and high denominator (Theorem 3.2). If  $s = p$  and  $r = q$ , then the condition (1.2) is also sufficient for  $q$  to be a period, which we also show (Theorem 3.1).

## 1.4 Relationship with symplectic geometry

We briefly remark that the triangles  $\mathcal{T}_{u,v}$  with  $u/v$  irrational satisfying (1.1) seem to have interesting relationships with symplectic geometry. For example, the triangle from (1.2) is closely related to a foundational result of McDuff and Schlenk [11] about “symplectic” embedding problems; see also [4]. Some of the other triangles from (1.1) also seem to be relevant in the context of symplectic embeddings. This is further explored in [5].

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## 2 Irrational triangles with Ehrhart quasipolynomials

### 2.1 Proof of the main theorem

Here we prove Theorem 1.1 in several steps.

*Proof. Step 1.* We first prove the “if” direction of the first bullet point.

To start, we want to show that the number of nonnegative integer solutions  $(x, y)$  to

$$ux + vy \leq t \tag{2.1}$$

is a quasipolynomial in  $t$  for nonnegative integer  $t$ . Let  $0 \leq m \leq \lfloor t/\alpha \rfloor$  be an integer. By (1.1), the number of solutions to (2.1) with  $x = m$  and  $y \geq m$  is  $1 + \lfloor \frac{t-m\alpha}{v} \rfloor$ . Similarly, the number of solutions to (2.1) with  $y = m$  and  $x > m$  is  $\lfloor \frac{t-m\alpha}{u} \rfloor$ . It follows that the number of solutions to (2.1) is

$$\sum_{m=0}^{\lfloor \frac{t}{\alpha} \rfloor} \left( 1 + \left\lfloor \frac{t-m\alpha}{v} \right\rfloor + \left\lfloor \frac{t-m\alpha}{u} \right\rfloor \right). \tag{2.2}$$

By (1.1), we know that

$$\begin{aligned} \left\lfloor \frac{t-m\alpha}{v} \right\rfloor &= \lfloor (t-m\alpha)(\beta - 1/u) \rfloor \\ &= \left\lfloor (t\beta - m\alpha\beta) - \frac{(t-m\alpha)}{u} \right\rfloor \\ &= (t\beta - m\alpha\beta) + \left\lfloor -\frac{(t-m\alpha)}{u} \right\rfloor. \end{aligned}$$

We can therefore rewrite (2.2) as

$$\sigma(t) + \sum_{m=0}^{\lfloor \frac{t}{\alpha} \rfloor} (t\beta - m\alpha\beta), \tag{2.3}$$

where

$$\sigma(t) := \begin{cases} 1 & \text{if } t \text{ is divisible by } \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

We now claim that (2.3) is a quasipolynomial in  $t$ , of period dividing  $\alpha$ . We can rewrite (2.3) as

$$\sigma(t) + (\lfloor t/\alpha \rfloor + 1) \left( t\beta - \frac{\alpha\beta\lfloor t/\alpha \rfloor}{2} \right). \quad (2.4)$$

Let  $z := t \pmod{\alpha}$ . Then  $\lfloor \frac{t}{\alpha} \rfloor = \frac{t-z}{\alpha}$ . Hence we can rewrite (2.4) as

$$\begin{aligned} & \sigma(t) + \frac{(t - (z - \alpha))(t\beta + z\beta)}{2\alpha} \\ &= \frac{t^2\beta + t\alpha\beta + z\beta(\alpha - z) + 2\alpha\sigma(t)}{2\alpha}. \end{aligned} \quad (2.5)$$

The coefficients of  $t^2$  and  $t$  in (2.5) do not depend on  $t$ , and the constant term only depends on the equivalence class of  $t$  modulo  $\alpha$ . This proves the pseudo-rational part of the “if” direction of the first bullet point.

*Step 2.* We now begin the proof of the “only if” direction (we will return to the rest of the proof of the “if direction” in a later step). Let  $(u, v)$  be any pair of positive numbers with  $v/u$  irrational.

We will use the following, which was explained to us by Bjorn Poonen:

**Lemma 2.1.** *If  $u/v$  is irrational, then  $I_{\mathcal{T}_{u,v}}(t) = \frac{1}{2uv}t^2 + \frac{1}{2}(\frac{1}{u} + \frac{1}{v})t + o(t)$  for  $t \in \mathbb{R}_{>0}$ .*

*Proof.* By scaling, we may assume that  $u = 1$ . By counting in each vertical line, we see that

$$I_{\mathcal{T}_{1,1/v}}(t) = \sum_{m=0}^{\lfloor t \rfloor} (\lfloor t/v - m/v \rfloor + 1). \quad (2.6)$$

It is convenient to define a function  $f(x)$  by

$$\lfloor x \rfloor + 1 = x + f(x).$$

We can then rewrite (2.6) as

$$I_{\mathcal{T}_{1,v}}(t) = \sum_{m=0}^{\lfloor t \rfloor} (t/v - m/v) + \sum_{m=0}^{\lfloor t \rfloor} f(t/v - m/v). \quad (2.7)$$

Now note that the  $m$ th term in the first sum computes the area of the trapezoid defined by  $m - 1/2 \leq x \leq m + 1/2, 0 \leq y \leq t/v - x/v$ . The

first sum is therefore within  $O(1)$  of the area of the triangle defined by  $-1/2 \leq x \leq t, 0 \leq y \leq t/v - x/v$ , and so we have

$$\sum_{m=0}^{\lfloor t \rfloor} (t/v - m/v) = (t^2 + t)/2v + O(1). \quad (2.8)$$

The second sum is a sum of values of a bounded, integrable, periodic function  $f$  at points that become equidistributed mod 1 as  $t \rightarrow \infty$  (by Weyl's criterion for uniform distribution), and so we find that

$$\sum_{m=0}^{\lfloor t \rfloor} f(t/v - m/v) = t \int_0^1 f(x) dx + o(t) = t/2 + o(t). \quad (2.9)$$

Lemma 2.1 now follows by combining (2.7), (2.8), and (2.9).  $\square$

*Step 3.* Now assume that  $I_{\mathcal{T}_{u,v}}(t)$  is a quasipolynomial in the positive integer  $t$ , of period  $C$ . Write  $I_{\mathcal{T}_{u,v}}(pt) = A(C^2 t^2) + B(Ct) + D$ . We know that the number  $I_{\mathcal{T}_{u,v}}(t)$  must always be an integer. Hence, the numbers  $A$  and  $B$  here must be rational. Lemma 2.1 now implies that  $u + v$  and  $1/u + 1/v$  must be rational as well.

