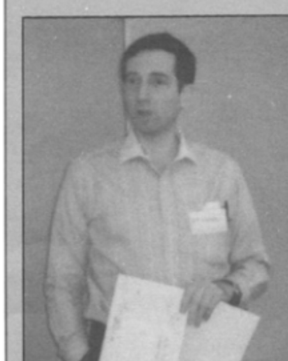
Professor Eubanks in Zetaland

Richard P. Stanley

Professor E. Pluribus Eubanks threw yet another of his notebooks into the garbage where it rightfully belonged. Frustration and despair filled his being, as they had so many times before. These feelings, he felt, he richly deserved for having the audacity to devote his life to the most awesomely difficult problem in the history of human thought. Since his graduate student days 20 years ago, his life had been dominated by an overwhelming desire to prove the celebrated Riemann hypothesis. He had long ago sublimated all of life's normal desires into an intense effort to resolve this insidious problem. Formulated in 1859 by Bernhard Riemann, this conjecture had resisted for over 120 years the efforts of the world's greatest mathematicians. Professor Eubank's paragon was the mathematician David Hilbert, who included the Riemann hypothesis as part of the eighth problem in his famous list of 23 unsolved problems delivered before the International Congress of Mathematicians in 1900. Hilbert, upon being asked what he would do after awakening from a sleep of 500 years, replied that he would ask, "Has somebody proved the Riemann hypothesis?"

Professor Eubanks slumped in his austere office chair in a state of wretched dejection. His mind seethed with the ideas and inspirations of his career. He recalled the countless hours spent investigating subtle variations of his earlier intricate arguments, which ultimately led to yet another dreary paper entitled "A note on the Riemann hypothesis." If only that integral had converged at $z = 1$! If only Siegel's method could be extended to essential singularities! If only. . . . His usually disciplined thoughts drifted aimlessly. Finally the pressure of 20 years of constant concentration on a single problem strained the very fabric of the space-time continuum. The objects about him became hazy and indistinct, and soon evaporated into nothingness. He found himself sitting in the midst of a vast plain, without the slightest variation in landscape as far as he could see. His initial reaction of disorientation and panic was soon replaced by in-



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Richard Stanley received his B.S. in mathematics from Caltech and Ph.D. in mathematics from Harvard University. During 1970–1971 he was an instructor at MIT and then for two years a Miller Research Fellow at Berkeley. He returned to MIT in 1973, where he has remained. His research interests are in combinatorics and algebra. In 1975 he was awarded the Pólya Prize in Applied Combinatorics from the Society for Industrial and Applied Mathematics.

creasing excitement as he turned around and surveyed the scene before him. He faced a spectacular, inconceivably vast mountain range, which was split in two by a kind of canyon whose sides seemed to blend into the mountains and whose interior was filled with a series of hills that grew taller and taller until they faded out of sight in the distance. Directly in front of him, at the boundary of the plain and mountains, rose a perfect spire that stretched upward to the limits of his vision.

To anyone but Professor Eubanks this scene would have been as alien as the surface of a neutron star. Professor Eubanks, however, almost instantly realized that he was on the contour surface of the Riemann zeta function. At some particular "sea level" lay the complex z-plane, and the height above sea level of a point z on the surface was the absolute value $|\zeta(z)|$ of the zeta function at that point. The vast plain was the surface for large real parts of z, where the zeta function sloped gently toward sea level. The hills and valleys inside the canyon were caused by the vanishing of $\zeta(z)$ at negative even integers, and the spire was the zeta function's only singularity, a simple pole at $z = 1$. A feeling of exaltation rapidly mounted in Professor Eubanks as he realized that he could check the validity of the Riemann hypothesis simply by walking down the Critical Strip, the area between the parallel lines $\text{Re } z = 0$ and $\text{Re } z = 1$. By a simple theorem in complex variable theory, the bottoms of valleys on the surface were the zeros of the zeta function. The Riemann hypothesis would be true if all valley bottoms in the Critical Strip lay on a straight line—necessarily the line $\text{Re } z = \frac{1}{2}$, the Critical Line.

