SOLUTIONS TO SUPPLEMENTARY EXERCISES
for Chapter 7 (symmetric functions) of
*Enumerative Combinatorics, vol. 2*
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**Notation.** The symbol ♣ denotes a request to the reader.

1. Let $\nu \vdash k \leq n$. Let $\lambda$ (respectively, $\mu$) be the partition obtained from $\nu$ by adding a part equal to $n - k$ (respectively, $n - k + 1$). This bijection $\nu \mapsto (\lambda, \mu)$ shows that $f(n) = p(0) + p(1) + \cdots + p(n)$. See also EC1, second ed., Exercise 1.71.

2. To get a partition of rank $r$, take an $r \times r$ square $S$ and two partitions $\lambda, \mu$ with largest part at most $r$. Attach (the diagram of) $\lambda$ to the right of $S$ and $\mu'$ below $S$. This construction yields

$$F_r(t) = \frac{t^r}{(1 - t)^2(1 - t^2)^2 \cdots (1 - t^r)^2}.$$

See also EC1, second ed., Prop. 1.8.6(b).

3. Note that $p_1 = e_1$.

4. Note that

$$\sum_{k=2}^{n} (-1)^k 2^k e_k e_1^{n-k} - e_1^n = \sum_{k=0}^{n} (-1)^k 2^k e_k e_1^{n-k}.$$

The proof is now immediate from the observation that

$$F_n(x) = \prod_{i=1}^{n} (e_1 - 2x_i).$$
5. We have
\[ e_2(x)^m = [(y_1 \cdots y_m)^2] \prod (1 + x_i y_j) \]
\[ = [(y_1 \cdots y_m)^2] \exp \sum_{n \geq 1} (-1)^{n-1} \frac{p_n(x)p_n(y)}{n} \]
\[ = [(y_1 \cdots y_m)^2] \exp \left( p_1(x)p_1(y) - \frac{1}{2} p_2(x)p_2(y) \right) \]
\[ = [(y_1 \cdots y_m)^2] e^{p_1(x)p_1(y)} e^{-\frac{1}{2} p_2(x)p_2(y)}, \]

etc.

Darij Grinberg points out that an easier solution is obtained by using the binomial theorem and the formula \( e_2 = \frac{1}{2}(p_1^2 - p_2) \).

6. By Exercise 7.48(f) we have
\[ F_{NC, n+1} = \frac{1}{n+1} [t^n] E(t)^{n+1}. \]
But \( E(t) = 1/H(-t) \), so
\[ F_{NC, n+1} = \frac{1}{n+1} [t^n] H(-t)^{-n-1}. \]

Now
\[ H(-t)^{-n-1} = (1 + (-h_1 t + h_2 t^2 - h_3 t^3 + \cdots))^{-n-1} \]
\[ = \sum_{k \geq 0} \binom{-n-1}{k} (-h_1 t + h_2 t^2 - h_3 t^3 + \cdots)^k. \]

It is now straightforward to expand each term by the multinomial theorem, etc.

7. Answer: \( a \geq 2 \). Suppose \( a \geq 2 \). Then \( 1 + at + t^2 = (1 + \alpha t)(1 + \beta t) \) where \( \alpha, \beta \geq 0 \). Now expand \( F(x) \) as in the solution to Problem 12 to get \( e \)-positivity. Conversely, suppose that \( a < 2 \). Now \( \alpha \) and \( \beta \) are not nonnegative real numbers. Since \( \arg(\alpha^n) = n \arg(\alpha) \), it is easy to see that for some \( n > 0 \) we have \( \Re(\alpha^n) < 0 \), where \( \Re \) denotes real part. Then the coefficient of \( e_{n+1} e_1 \) in \( F(x) \) is
\[ [e_{n+1} e_1] F(x) = \alpha \beta (\alpha^n + \beta^n) = \alpha^n + \beta^n = 2 \Re(\alpha^n) < 0. \]
Note the subtlety of this problem: there is no $\lambda$ for which the coefficient of $e^{\lambda}$ in $F(x)$ is negative for all $a < 2$. For the generalization to $\prod P(x_i)$ where $P(t)$ is any polynomial satisfying $P(0) = 1$, see Exercise 7.91(e). For the even more general (and considerably more difficult) generalization when $P(t)$ is an arbitrary power series satisfying $P(0) = 1$, see the references to Edrei and Thoma in the solution to Exercise 7.91(e).