So, if  $\mathcal{T}_{u,v}$  is pseudo-rational, we know that  $u + v$  and  $1/u + 1/v$  must be rational; write  $u + v = \tilde{\alpha}, 1/u + 1/v = \tilde{\beta}$ . The “only if” direction will then follow from the following claim, which is closely related to Theorem 1.3:

**Claim 2.2.** *Unless  $\tilde{\beta}$  and  $\tilde{\alpha}\tilde{\beta}$  are both integers, the Ehrhart function of  $\mathcal{T}_{u,v}$  is not  $\mathcal{P}$ -recursive.*

*Proof of claim.* By (2.2) (which still holds even though  $\tilde{\alpha}, \tilde{\beta}$  are no longer assumed integral), we can write  $I_{\mathcal{T}_{u,v}}(t)$  as

$$\sum_{m=0}^{\lfloor \frac{t}{\tilde{\alpha}} \rfloor} \left( 1 + \left\lfloor \frac{t - m\tilde{\alpha}}{v} \right\rfloor + \left\lfloor \frac{t - m\tilde{\alpha}}{u} \right\rfloor \right).$$

We know that  $\lfloor \frac{t - m\tilde{\alpha}}{v} \rfloor = \lfloor \tilde{\beta}(t - m\tilde{\alpha}) - \frac{t - m\tilde{\alpha}}{u} \rfloor$ . Hence we can rewrite this sum as

$$\text{err}(t) + \sigma(t) + \sum_{m=0}^{\lfloor \frac{t}{\tilde{\alpha}} \rfloor} \lfloor \tilde{\beta}(t - m\tilde{\alpha}) \rfloor, \quad (2.10)$$

where

$$\sigma(t) := \begin{cases} 1, & \text{if } t/\tilde{\alpha} \text{ is an integer,} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\text{err}(t) := \# \left\{ 0 \leq m \leq \lfloor t/\tilde{\alpha} \rfloor \mid \{\tilde{\beta}(t - m\tilde{\alpha})\} > \left\{ \frac{t - m\tilde{\alpha}}{u} \right\} \right\},$$

where  $\{\cdot\}$  denotes the fractional part function.

Now assume that  $I_{\mathcal{T}_{u,v}}(t)$  is  $\mathcal{P}$ -recursive, and write the recurrence as

$$p_s(t) \left( \text{err}(t) + q(t) \right) + \cdots + p_0(t) \left( \text{err}(t - s) + q(t - s) \right) = 0, \quad (2.11)$$

where  $q$  is given by  $q := \sigma(t) + \sum_{m=0}^{\lfloor \frac{t}{\tilde{\alpha}} \rfloor} \lfloor \tilde{\beta}(t - m\tilde{\alpha}) \rfloor$ . The function  $q$  is a quasipolynomial in  $t$ . Now write  $\tilde{\alpha} = \frac{p}{q}$  in lowest terms and introduce the function

$$\text{head}(t) = \# \left\{ 0 \leq m \leq (q-1) \mid \{\tilde{\beta}(t - m\tilde{\alpha})\} > \left\{ \frac{t - m\tilde{\alpha}}{u} \right\} \right\}.$$

We know by its definition that  $\text{err}(t) - \text{err}(t - p) = \text{head}(t)$  for all  $t \geq p$ . Then for  $t \geq p + s$ , by applying (2.11) twice, we have

$$\begin{aligned} & p_s(t) \text{head}(t) + \cdots + p_0(t) \text{head}(t - s) + p_s(t)q(t) \\ & - p_s(t - p)q(t - p) + \cdots + p_0(t)q(t - s) - p_0(t - p)q(t - p - s) \\ & + \text{err}(t - p)(p_s(t) - p_s(t - p)) + \cdots + \text{err}(t - s - p)(p_0(t) - p_0(t - p)) = 0. \end{aligned} \quad (2.12)$$

We will now derive a contradiction.

*Step 4.* Assume first that  $\tilde{\beta}$  is not an integer, and write  $\tilde{\beta} = k/l$ . Let  $C$  be the period of the quasipolynomial  $q$ . Introduce the set  $S = \{1 + iCl \mid i \in \mathbb{Z}_{\geq 0}\}$ . Then for any  $t \in S$ ,  $\{t\tilde{\beta}\}$  is some fixed nonzero number  $a_0$  independent of  $t$ . We now claim there is some  $\varepsilon > 0$  with the property that if  $\{\frac{t}{u}\}$  is in  $(a_0 - \varepsilon, a_0 + \varepsilon)$ , then  $\text{head}(t)$  is determined by whether or not  $a_0 > \{\frac{t}{u}\}$ , and  $\text{head}(t)$  will differ depending on whether or not this condition is met.

