CALIFORNIA INSTITUTE OF TECHNOLOGY

Department of Mathematics

Ma 108b Section C (Knuth) Final Exam Winter, 1964

Spend no more than one hour on each of the five portions of this test. You may use Apostol, Knopp, and class notes (but not integral tables). Use blue books and return them to the slot near Room 255 Sloan by 4:00 p.m., Thursday, March 19, 1964. Don't panic.

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1. (Differentiation and integration of real-valued functions.)

The function $f(x)$ is defined for $x \in [0,1]$ as follows:

Let the <u>ternary</u> representation of x be $.a_1a_2a_3\ldots$ and let the <u>binary</u> representation of $f(x)$ be $.b_1b_2b_3\ldots$. Then $b_n = 0$ if $a_n = 0$ or if at least one of $a_1, a_2, \ldots, a_{n-1} = 1$; $b_n = 1$ otherwise. (Thus, in particular, $f(x) = \frac{1}{2}$ if $\frac{1}{3} \leq x \leq \frac{2}{3}$.)

1 (a) Show that f is well-defined, i.e. the ambiguity of the ternary representation of x makes no difference in this definition.

4/5 (b) Determine the set of points at which $f'(x)$ exists; show in particular that $f'(x)$ exists almost everywhere and $f'(x) = 0$ wherever it exists (yet $f(x)$ is certainly not constant).

1-ε (c) Show that $\int_0^1 f(x)dx$ exists and determine its value. 1/2

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2. (Uniform convergence.)

1 (a) Let $\{x\}$ denote the fractional part of x, i.e. $\{x\} = x - [x]$. Show that $f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2}$ is continuous almost everywhere on E_1.

Compute $\int_0^1 f(x)dx$. $\pi^2/12$

1 (b) Let $f_n(z)$, $g_n(z)$ be sequences of complex valued functions defined on a set $G \subseteq E_2$, such that

(i) $|f_1(z) + \cdots + f_n(z)| < M$ for all $z \in G$.

(ii) $\lim_{n\to\infty} g_n(z) = 0$ <u>uniformly</u> in G.

(iii) $\sum_{n=1}^{\infty} |g_{n+1}(z) - g_n(z)|$ converges <u>uniformly</u> in G.

Show that $\sum f_n(z)\, g_n(z)$ converges <u>uniformly</u> in G.

2/3 (c) If $\sum a_n/n^x$ converges for all $x > \lambda$, then $\sum a_n/n^z$ converges <u>uniformly</u> in the set $\{z \mid \operatorname{Re}(z) \geq \lambda_1 > \lambda,\ |z| \leq R\}$.

[Hint: Apply part (b) with $f_n(z) = a_n/n^{\lambda+\delta}$, $g_n(z) = 1/n^z$, where $\operatorname{Re}(z) \geq \delta > 0$; show that $|g_n(z) - g_{n+1}(z)| \leq K/n^{1+\delta}$ for some K independent of n and z.]