3. (Singularities.)

(a) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence 1, and if $a_n \geq 0$ for all large n, then show that $z = +1$ is a singular point of $f(z)$.

[Hint: Expand the series about $z = +\frac{1}{2}$. Assuming the new series converges for $z = 1 + \delta$ where $\delta > 0$, obtain a contradiction.]

(b) Prove that even though the function $\sum_{n=1}^{\infty} \frac{z^{n!}}{n^2}$ converges on its entire circle of convergence, every point on this circle is a singularity.

[Hint: Show that $e^{2\pi i r}$, for rational r, is a singularity, by using part (a).]

---

4. (Residues.)

(a) Let g, h be analytic in a neighborhood of z_o, and assume $h(z_o) \neq 0$, $g(z_o) = 0$, $g'(z_o) \neq 0$. Show that the residue of $f(z) = h(z)/g(z)^2$ at z_o is

$$\frac{h'(z_o)g'(z_o) - h(z_o)g''(z_o)}{g'(z_o)^3} .$$

(b) Evaluate $\int_0^{\infty} \frac{x^6\,dx}{(x^4+16)^2}$ $\frac{3\pi\sqrt{2}}{16}$

(c) Evaluate $\int_{-\infty}^{\infty} \frac{\cos \alpha x\,dx}{x^2 + a^2}$ if α is real and $a > 0$. $\frac{\pi}{a} e^{-\alpha a}$

$\frac{1}{2}$ (d) Find the partial fraction expansions of $\frac{1}{z^n - 1}$ and $\frac{1}{(z^n-1)^2}$.

---

5. (Applied complex variable theory.)

(a) $\int_{|z| = \frac{1}{2}} dz/(1-z^r)^{m+1} z^n$ where r, m, n > 0.

$2\pi i \binom{m+k}{k}$, if $\exists_k$ $kr = m-1$
0, otherwise

Give your answer in terms of binomial coefficients.

(b) Let $m > 1$ be an integer. Show that there is precisely one value of z inside the circle $|z| = \frac{1}{2}$ for which $z^m = z - \alpha$, when $|\alpha| < \frac{1}{2} - (\frac{1}{2})^m$.

Hint: Use Rouche's theorem.

(c) The value of z mentioned in part (b) is $\frac{1}{2\pi i} \int_c \frac{\zeta F'(\zeta)d\zeta}{F(\zeta)}$ where $F(\zeta) = \zeta - \zeta^m - \alpha$ and c is the circle $|z| = \frac{1}{2}$. If $|\alpha| \leq \rho < \frac{1}{2} - (\frac{1}{2})^m$, show that the series $\sum_{n=0}^{\infty} \frac{\zeta F'(\zeta)\alpha^n}{(\zeta - \zeta^m)^{n+1}}$ converges uniformly to $\frac{\zeta F'(\zeta)}{F(\zeta)}$ on C.

(d) We have the following power series expansion for this value of z:

$$z = \alpha + \sum_{k=1}^{\infty} \frac{1}{k} \binom{m\cdot k}{k-1} \alpha^{(m-1)k+1} . \text{ (Prove it)}$$

$\in$ (e) Give a similar power series expansion for z^r, $r = 2,3,\ldots$.