To see this, introduce the two homeomorphisms  $f_1, f_2$  from  $[0, 1] \pmod{1}$  to itself given by:

$$f_1(x) = \{x - \tilde{\alpha}\tilde{\beta}\}, \quad f_2(x) = \left\{ x - \frac{\tilde{\alpha}}{u} \right\}.$$

Since  $\tilde{\alpha}$  is rational, while  $u$  is irrational, for rational  $x$  we can never have  $f_1^m(x) = f_2^m(x)$  for any positive integer  $m$ . With this in mind, consider  $f_1^m(a_0), \dots, f_1^{q-1}(a_0)$ . If we take  $\varepsilon$  sufficiently small, we can guarantee that for any  $y \in (a_0 - \varepsilon, a_0 + \varepsilon)$ ,  $f_2^i(y) \neq f_1^i(a_0)$  for any  $1 \leq i \leq q-1$ . By shrinking  $\varepsilon$  if necessary, we can also conclude that for any  $y$  in this interval,  $f_2^i(y) \neq 0$ . It follows that if  $\{\frac{t}{u}\}$  is in  $(a_0 - \varepsilon, a_0 + \varepsilon)$ , then  $\text{head}(t)$  is determined by whether or not  $a_0 > \{\frac{t}{u}\}$ ; to emphasize,  $\text{head}(t)$  will be different depending on whether or not this condition is met, as claimed.

*Step 5.* Assume now that  $t \in S$ , and  $t \geq s + p$ .

*Claim 1.* By decreasing  $\varepsilon$  if necessary, if  $\{\frac{t}{u}\}$  is in  $(a_0 - \varepsilon, a_0 + \varepsilon)$ , then  $\text{head}(t \pm 1), \dots, \text{head}(t \pm s)$  do not depend on  $t$ .

*Proof of Claim.* Let  $a_1^\pm, \dots, a_s^\pm$  be the rational numbers (mod 1) defined by  $a_i^\pm := \{\tilde{\beta}(t \pm i)\} = \{a_0 \pm i\tilde{\beta}\}$ . For any  $1 \leq i \leq s$ , we can never have  $\{a_0 \pm \frac{i}{u}\} = a_i^\pm$ , since  $\{a_0 \pm \frac{i}{u}\}$  is irrational. Consider then  $a_i^\pm$  and  $z_i^\pm = \{a_0 \pm \frac{i}{u}\}$ . Essentially the same argument<sup>1</sup> as in Step 5 allows us to conclude that if  $\tilde{\varepsilon}$  is sufficiently small, and  $x_i$  is some irrational number in  $(z_i^\pm - \tilde{\varepsilon}, z_i^\pm + \tilde{\varepsilon})$ , then  $\#\left\{0 \leq m \leq (q-1) \mid \{a_i^\pm - m\tilde{\alpha}\tilde{\beta}\} > \{x_i - \frac{m\tilde{\alpha}}{u}\}\right\}$  does not depend on  $x_i$ . Since we can make  $\frac{t \pm i}{u}$  arbitrarily close to  $z_i^\pm$  by making  $\{\frac{t}{u}\}$  sufficiently close to  $a_0$ , the claim follows.

We can now complete the proof, in the case where  $\tilde{\beta}$  is not an integer. Let  $p_{s-\tilde{s}}$  be one of the polynomials  $p_j$ , with the property that no other  $p_j$  has higher degree. As  $t$  ranges over  $S$ ,  $\{t/u\}$  is dense in  $(0, 1)$ . Take an infinite sequence  $t_i$  such that  $\{(t_i - \tilde{s})/u\} \in (a_0, a_0 + \varepsilon)$  and  $t_i - \tilde{s} \in S$ . Now it follows from Claim 1, (2.12), and the fact that we are fixing  $t$ , mod  $C$ , that if  $i$  is sufficiently large (so that  $p_{s-\tilde{s}}(t_i) \neq 0$ ) then

$$\text{head}(t_i - \tilde{s}) = a(t_i)/p_{s-\tilde{s}}(t_i) - R(t_i)/p_{s-\tilde{s}}(t_i), \quad (2.13)$$

where  $a$  is some fixed polynomial of  $t_i$ , whose coefficients do not depend on  $i$ . Meanwhile, the term  $R(t)$  is given by

$$R(t_i) = \sum_{j=0}^s \text{err}(t_i - p - j)(p_{s-j}(t_i) - p_{s-j}(t_i - p)).$$

The term  $R(t_i)/p_{s-\tilde{s}}(t_i)$  is controlled by the following:

*Claim 2.* Let  $x_l \in S$  be a sequence of points tending to  $+\infty$ . Then

$$\lim_{l \rightarrow \infty} R(x_l)/p_{s-\tilde{s}}(x_l) \rightarrow M,$$

where  $M$  is some constant.

*Proof of Claim.* We know that the degree of  $p_{s-\tilde{s}}$  is strictly greater than the degree of any of the polynomials  $p_{s-j}(t) - p_{s-j}(t - p)$ . Thus, the claim will follow if we can show that

$$\text{err}(t) = l(t) + o(t), \quad (2.14)$$

for  $t \in S$ , for some linear polynomial  $l$ . This follows by combining (2.10) and Lemma 2.1 (note that we are fixing the equivalence class of  $t$ , mod  $C$ ).  $\square$

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<sup>1</sup>To elaborate slightly on this, it is probably worth emphasizing that we can never have  $\{a_i^\pm - m\tilde{\alpha}\tilde{\beta}\} = \{z_i^\pm - \frac{m\tilde{\alpha}}{u}\}$  for  $0 \leq m \leq (q-1)$ .

Given this claim, we can complete the proof. Since  $\frac{t_i - \tilde{s}}{u} \in (a_0, a_0 + \varepsilon)$ , by Claims 1 and 2, and (2.13), the rational function  $a(t_i)/p_{s-\tilde{s}}(t_i)$  must have a horizontal asymptote as  $t_i \rightarrow \infty$ . Now choose some collection of  $\hat{t}_i - \tilde{s} \in S$  such that  $\{(\hat{t}_i - \tilde{s})/u\} \in (a_0 - \varepsilon, a_0)$  and  $p_{s-\tilde{s}}(\hat{t}_i) \neq 0$ . By again appealing to Claims 1 and 2 and (2.13), the rational function  $a(\hat{t}_i)/p_{s-\tilde{s}}(\hat{t}_i)$  has a horizontal asymptote as  $\hat{t}_i$  goes to  $+\infty$ . Since  $a_0 > \{\frac{\hat{t}_i - \tilde{s}}{u}\}$  while  $a_0 < \{\frac{t_i - \tilde{s}}{u}\}$ , these two asymptotes are different, by Step 5; this can not happen for a rational function. This is a contradiction.