8. It is not difficult to check that the coefficient of $h_{21}^{n-2}$ in $e^{\lambda}$ is $\ell(\lambda) - n$. Since this number is negative unless $\lambda = \langle 1^n \rangle$, it follows that we must have $f = \alpha e_1^n = \alpha h_1^n$ for $\alpha \geq 0$.

9. Open.

10. Since $\omega^2 = 1$, the eigenvalues of $\omega$ are $\pm 1$. But if $f \neq 0$ and $\omega f = 2f$, then $f$ is an eigenvector for the eigenvalue 2. Hence $f = 0$.

11. (a) We have $\langle e_{\lambda}, h_{\mu} \rangle = M_{\lambda\mu}$ and $\langle h_{\lambda}, h_{\mu} \rangle = N_{\lambda\mu}$ (as defined in Propositions 7.4.1 and 7.5.1). Clearly from the definitions we have $M_{\lambda\mu} \leq N_{\lambda\mu}$.

(b) We want to characterize those $\lambda, \mu$ for which every $\mathbb{N}$-matrix with row sum vector $\lambda$ and column sum vector $\mu$ is a $(0,1)$-matrix. For any $\lambda, \mu \vdash n$ it is easy to find an $\mathbb{N}$-matrix $A = (a_{ij})$ with $\text{row}(A) = \lambda$, $\text{col}(A) = \mu$, and $a_{11} = \min\{\lambda_1, \mu_1\}$. Hence a necessary condition is that either $\lambda = \langle 1^n \rangle$ or $\mu = \langle 1^n \rangle$. It is easy to see that this condition is also sufficient.

12. Let $P(x) = (1 + \alpha x)(1 + \beta x) \cdots$. Then

$$\omega \prod_i P(x_i) = \omega \prod_i (1 + \alpha x_i)(1 + \beta x_i) \cdots$$

$$= \omega \left( \sum_{n \geq 0} \alpha^n e_n \right) \left( \sum_{n \geq 0} \beta^n e_n \right) \cdots$$

$$= \left( \sum_{n \geq 0} \alpha^n h_n \right) \left( \sum_{n \geq 0} \beta^n h_n \right) \cdots$$

$$= \frac{1}{\prod_i (1 - \alpha x_i)(1 - \beta x_i) \cdots}$$

$$= \frac{1}{\prod_i P(-x_i)}.$$
One could also take logarithms and expand in terms of power sums, etc.

13. Note that $m_{\langle k \rangle} = e_j(x^k_1, x^k_2, \ldots)$. Hence by Proposition 7.7.6,

$$m_{\langle k \rangle} = \sum_{\lambda \vdash j} \varepsilon_\lambda z_\lambda^{-1} p_{k\lambda},$$

where $k\lambda = (k\lambda_1, k\lambda_2, \ldots)$.

14. (a) By Corollary 7.7.2 the transition matrix $(R_{\lambda\mu})$ between the $m_\mu$'s and $p_\lambda$'s is upper triangular with respect to any linear extension of dominance order, with diagonal entries $R_{\mu\mu} = d_\mu$. An easy combinatorial argument shows that $R_{\lambda\mu}$ is divisible by $d_\mu$. We can perform integral elementary row operations on the matrix $(R_{\lambda\mu})$, except for multiplying a row by a scalar, without changing the abelian group generated by the rows. Since $d_\mu$ divides $R_{\lambda\mu}$ we can obtain the diagonal matrix $(d_\mu)$ by such row operations, and the proof follows.