Professor Eubanks enthusiastically commenced his walk through the Critical Strip. He quickly passed a few thousand zeros all neatly arranged on a line. As he continued walking, he became aware of a faint but pungent odor that gradually grew stronger. After a few hundred more zeros he observed a hint of movement in the horizon ahead of him. The movement quickly took the shape of a hideous sluglike creature that was lumbering down the Critical Strip toward Professor Eubanks. The stench emanating from this creature was becoming unbearable. Only Professor Eubank's extraordinary determination, fueled by a lifetime of broken dreams, kept him from bolting away from the repulsive horror drawing near to him. As the unspeakable abomination reached the professor, a barrier of festering slime oozed out of an odious orifice and completely blocked the Critical Strip.

"I knew the proof couldn't be that easy," thought Professor Eubanks. "This creature can only be the Guardian of the Critical Strip, the bane of all mathematicians since Riemann." After a few more perfunctory effects to circumnavigate the Guardian of the Critical Strip, Professor Eubanks withdrew to the vicinity of the pole in abject defeat.

The encounter with the Guardian had so repulsed him that several hours passed before he was once again able to concentrate on mathematics. Eventually Professor Eubanks felt revived enough to make another attempt. This time he would walk down the peaceful terrain of the line $\text{Re } z = 100$ or so, until $\text{Im } z$ was as large as he had previously encountered, and then turn left and sneak across the line $\text{Re } z = 1$ into the Critical Strip. Perhaps the Guardian would be oblivious to such an oblique approach. After a short spell of uneventful progress, Professor Eubanks turned toward the Critical Strip. Only his unparalleled knowledge of the Riemann zeta function enabled him to realize that he had reached the same value of $\text{Im } z$ as on his previously unsuccessful sojourn down the Strip. To his great disappointment he soon noticed a familiar unpleasant odor. Surprisingly, though, this noxious scent did not seem as unbearable as before. As he approached the line $\text{Re } z = 1$, he was confronted not by the repugnant monstrosity he expected, but by a considerably less awesome creature, no bigger than a sheep and roughly resembling a cross between a cockroach and a sea anemone. "Aha!" thought the professor. "This pale imitation of the Guardian of the Critical Strip is obviously the Guardian of the Boundary of the Critical Strip." By a very clumsy feint to the left followed by a dash to the right, Professor Eubanks easily eluded this Lesser Guardian's feeble efforts to ensnare him and entered the exhilarating terrain of the Critical Strip. "After all," he reasoned, "as long ago as 1896 Jacques Hadamard and Charles Jean de la Vallée Poussin proved the Prime Number Theorem, and that is equivalent to crossing the Boundary of the Critical Strip."

Unfortunately, his optimism that he was now free to explore the Strip at his leisure proved short-lived. After examining only a few dozen additional zeros he sensed an all-too-familiar odor, quickly followed by the sight of the Guardian of the Critical Strip itself. Professor Eubank's efforts this time to bypass the Guardian were as futile as before. In fact, a small drop of slime from the creature happened to fall on his arm, and the ensuing agony was excruciating beyond belief. His entire arm felt as if it had been dipped into a vat of boiling hydrochloric acid. The professor raced toward the pole faster than he had ever run in his life, once again to sit for several hours in miserable despair.

He had stumbled upon the opportunity of a lifetime —of a thousand lifetimes—yet he seemed helpless to pursue it. There must be some way of exploring the terrain without this execrable obstruction. If he could only survey a large enough area of Zetaland, he was convinced that he would understand the subtle mysteries of the zeta function well enough to resolve this absurd conjecture of Riemann! What more could he do? Clearly it was hopeless to try to sneak into the

Critical Strip from any value of Im z. Perhaps he could simply examine very carefully the surface near him and invoke analytic continuation. No, that was ridiculous. Perhaps a poet was able "to see a world in a grain of sand," but he was a mere mathematician.