*Step 6.* We can therefore assume that  $\tilde{\beta}$  is an integer. We can also assume that  $q \geq 2$ , or else  $\tilde{\alpha}\tilde{\beta}$  would be an integer. We know that  $\{\tilde{\beta}(t - \tilde{s})\} = 0$ , and  $\{\tilde{\beta}(t - \tilde{s}) - \tilde{\alpha}\}$  is some nonzero number  $b_0$  independent of  $t$ . Essentially the same argument in the previous step can now be used to conclude that there is some small  $\varepsilon > 0$ , such that if  $\{\frac{t - \tilde{s} - \tilde{\alpha}}{u}\} \in (b_0 - \varepsilon, b_0 + \varepsilon)$ , then  $\text{head}(t - \tilde{s})$  will differ based on whether or not  $b_0 > \{\frac{t - \tilde{s} - \tilde{\alpha}}{u}\}$ , and  $\text{head}(t - \tilde{s} \pm 1), \dots, \text{head}(t - \tilde{s} \pm s)$  do not depend on  $t$ . The fact that  $\{\frac{t - \tilde{s} - \tilde{\alpha}}{u}\}$  is dense as  $t - \tilde{s}$  ranges over  $S$  now gives a contradiction by essentially repeating the arguments from the end of the previous step. This proves Claim 2.2.  $\square$

*Step 7.* The arguments in the previous steps have proved the first bullet point, since the sequence of values of a quasipolynomial is  $\mathcal{P}$ -recursive. This, together with Claim 2.2 now implies Theorem 1.3.

For the second bullet point, if  $I_{\mathcal{T}_{u,v}}(t)$  is primitive and pseudo-integral, then  $(u, v)$  must be conjugate admissible quadratic irrationalities, and in particular  $I_{\mathcal{T}_{u,v}}(t)$  must be given by (2.5). Then setting  $t = 0$  (or any multiple of  $\alpha$ ) gives

$$\frac{z\beta(\alpha - z) + 2\alpha\sigma(t)}{2\alpha} = 1 \quad (2.15)$$

for all equivalence classes  $z$ , by (2.5). Now choose  $t$  such that  $t \equiv 1 \pmod{\alpha}$ . By (2.15), this gives

$$\frac{\beta(\alpha - 1)}{2\alpha} = 1,$$

so  $\beta = 2\alpha/(\alpha - 1)$ . The only solutions to this equation with  $\alpha > 1$  and  $\beta$  an integer are (3, 3) and (2, 4).

Conversely, if  $\alpha = 1$ , or  $(\alpha, \beta) \in \{(3, 3), (2, 4)\}$ , then  $\frac{z\beta(\alpha - z) + 2\alpha\sigma(t)}{2\alpha} = 1$  for all equivalence classes  $z$ , hence the result follows, again by (2.5).

This completes the proof of Theorem 1.1.  $\square$

We can now give the proof that was owed for Example 1.2.

*Proof.* Let  $(u, v)$  be admissible. Then  $uv = \frac{\alpha}{\beta}$ . To minimize the area of  $\mathcal{T}_{u,v}$ , we would like to maximize  $\frac{\alpha}{\beta}$ . By Theorem 1.1, if  $\alpha > 1$ , then  $(\alpha, \beta) \in$

$\{(3, 3), (2, 4)\}$ . The largest possible value in these two cases is  $\frac{\alpha}{\beta} = 1$ , which is uniquely obtained by the “golden mean” triangle.

For  $\alpha = 1$ , the only possible value of  $\beta$  that could give an equally large  $\frac{\alpha}{\beta}$  is when  $\beta = 1$ . There are no real numbers satisfying  $u + v = 1, 1/u + 1/v = 1$ , however.  $\square$

**Remark 2.3.** Similar arguments can be used to give another characterization of the “golden mean” triangle: it is the only pseudo-integral triangle in the family  $\mathcal{T}_{u,v}$  where  $u$  and  $v$  are quadratic irrational algebraic integers. We omit the proof for brevity.

## 2.2 Properties of admissible irrational triangles

We now prove some properties of the Ehrhart functions of admissible triangles that mirror properties from the rational case; compare [1, §3], [3]. For simplicity, we state some of the results for pseudo-integral triangles, although we expect they should hold more generally. Along those lines, recall from [1, Lem. 3.9] that if  $\mathcal{T}_{u,v}$  is pseudo-integral, then we can write

$$\sum_{t \geq 0} I_{\mathcal{T}_{u,v}}(t) z^t = \frac{g_{u,v}(z)}{(1-z)^3},$$

where  $g_{u,v}(z)$  is a polynomial of degree at most 2.

**Proposition 2.4.** *Let  $\mathcal{T}_{u,v}$  be admissible. Then the Ehrhart function of  $\mathcal{T}_{u,v}$  satisfies*

- (Reciprocity) *If  $t$  is positive, then*

$$I_{\mathcal{T}_{u,v}}(-t) = \{\#(t\mathcal{T}_{u,v}^o \cap \mathbb{Z}^2)\} + \mu(t),$$

*where the superscript  $o$  denotes the interior, and the function  $\mu(t)$  is defined by  $\mu(t) = 0$  if  $\alpha|t$ , and 1 otherwise.*

- (Nonnegativity) *If  $\mathcal{T}_{u,v}$  is pseudo-integral, then each coefficient of  $g_{u,v}$  is nonnegative.*
- (Monotonicity) *If  $\mathcal{T}_{u,v}$  and  $\mathcal{T}_{u',v'}$  are both pseudo-integral, and  $\mathcal{T}_{u,v} \subset \mathcal{T}_{u',v'}$ , then each coefficient of  $g_{u,v}$  is less than or equal to the corresponding coefficient of  $g_{u',v'}$ .*

*Proof.* For the first bullet point, note that the number of lattice points on the boundary of the triangle with vertices  $(t/u, 0)$ ,  $(0, t/v)$  and  $(0, 0)$  is  $\lfloor t/u \rfloor + \lfloor t/v \rfloor$ , plus the number of points on the slant edge. We know that

$$\lfloor t/u \rfloor + \lfloor t/v \rfloor = t\beta - 1,$$

and we know that  $ux + vy = t$  only if  $x = y$ . The first bullet point now follows by subtracting the number of lattice points on the boundary from the formula in (2.5).