(b) The matrix $X_n$ is the transition matrix between the $s_{\lambda}$'s and $p_\mu$'s. Since the set $\{s_\lambda\}_{\lambda \vdash n}$ is a basis for $\Lambda^n_Z$ (see the first sentence after the proof of Corollary 7.10.6), the proof follows from (a) and the definition of Smith normal form.

15. We have

$$\prod_i (1 + x_i)^\alpha = \exp \alpha \sum_i \log(1 + x_i)$$

$$= \exp \alpha \sum_i \sum_{n \geq 1} (-1)^{n-1} x_i^n \frac{1}{n}$$

$$= \exp \alpha \sum_{n \geq 1} (-1)^n \frac{p_n}{n}$$

$$= \sum_{\lambda} \varepsilon_\lambda \alpha^{\ell(\lambda)} z_\lambda^{-1} p_\lambda,$$

by the exponential formula, permutation version (Cor. 5.1.9).

16. Let $C(x, y) = \prod_{i,j} (1 - x_i y_j)^{-1}$. By (7.20), we have

$$C(x, y) = \exp \sum_{n \geq 1} \frac{1}{n} p_n(x) p_n(y),$$

4
from which it is immediate that
\[ \frac{\partial}{\partial p_k(x)} C(x, y) = \frac{p_k(y)}{k} C(x, y). \]

17. For fixed \( u \in G \), the number of \( v \in G \) satisfying \( uv = vu \) is (by definition) \( \#C(u) \), where \( C(u) \) is the centralizer of \( u \). By elementary group theory, \( \#C(u) = \#G/\#K_u \), where \( K_u \) is the conjugacy class of \( G \) containing \( u \). Hence if \( C(G) \) denotes the set of conjugacy classes of \( G \), then
\[
\# \{(u, v) \in G : uv = vu\} = \sum_{u \in G} \#C(u) = \sum_{u \in G} \frac{\#G}{\#K_u} = (\#G) \sum_{K \in C(G)} \sum_{u \in K} \frac{1}{\#K} = (\#G)k(G),
\]
where \( k(G) \) denotes the number of conjugacy classes in \( G \). For \( G = \mathfrak{S}_n \) the answer becomes \( p(n)n! \).

18. Since \( z_\lambda = 1^{m_1!} 2^{m_2!} \cdots \), the condition that \( z_\lambda \equiv 0 \pmod{p} \) is equivalent to \( \lambda \) having no parts divisible by \( p \) and no parts with multiplicity at least \( p \). Hence
\[
F_p(x) = \prod_{k \geq 1, \frac{p^k}{k}} (1 + x^k + x^{2k} + \cdots + x^{(p-1)k}) = \prod_{k \geq 1, \frac{p^k}{k}} \frac{1 - x^{pk}}{1 - x^k}.
\]

Note that \([why?] \) \( f_2(n) \) is the number of self-conjugate partitions of \( n \).

20. (a) Let \( H(t) = \sum h_n t^n = \prod (1 - x_i t) \). Then

\[
T = \frac{H(1) - H(-1)}{H(1) + H(-1)}
\]

\[
= \exp \left( \sum \frac{p_n}{n} \right) - \exp \left( \sum (-1)^n \frac{p_n}{n} \right) \\
\exp \left( \sum \frac{p_n}{n} \right) + \exp \left( \sum (-1)^n \frac{p_n}{n} \right)
\]

(b) Let \( \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \sum_{n \geq 0} (-1)^n E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} \),

where \( E_{2n+1} \) is an Euler number (the number of alternating permutations of \( 1, 2, \ldots, 2n + 1 \)). (Everyone in combinatorics should be familiar with these creatures; see e.g. EC1, second ed., §1.6.)

Let

\[
y = \sum_{n \text{ odd}} \frac{p_n}{n}.
\]

Then by (a),

\[
T = \tanh y
\]

\[
= \sum_{n \geq 0} (-1)^n E_{2n+1} \frac{y^{2n+1}}{(2n+1)!}.
\]

From this it follows easily that if \( \lambda = (1^m 3^m 5^m \cdots) \) where \( \ell(\lambda) = \sum m_i = 2m + 1 \), then

\[
[p_\lambda] T = \frac{(-1)^m E_{2m+1}}{z_\lambda},
\]

and otherwise this coefficient is 0. For the algebraic significance of this problem, see Exercise 7.64.