The inspiration struck faster than any bolt of lightning. How trivial it all seemed now! There was no need to venture into the Critical Strip at all—he could simply climb the pole! The higher he climbed, the more zeros he would see, and the vast lumbering bulk of the Guardian would prevent it from reaching him. He immediately commenced his ascent of the pole. Although he had never climbed anything more difficult than a flight of stairs before, such was his determination that he made remarkably fast progress. As a greater and greater extent of the stunning panorama of Zetaland became visible, he could see far beyond the measly five or six thousand zeros he had previously encountered. Hundreds of thousands of zeros lay below him, neatly strung out on a line, with no sign of a ruinous pair of zeros symmetric about the Critical Line. His extraordinary exhilaration rapidly deserted him as a familiar scent assailed his nostrils. A glance downward revealed that the accursed Guardian of the Critical Strip was following him up the pole. For a creature of its size, it was climbing remarkably quickly, and Professor Eubanks saw that the Guardian would soon overtake him. Professor Eubanks increased his rate of climbing, but the Guardian continued to gain on him. The professor strained his tortured muscles even further. His heart and lungs screamed in agony as he forced himself up the spire, which by now had become so thin that it was almost impossible to grasp. As the Guardian was breathing down his neck, Professor Eubanks was becoming aware of a kind of pattern in the procession of zeros spread out before him. Just as he was feeling the first faint glimmerings of the logical necessity for the not-yet-seen zeros to stay on the Critical Line, a burning mass of slime reached his body. He screamed in horror as the Guardian reached upward to devour him. He released his grip on the pole and fell together with the Guardian a seemingly endless distance onto the alien surface below.

Professor Eubanks returned to consciousness sprawled on the floor of his office. Hideous burns scarred vast areas of his body, and his fingers were practically sliced in two by a series of exceedingly fine cuts. But he barely noticed these intrusions on his well-being, for his mind held the solution to the greatest problem devised by human intellect.

Two years later, after Professor Eubanks was awarded a Fields Medal at the International Congress of Mathematicians, a reporter asked him how he felt upon receiving mathematics' greatest honor. "I'm just happy that I didn't devote my life to the theory of partitions," he replied cryptically. "The generating function for partitions has the unit circle for a natural boundary, and I would never have been able to get back."

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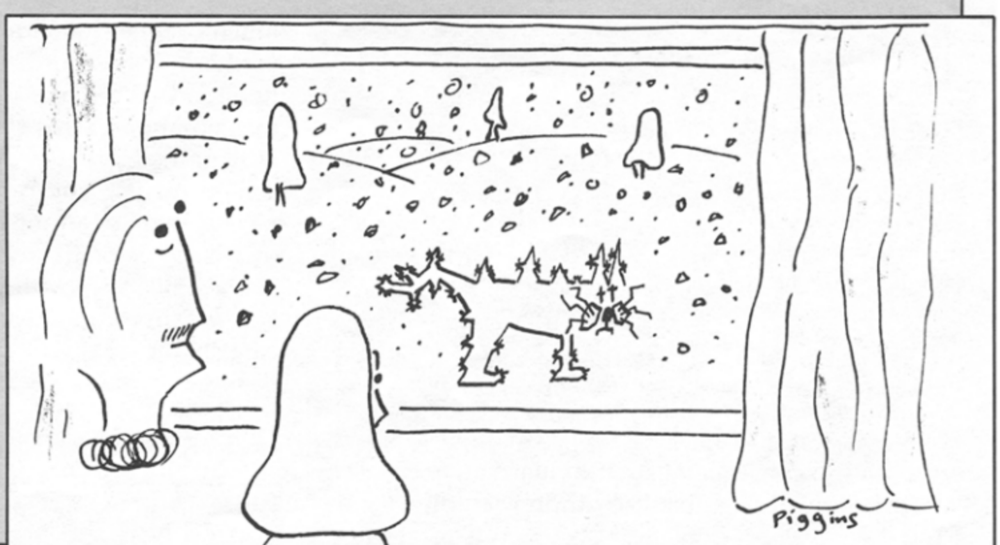
Cartoon Contest

This cartoon was selected as the winner of the *Mathematical Intelligencer* cartoon contest (announced in Vol. 9, No. 3 and Vol. 9, No. 4). David Piggins (University of Guelph, Canada), the creator of this cartoon, will receive a Springer-Verlag book of his choice.



"Oh, he always goes a bit fractal in this weather"

David Piggins