For the second and third bullet points, note that if  $g_{u,v}(z) = a_0 + a_1z + a_2z^2$ , then  $a_0 = 1$ ,  $a_1 = I_{\mathcal{T}_{u,v}}(1) - 3$ , and  $a_2 = 3 - 3I_{\mathcal{T}_{u,v}}(1) + I_{\mathcal{T}_{u,v}}(2)$  (here, we are implicitly using the fact that  $I_{\mathcal{T}_{u,v}}(0) = 1$ , as can be seen by (2.15).) To show the second bullet point, we therefore first have to show that  $I_{\mathcal{T}_{u,v}}(1) \geq 3$ , or equivalently, by (2.5), that  $\beta/\alpha + \beta \geq 4$ . If  $(\alpha, \beta) \in \{(3, 3), (2, 4)\}$ , then this holds, so by Theorem 1.1 we can assume that  $\alpha = 1$ . Since there are no quadratic irrationalities with  $\alpha = 1$  and  $\beta \leq 3$ , nonnegativity for  $a_1$  follows. For nonnegativity of  $a_2$ , we need to show that  $(\beta - \alpha\beta)/\alpha \geq -2$ . If  $\alpha = 1$ , then this is automatic; if  $(\alpha, \beta) \in \{(3, 3), (2, 4)\}$ , then it holds as well. This proves the second bullet point.

For the third bullet point, we are given that  $1/u \leq 1/u'$  and  $1/v \leq 1/v'$ . That  $a_0 \leq a'_0$  and  $a_1 \leq a'_1$  are immediate; to see that  $a_2 \leq a'_2$ , we need to show that  $(1/\alpha)(\beta - 1) \leq (1/\alpha')(\beta' - 1)$ . This follows from  $1/u \leq 1/u'$ ,  $1/v \leq 1/v'$ .  $\square$

### 2.3 Examples in other dimensions

Here we briefly mention some examples of polytopes in other dimensions that are not rational, but nevertheless have Ehrhart functions that are polynomials.

In dimension 1, such examples are easy to come by:

**Example 2.5.** Let  $\mathcal{P} = [u, v] \subset \mathbb{R}$ , where  $u$  and  $v$  are irrational numbers with  $u - v = m$  and  $m$  is an integer. Then for positive integer  $t$ ,  $I_{\mathcal{P}}(t) = tm$

*Proof.* We know that  $I_{\mathcal{P}}(t) = \lfloor tu \rfloor - \lfloor tv \rfloor = tm$ .  $\square$

In higher dimensions, we do not currently know many examples of truly different character. However, one has:

**Example 2.6.** Let  $\mathcal{P}$  denote the polytope with vertices

$$(0, 0, \dots, 0), (1, 0, \dots, 0), (0, 1, \dots, 0), \\ (0, 0, 1, \dots, 0), \dots, (0, \dots, u, 0), (0, \dots, 0, v),$$

where  $(u, v)$  are positive admissible quadratic irrationalities. Then  $I_{\mathcal{P}}(t)$  is a polynomial in  $t$ .

A more interesting example is given in dimension three:

**Example 2.7.** The polytope with vertices  $(0, 0, 0)$ ,  $(\frac{1}{2}, 0, 0)$ ,  $(0, 2 + \sqrt{2}, 0)$  and  $(0, 0, 2 - \sqrt{2})$  has an Ehrhart function which is a polynomial.

The proof of Example 2.6 is immediate; we defer the proof of Example 2.7 to §3.3.

### 3 Rational examples

We now give the proof of Theorem 1.4.

*Proof.* Firstly, it is easy to see that  $\gcd(rq, ps) = 1$ . Let  $\xi_m$  denote a primitive  $m$ th root of unity. By [1, Theorem 2.10], we have

$$\begin{aligned}
I_{\mathcal{T}_{q/p,s/r}} &= \frac{1}{2 \cdot rq \cdot ps} (t \cdot pr)^2 + \frac{1}{2} (t \cdot pr) \left( \frac{1}{rq} + \frac{1}{ps} + \frac{1}{rq \cdot ps} \right) \\
&\quad + \frac{1}{4} \left( 1 + \frac{1}{rq} + \frac{1}{ps} \right) + \frac{1}{12} \left( \frac{rq}{ps} + \frac{ps}{rq} + \frac{1}{rq \cdot ps} \right) \\
&\quad + \frac{1}{rq} \sum_{j=1}^{rq-1} \frac{\xi_{rq}^{-jt \cdot pr}}{(1 - \xi_{rq}^{j \cdot ps})(1 - \xi_{rq}^j)} + \frac{1}{ps} \sum_{l=1}^{ps-1} \frac{\xi_{ps}^{-lt \cdot pr}}{(1 - \xi_{ps}^{l \cdot rq})(1 - \xi_{ps}^l)} \\
&= \frac{pr}{2qs} \cdot t^2 + \frac{1}{2} \left( \frac{p}{q} + \frac{r}{s} + \frac{1}{qs} \right) t + \frac{1}{4} \left( 1 + \frac{1}{rq} + \frac{1}{ps} \right) \\
&\quad + \frac{1}{12} \left( \frac{rq}{ps} + \frac{ps}{rq} + \frac{1}{rq \cdot ps} \right) \\
&\quad + \frac{1}{rq} \sum_{j=1}^{rq-1} \frac{\xi_q^{-jtp}}{(1 - \xi_{rq}^{j \cdot ps})(1 - \xi_{rq}^j)} + \frac{1}{ps} \sum_{l=1}^{ps-1} \frac{\xi_s^{-ltr}}{(1 - \xi_{ps}^{l \cdot rq})(1 - \xi_{ps}^l)}. \quad (3.1)
\end{aligned}$$