21. Since the coefficient of \( h_\mu \) for \( \mu = (\mu_1, \ldots, \mu_k) \) (so \( \ell(\mu) \leq k \)) in \((1 + h_1 + h_2 + \cdots)^k \) is just the number of permutations of the entries of the vector \( \mu \), it follows that the desired scalar product is just the total
number of N-matrices with k rows and with column sums $\lambda_1, \lambda_2, \ldots$. The number of sequences $(a_1, \ldots, a_k)$ with sum $s$ is $\binom{s+k-1}{s}$. Hence

$$\langle (1 + h_1 + h_2 + \cdots)^k, h_\lambda \rangle = \binom{\lambda_1 + k - 1}{\lambda_1} \binom{\lambda_2 + k - 1}{\lambda_2} \cdots.$$  

22. (a) Without loss of generality (since $F_p$ is symmetric), consider the coefficient of the monomial $M = x_1^{a_1} \cdots x_k^{a_k}$, where each $a_i > 0$. If a term $T = (\sum_{i \in S} x_i)^{p-1}$ contains $M$, then the coefficient of $M$ is the multinomial coefficient $\binom{p-1}{a_1, \ldots, a_k}$. The number of $p$-subsets $S$ for which the term $T$ contains $M$ is $\binom{2p-1-k}{p-k} - \binom{2p-1-k}{p-k}$, the number of $(p-k)$-element subsets of $[2p-1] - [k]$. Now

$$\binom{2p-1-k}{p-k} = \frac{(2p-1-k)!}{(p-k)!(p-1)!}.$$  

Since $1 \leq k \leq p-1$, the numerator is divisible by $p$ but not the denominator, and the proof follows.

(b) Let $a_1, \ldots, a_{2p-1} \in \mathbb{Z}$. We have by (a) and elementary properties of congruences that

$$\# \{ S \subseteq [2p-1] : \sum_{i \in S} i \equiv 0 \pmod{p} \text{ and } \#S = p \}$$

$$\equiv \sum_{\substack{S \subseteq [2p-1] \\#S = p}} \left( 1 - \left( \sum_{i \in S} a_i \right)^{p-1} \right) \equiv \binom{2p-1}{p} \equiv 1 \pmod{p},$$


(c) It is not hard to see that if $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$ then

$$(m_1! m_2! \cdots) m_\lambda \in \mathbb{Z}[p_1, p_2, \ldots].$$

Let $\ell(\lambda) = k$ and $\lambda \vdash p-1$. By the solution to (a) the coefficient $C$ of $m_\lambda$ in $F_p$ is

$$C = \binom{2p-1-k}{p-k} \frac{(p-1)!}{\prod \lambda_i!}.$$
Now
\[ \frac{(p-1)!}{(\prod \lambda_i!)(\prod m_i!)} \]
is just the number of partitions of \([p-1]\) with block sizes \(\lambda_1, \lambda_2, \ldots\)
and is hence an integer. Thus
\[ \frac{(p-1)!}{\prod \lambda_i!} m_{\lambda} \in \mathbb{Z}[p_1, p_2, \ldots]. \]

Since \(2^{p-1-k} \equiv 0 \pmod{p}\), the proof follows.

23. (a) By linearity it suffices to do this for a fixed basis \(u_{\lambda}\). The easiest choice is \(u_{\lambda} = p_{\lambda}\). If \(\lambda \vdash n\), then
\[ p_{\lambda}(kx) = k^{\ell(\lambda)}p_{\lambda}(x). \]

(b) Again it suffices to choose \(f(x) = p_{\lambda}(x)\), for which the computation is trivial. Note in fact that \(fg(kx) = f(kx)g(kx)\), so the operation \(f(x) \mapsto f(kx)\) is in fact an endomorphism (in fact, an automorphism) of \(\Lambda_{\mathbb{Q}}\). Hence it actually suffices to take \(f = p_n\).