Then it suffices to show that

$$\frac{1}{ps} \sum_{l=1}^{ps-1} \frac{\xi_s^{-ltr}}{(1 - \xi_{ps}^{l \cdot rq})(1 - \xi_{ps}^l)}$$

is a constant function in  $t$ . In fact, writing  $l = is + u : 0 \leq i < p, 0 \leq u < s$  and using the fact that  $rq \equiv -1 \pmod{p}$ , we have

$$\begin{aligned}
&\frac{1}{ps} \sum_{l=1}^{ps-1} \frac{\xi_s^{-ltr}}{(1 - \xi_{ps}^{l \cdot rq})(1 - \xi_{ps}^l)} \\
&= \frac{1}{ps} \sum_{i=1}^{p-1} \frac{1}{(1 - \xi_p^{irq})(1 - \xi_p^i)} + \frac{1}{ps} \sum_{u=1}^{s-1} \xi_s^{-utr} \sum_{i=0}^{p-1} \frac{1}{(1 - \xi_{ps}^{(is+u)rq})(1 - \xi_{ps}^{is+u})} \\
&= \frac{1}{ps} \sum_{i=1}^{p-1} \frac{1}{(1 - \xi_p^{irq})(1 - \xi_p^i)} + \frac{1}{ps} \sum_{u=1}^{s-1} \xi_s^{-utr} \sum_{i=0}^{p-1} \frac{1}{(1 - \xi_{ps}^{urq-is})(1 - \xi_{ps}^{u+is})} \quad (3.2)
\end{aligned}$$

Keeping in mind that  $s|(rq+1)$  and  $\gcd\left(\frac{rq+1}{p}, s\right) = 1$ , we find that

$$ps \nmid u(rq+1), \quad \text{and} \quad ps|u(rq+1)p$$

for any  $1 \leq u \leq s-1$ . By [4, Lemma 2.1], we deduce that

$$\sum_{i=0}^{p-1} \frac{1}{(1 - \xi_{ps}^{urq-is})(1 - \xi_{ps}^{u+is})} = 0$$

for any  $1 \leq u \leq s-1$ . Therefore, we have

$$\frac{1}{ps} \sum_{l=1}^{ps-1} \frac{\xi_s^{-ltr}}{(1 - \xi_{ps}^{l,rq})(1 - \xi_{ps}^l)} = \frac{1}{ps} \sum_{i=1}^{p-1} \frac{1}{(1 - \xi_p^{irq})(1 - \xi_p^i)},$$

which is a constant function in  $t$ . It follows from (3.1) that  $q$  is a quasi-period of  $I_{\mathcal{T}_{q/p, s/r}}(t)$ . □

Next we prove Corollary 1.5.

*Proof.* By Theorem 1.4, condition (1.3) implies that  $q$  is a quasi-period of  $I_{\mathcal{T}_{q/p, s/r}}(t)$ . On the other hand, it is obvious that

$$(x, y) \mapsto (y, x)$$

is a bijection between lattice points in triangles  $\mathcal{T}_{q/p, s/r}$  and  $\mathcal{T}_{s/r, q/p}$ . So we have

$$I_{\mathcal{T}_{q/p, s/r}}(t) = I_{\mathcal{T}_{s/r, q/p}}(t).$$

By Theorem 1.4 again, condition (1.4) means that  $s$  is also a quasi-period of  $I_{\mathcal{T}_{q/p, s/r}}(t)$ . Clearly,  $\gcd(q, s) = 1$ . It follows that 1 is a quasi-period of  $I_{\mathcal{T}_{q/p, s/r}}(t)$ . Therefore, the triangle  $I_{\mathcal{T}_{q/p, s/r}}(t)$  is a pseudo-integral triangle. □

### 3.1 The case where $u = 1/v$

As mentioned in the introduction, for when  $s = p$  and  $r = q$ , one can also give a sufficient condition for a version of Theorem 1.4. Specifically, we have:

**Theorem 3.1.** *Suppose that  $p, q$  are relatively prime positive integers. Then  $q$  is a quasiperiod of  $I_{\mathcal{T}_{q/p, p/q}}(t)$  if and only if*

$$p|(q^2 + 1) \quad \text{and} \quad \gcd\left(\frac{q^2 + 1}{p}, p\right) = 1. \quad (3.3)$$

*Proof.* Clearly, the “if ” part follows from the proof of Theorem 1.4. We now proceed to the proof of the “only if” part.

Suppose that  $q$  is a quasi-period of  $I_{\mathcal{T}_{q/p,p/q}}(t)$ . By (3.1), we deduce that

$$f_{p,q}(t) := \frac{1}{p^2} \sum_{l=1}^{p^2-1} \frac{\xi_p^{-ltq}}{(1 - \xi_p^{lq^2})(1 - \xi_p^l)}$$

is a periodic function of  $t$  with period  $q$ . Clearly,  $p$  is also a period of  $f_{p,q}(t)$ . Since  $(p, q) = 1$ , we deduce that  $f_{p,q}(t)$  is a constant function of  $t$ . It follows from (3.2) that

$$\frac{1}{p^2} \sum_{u=1}^{p-1} \xi_p^{-utq} \sum_{i=0}^{p-1} \frac{1}{(1 - \xi_p^{uq^2-ip})(1 - \xi_p^{u+ip})} = C$$

for some constant  $C$ . Keeping in mind the fact that  $\gcd(p, q) = 1$ , we have