(c) Since both the operations \(f(x) \mapsto f(kx)\) and \(\omega\) are linear, it suffices to take \(f(x) = p_{\lambda}(x)\) (or even \(p_n(x)\) as discussed in (b)). Now for \(\lambda \vdash n\) we have
\[ p_{\lambda}(-x) = (-1)^{\ell(\lambda)}p_{\lambda}(x) = (-1)^np_{\lambda}(x). \]
Hence \(f(-x) = (-1)^n\omega f(x)\) for all \(f \in \Lambda_{\mathbb{Q}}^n\).

Note in particular the “reciprocity” \(e_n(-x) = (-1)^n p_{\lambda}(x)\), an extension of \((-N)_n\).

24. Since \(\langle h_{\lambda}, m_{\mu} \rangle = \delta_{\lambda\mu}\), it follows from Proposition 7.5.1 that \(\langle h_{\lambda}, h_{\mu} \rangle = N_{\lambda\mu}\). Hence we want the number of \(N\)-matrices \(A\) with row and column sum vector \((2, 1^{n-2}, 0, 0, \ldots)\). If \(A_{11} = 2\), then subtracting 1 from \(A_{11}\) yields an \((n-1) \times (n-1)\) permutation matrix, of which there are \((n-1)!\). If \(A_{11} = 0\) there are \(\binom{n-2}{2}(n-4)!\) matrices. Adding these two numbers gives \((n^2 - n + 2)/(n-2)!!/4\).

Another method: use \(h_2 = \frac{1}{2}(p_1^2 + p_2)\). Hence
\[ \langle h_2 h_1^{n-2}, h_2 h_1^{n-2} \rangle = \frac{1}{4}(p_1^{n} + p_2 p_1^{n-2}, p_1^{n} + p_2 p_1^{n-2}) \]
\[ = \frac{1}{4}(z(1^n) + z(2,1^{n-2})), \]
25. The linear transformation defined by \( e_\lambda \mapsto h_\lambda + e_\lambda \) is just \( \omega + I \) (where \( I \) is the identity transformation), so we want to compute \( \text{rank}(\omega + I) \). Now \( \omega \) is an involution and hence diagonalizable (its minimal polynomial has distinct roots). The eigenvalues of \( \omega \) are \( \pm 1 \). Hence \( \text{rank}(\omega + I) \) is the number of eigenvalues of \( \omega \) equal to 1. By the top of page 301, this is equal to the number \( e(n) \) of even conjugacy classes of \( S_n \). See Exercise 1.22(b) (together with Proposition 1.8.4) for the generating function of \( e(n) \).

26. (a) The matrix of \( \varphi^{-1} \) with respect to the basis \( \{ m_\lambda \} \) is the matrix \( M \) of Section 7.4. By a simple result of linear algebra, \( M \) and \( M^{-1} \) have the same Jordan block sizes. By Theorem 7.4.4 the matrix \( M \) is upper triangular with 1’s on the main diagonal. Hence the size of the largest Jordan block of \( \varphi \) is the least integer \( m \) for which \( (M - I)^m = 0 \). Again by Theorem 7.4.4 we have \( M_{\lambda \mu} = 0 \) unless \( \mu \leq \lambda' \). By Proposition 7.4.1 and the Gale-Ryser theorem (a basic combinatorial result which everyone should know) the converse is true, i.e., \( M_{\lambda \mu} > 0 \) if \( \mu \leq \lambda' \). Since all entries of \( M \) are nonnegative, there is no cancellation in the computation of powers of \( M - I \). It follows that \( m \) is the size of the longest chain in the poset \( \text{Par}(n) \), ordered by dominance. Now use Exercise 7.2(f). (This exercise gives the length of the longest chain, so you need to add 1 to get the size.) For some further work on chains in \( \text{Par}(n) \), see E. Early, *Discrete Math.* **313** (2013), 2168–2177.

(b) Define \( \mu_1, \mu_2, \ldots \) by setting \( \mu_1 + \mu_2 + \cdots + \mu_i \) equal to the number of elements in the largest union of \( i \) chains of \( \text{Par}(n) \) (with the dominance order). If the nonzero entries of \( M \) above the main diagonal were generic, then a theorem of Gansner and Saks (independently) shows that the Jordan block sizes of \( M \) are \( \mu_1, \mu_2, \ldots \). It seems plausible that \( M \) is “sufficiently generic” for the Gansner-Saks result to hold for \( M \) itself, but I don’t know how to show this. (An interesting consequence: \( \text{corank}(M - I) \) \((= p(n) - \text{rank}(M - I))\) is the size of the largest antichain in \( \text{Par}(n) \). I don’t know whether anyone has looked at the problem of determining this size.) Regardless, is there an explicit formula for the numbers \( \mu_1, \mu_2, \ldots ? \)
27. (a) Since $\omega$ is an isometry and an involution, we have
\[
\langle \omega f, g \rangle = \langle \omega^2 f, \omega g \rangle = \langle f, \omega g \rangle.
\]
Hence $\omega$, and thus also $\omega + aI$, is self-adjoint.

(b) It is easy to verify by considering the basis $\{p_\lambda\}$ that the adjoint $p_j^\perp$ to $M_j$ is $j \frac{\partial}{\partial p_j}$.

28. We want to count SSYT’s of shape $(k, k, k)$ with $k - 1$ 1’s, $k - 1$ 2’s, and one $3, 4, \ldots, k + 4$. The 1’s must go into the first row. If all the 2’s are in the second row, then there are \(\binom{k+1}{2}\) choices for the remaining numbers. Suppose there is one 2 in the first row and $k - 2$ in the second row. The hook-length formula (Corollary 7.21.6) makes it easy to count the possibilities, but we can also give a naive argument. Given an SSYT of shape $(k, k, k)$ and type $(k - 1, k - 1, 1^{k+2})$ with all the 2’s in the second row (as counted above), interchange the last 2 in the second row with the last element of the first row. This will give an SSYT with one exception, when the last column is $(k+2, k+3, k+4)$, and the correspondence is reversible. Hence
\[
K_{(k,k,k),(k-1,k-1,1^{k+2})} = \binom{k+1}{2} + \binom{k+1}{2} - 1 = k^2 + k - 1.
\]

29. (a) Let $T$ be the $180^\circ$ rotation of a semistandard tableau of shape $\langle kn \rangle / \lambda$ and content $\langle (k - 1)n \rangle$. (Note that by Exercise 7.56(a), for any skew $\theta$ and $180^\circ$ rotation $\theta^r$, and for any partition $\mu$, we have $K_{\theta, \mu} = K_{\theta^r, \mu}$.) Let $T'$ be the tableau of shape $\lambda$ whose $i$th column consists of the elements of $[n]$ not in column $k + 1 - i$ of $T$, arranged in increasing order. For instance, suppose that $k = 4$, $n = 6$, $\lambda = (4, 2)$, and
\[
T = \begin{array}{ccccc}
1 & 1 & 1 & 3 \\
2 & 2 & 2 & 4 \\
3 & 3 & 4 & 5 \\
4 & 5 & 6 & 6 \\
5 & 6 \\
\end{array}
\]
Then
\[
T' = \begin{array}{ccccc}
1 & 3 & 4 & 6 \\
2 & 5 \\
\end{array}
\]
It is easy to check that the map $T \mapsto T'$ is a bijection from SSYT of shape $\langle k^n \rangle / \lambda$ and content $\langle (k - 1)^n \rangle$ to SYT of shape $\lambda$.