$$\{\xi_p^{-utq} : 1 \leq u \leq p-1\} = \{\xi_p^{jt} : 1 \leq j \leq p-1\}.$$

So

$$\{1, \xi_p^{-tq}, \xi_p^{-2tq}, \dots, \xi_p^{-(p-1)tq}\} = \{1, \xi_p^t, \xi_p^{2t}, \dots, \xi_p^{(p-1)t}\},$$

which consists of  $p$  linearly independent functions from  $\mathbb{N}$  to  $\mathbb{C}$ . Hence we have

$$\sum_{i=0}^{p-1} \frac{1}{(1 - \xi_p^{uq^2-ip})(1 - \xi_p^{u+ip})} = 0$$

for any  $1 \leq u \leq p-1$ . By applying [4, Lemma 2.1], we deduce that

$$p|u(q^2 + 1) \quad \text{and} \quad p^2 \nmid u(q^2 + 1)$$

for each  $1 \leq u \leq p-1$ . Now we can conclude immediately that

$$p|(q^2 + 1) \quad \text{and} \quad \gcd\left(p, \frac{q^2 + 1}{p}\right) = 1.$$

□

### 3.2 The k-Fibonacci numbers

In [4], it was shown that  $\mathcal{T}_{q/p,p/q}$  is pseudo-integral if and only if  $p = q = 1$  or  $\{p, q\} = \{F_{2k-1}, F_{2k+1}\}$  for some  $k \geq 1$ , where  $p$  and  $q$  are relatively prime positive integers,  $F_m$  denotes the  $m$ th Fibonacci number. We now further study the relationship between the period collapse problem and recursive sequences, by proving a similar result, involving two consecutive terms in the sequence of generalized Fibonacci numbers.

Recall first for any integer  $k \geq 1$ , the  $k$ -Fibonacci sequence  $\{F_n(k)\}$ , defined recursively as follows:

$$F_0(k) = 0, F_1(k) = 1, F_n(k) = kF_{n-1}(k) + F_{n-2}(k) \quad (n \geq 2).$$

Clearly, when  $k = 1$ , we get the Fibonacci sequence. For notational simplicity, for any  $k \geq 1, n \geq 2$ , we let

$$I_{k,n}(t) := I_{\mathcal{T}_{F_n(k)/F_{n-1}(k), F_{n-1}(k)/F_n(k)}}(t).$$

In the following we shall consider quasi-period collapse in  $I_{k,n}(t)$ . To this end, we need the following immediate facts:

**Fact 1:** For any  $k, n \geq 1$ ,  $\gcd(F_n(k), F_{n-1}(k)) = 1$  and  $\gcd(F_n(k), k) = 1$ .

**Fact 2:** For any  $k, n \geq 1$ , we have

$$F_n(k)^2 - kF_{n-1}(k)F_n(k) - F_{n-1}(k)^2 + (-1)^n = 0.$$

Both Fact 1 and Fact 2 can be verified immediately by induction on  $n$ . We only give the proof of Fact 2 here. Clearly, the fact holds for  $n = 1$ . Assume  $n \geq 2$  and Fact 2 is true for  $n - 1$  and then we have

$$\begin{aligned} & F_n(k)^2 - kF_{n-1}(k)F_n(k) - F_{n-1}(k)^2 + (-1)^n \\ &= (kF_{n-1}(k) + F_{n-2}(k))^2 - kF_{n-1}(k)(kF_{n-1}(k) + F_{n-2}(k)) - F_{n-1}(k)^2 + (-1)^n \\ &= kF_{n-1}(k)F_{n-2}(k) + F_{n-2}(k)^2 - F_{n-1}(k)^2 + (-1)^n \\ &= -(F_{n-1}(k)^2 - kF_{n-1}(k)F_{n-2}(k) - F_{n-2}(k)^2 + (-1)^{n-1}) \\ &= 0. \end{aligned}$$

It follows from Fact 1 and Fact 2 that, when  $n$  is even, both  $(p, q) = (F_{n-1}(k), F_n(k))$  and  $(p, q) = (F_{n+1}(k), F_n(k))$  satisfy condition (3.3). We therefore get:

**Theorem 3.2.** *For any  $k \geq 1$  and even integer  $n \geq 2$ ,  $F_n(k)$  is a common quasi-period of  $I_{k,n}(t)$  and  $I_{k,n+1}(t)$ .*

### 3.3 Tetrahedra

We now give a few higher dimensional examples of period collapse.

Recall first the sequence given by  $a_1 = 2, a_2 = 3, a_3 = 10, a_4 = 17$  and

$$a_n = 6a_{n-2} - a_{n-4}. \tag{3.4}$$

It follows from [4, Thm. 1.6.i] that for each  $n \geq 1$ , the triangle with vertices  $(0, 0)$ ,  $\left(\frac{a_{2n+1}}{a_{2n}}, 0\right)$  and  $\left(0, \frac{2a_{2n}}{a_{2n+1}}\right)$  is a pseudo-integral triangle with Ehrhart polynomial

$$I_n(t) = (t + 1)^2.$$

Using this we can show:

**Theorem 3.3.** *Let  $\{a_n\}$  be the sequence defined by (3.4). Then for any  $n \geq 1$ , the tetrahedron  $T_n$  with vertices  $(0, 0, 0)$ ,  $\left(\frac{1}{2}, 0, 0\right)$ ,  $\left(0, \frac{a_{2n+1}}{a_{2n}}, 0\right)$  and  $\left(0, 0, \frac{2a_{2n}}{a_{2n+1}}\right)$  is a pseudo-integral tetrahedron.*

*Proof.* Let  $f_n(t)$  denote the Ehrhart function for  $T_n$ . Then for any positive integer  $t$ , we have

$$\begin{aligned} f_n(t) &= \#\{(x, y, z) \in \mathbb{Z}^3 \mid 2x + \frac{a_{2n}}{a_{2n+1}}y + \frac{a_{2n+1}}{2a_{2n}}z \leq t, x, y, z \geq 0\} \\ &= \sum_{x=0}^{\lfloor \frac{t}{2} \rfloor} \#\{(y, z) \in \mathbb{Z}^2 \mid \frac{a_{2n}}{a_{2n+1}}y + \frac{a_{2n+1}}{2a_{2n}}z \leq t - 2x, y, z \geq 0\} \\ &= \sum_{x=0}^{\lfloor \frac{t}{2} \rfloor} I_n(t - 2x) = \sum_{x=0}^{\lfloor \frac{t}{2} \rfloor} (t - 2x + 1)^2 \\ &= \frac{1}{6}t^3 + t^2 + \frac{11}{6}t + 1, \end{aligned}$$

where  $I_n(t)$  denotes the Ehrhart function of the triangle with vertices  $(0, 0)$ ,  $\left(\frac{a_{2n+1}}{a_{2n}}, 0\right)$  and  $\left(0, \frac{2a_{2n}}{a_{2n+1}}\right)$ .  $\square$

We now give the proof of Example 2.7. Note that Example 2.7 is natural to consider, in view of Theorem 3.3. It is easy to show that

$$\lim_{n \rightarrow \infty} \frac{a_{2n+1}}{a_{2n}} = 2 + \sqrt{2}, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{2a_{2n}}{a_{2n+1}} = 2 - \sqrt{2}.$$

Theorem 3.3 states that for any  $n \geq 1$ , the tetrahedron with vertices

$$(0, 0, 0), \left(\frac{1}{2}, 0, 0\right), \left(0, \frac{a_{2n+1}}{a_{2n}}, 0\right) \quad \text{and} \quad \left(0, 0, \frac{2a_{2n}}{a_{2n+1}}\right)$$

is a pseudo-integral tetrahedron with the Ehrhart polynomial

$$f(t) = \frac{1}{6}t^3 + t^2 + \frac{11}{6}t + 1,$$

which is independent of  $n$ . Thus, it is reasonable to expect that the Ehrhart function of the limiting tetrahedron  $T$  with vertices  $(0, 0, 0)$ ,  $(\frac{1}{2}, 0, 0)$ ,  $(0, 2 + \sqrt{2}, 0)$  and  $(0, 0, 2 - \sqrt{2})$  is also equal to the polynomial  $f(t)$ , and one can indeed show this by using Theorem 3.3 plus a limiting argument. We instead give a more direct proof that does not require Theorem 3.3:

**Proposition 3.4.** *The Ehrhart function of the irrational tetrahedron  $T$  with vertices  $(0, 0, 0)$ ,  $(\frac{1}{2}, 0, 0)$ ,  $(0, 2 + \sqrt{2}, 0)$  and  $(0, 0, 2 - \sqrt{2})$  is the polynomial  $f(t) = \frac{1}{6}t^3 + t^2 + \frac{11}{6}t + 1$ .*

*Proof.* Let  $g(t)$  denote the Ehrhart function of  $T$ . Then we have

$$\begin{aligned}
g(t) &= \#(tT \cap \mathbb{Z}^3) \\
&= \#\{(x, y, z) \in \mathbb{Z}^3 \mid 2x + \left(1 - \frac{\sqrt{2}}{2}\right)y + \left(1 + \frac{\sqrt{2}}{2}\right)z \leq t, x, y, z \geq 0\} \\
&= \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \#\{(i, y, z) \mid (i, y, z) \in tT \cap \mathbb{Z}^3\} \\
&= \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \#\{(i, y, z) \mid (i, y, z) \in tT \cap \mathbb{Z}^3, y \geq z\} \\
&\quad + \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \#\{(i, y, z) \mid (i, y, z) \in tT \cap \mathbb{Z}^3, y < z\} \\
&= \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-2i}{2} \rfloor} \#\{(i, y, j) \mid (i, y, j) \in tT \cap \mathbb{Z}^3, y \geq j\} \\
&\quad + \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{t-2i-1+\sqrt{2}}{2} \rfloor} \#\{(i, k, z) \mid (i, k, z) \in tT \cap \mathbb{Z}^3, k < z\} \\
&= \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-2i}{2} \rfloor} \left( \left\lfloor \frac{t-2i-2j}{1 - \frac{\sqrt{2}}{2}} \right\rfloor + 1 \right) + \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{t-2i-1+\sqrt{2}}{2} \rfloor} \left\lfloor \frac{t-2k-2i}{1 + \frac{\sqrt{2}}{2}} \right\rfloor \\
&= \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-2i}{2} \rfloor} \left( \left\lfloor \frac{t-2i-2j}{1 - \frac{\sqrt{2}}{2}} \right\rfloor + 1 + \left\lfloor \frac{t-2i-2j}{1 + \frac{\sqrt{2}}{2}} \right\rfloor \right)
\end{aligned}$$

Keeping in mind that

$$\frac{1}{1 - \frac{\sqrt{2}}{2}} = 2 + \sqrt{2}, \quad \frac{1}{1 + \frac{\sqrt{2}}{2}} = 2 - \sqrt{2},$$

we deduce that

$$\begin{aligned} \left\lfloor \frac{t - 2i - 2j}{1 - \frac{\sqrt{2}}{2}} \right\rfloor &= \lfloor (t - 2i - 2j)(2 + \sqrt{2}) \rfloor \\ &= \lfloor (t - 2i - 2j)(4 - (2 - \sqrt{2})) \rfloor \\ &= 4t - 8i - 8j + \left\lfloor -\frac{t - 2i - 2j}{1 + \frac{\sqrt{2}}{2}} \right\rfloor. \end{aligned}$$

So we have

$$\begin{aligned} g(t) &= \begin{cases} \frac{t}{2} + 1 + \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-2i}{2} \rfloor} (4t - 8i - 8j), & t \text{ is even;} \\ \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-2i}{2} \rfloor} (4t - 8i - 8j), & t \text{ is odd} \end{cases} \\ &= \frac{1}{6}t^3 + t^2 + \frac{11}{6}t + 1. \end{aligned}$$

□

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