(b) We have

$$K_{\langle k^n \rangle, \langle (k-1)^n, 1^n \rangle} = \sum_{\lambda \subseteq \langle k^n \rangle \atop \lambda \not\vdash n} K_{\lambda, \langle 1^n \rangle} K_{\langle k^n \rangle / \lambda, \langle (k-1)^n \rangle}$$

$$= \sum_{\lambda \subseteq \langle k^n \rangle \atop \lambda \not\vdash n} (f^\lambda)^2$$

$$= \sum_{\lambda \not\vdash n \atop \lambda_1 \leq k} (f^\lambda)^2.$$

By Corollary 7.23.12 (using $f^\lambda = f^{\lambda'}$), the last expression is equal to the number of permutations in $S_n$ with no increasing subsequence of length $k + 1$.

30. See math.mit.edu/~rstan/papers/sfcong.pdf. There are a number of open problems and further directions for research in this area that may be interesting to pursue.

31. Answer: $\binom{2}{1} \binom{4}{2} \cdots \binom{2(n - 1)}{n - 1}$

32. Since $h_\mu = \sum_{\lambda} K_{\lambda, \mu} s_\lambda$ we have

$$\sum_{\lambda} g(\lambda) s_\lambda = \sum_{\mu} h_\mu$$

$$= \prod_{n \geq 1} (1 - h_n)^{-1}.$$

See MathOverflow #18597.

33. (a) Let $E(i)$ denote the expected number of $i$’s among all SSYT of shape $\lambda$ and largest part at most $n$, for $1 \leq i \leq n$. Then

$$|\lambda| = d = E(1) + E(2) + \cdots + E(n).$$

Since $s_\lambda$ is a symmetric function we have $E(1) = E(2) = \cdots = E(n)$. Hence $E(i) = d/n$ for all $1 \leq i \leq n$. 

11
(b) First note that

\[ B_{\lambda,n}(k) = \frac{\partial^{k} s_{\lambda}(x_1, \ldots, x_n)}{\partial x_1^{k}} \bigg|_{x_1=\cdots=x_n=1}. \]

Thus

\[ \sum_{k \geq 0} B_{\lambda,n}(k) t^{k} = \frac{s_{\lambda}(t+1, 1^{n-1})}{s_{\lambda}(1^n)}, \]

by Taylor’s formula (Exercise 1.167 of EC1, second ed.).

Now in equation (24) put \( x_1 = t \) and \( x_2 = \cdots = x_n = 0 \). Since \( s_{\mu}(t, 0, 0, \ldots) = 0 \) unless \( \mu = k \) (where we abbreviate the partition \( (k) \) as \( k \)), we get

\[ s_{\lambda}(t+1, 1^{n-1}) = \sum_{k} d_{\lambda k} s_{k}(t, 0, 0, \ldots) = \sum_{k} d_{\lambda k} t^{k}. \]

Hence

\[ B_{\lambda,n}(k) = \frac{d_{\lambda k}}{s_{\lambda}(1^n)}. \]

The solution to Problem 84 shows that

\[ d_{\lambda k} = \frac{f^{\lambda/k}}{(d-k)!} \prod_{u \in \lambda/k} (n + c(u)) \]

Then \( B_{\lambda,n}(k) \) simplifies straightforwardly to the claimed formula using Corollary 7.21.4.

(c) Using (a) and (b), the expected value of \( m_1(T)^2 \) is given by

\[ 2B_{\lambda,n}(2) + B_{\lambda,n}(1) = \frac{2d(d-1)}{n(n+1)} \frac{f^{\lambda/(2)}}{f^{\lambda}} + \frac{d}{n}. \]

(d) By the combinatorial definition of skew Schur functions (Definition 7.10.1) the right-hand side is equal to \( B_{\lambda,n}(k) \), so the proof follows from (b).

35. Follows from Theorem 10.3 of J. S. Kim, K.-H. Lee, and S.-J. Oh, arXiv:1703.10321. (There is a typo in Theorem 10.3: \( |S^{(3,3)}_{2m-1}| \) should be \( |S^{(3,3)}_{2m+1}| \).) (Private communication from Jang Soo Kim, 30 August 2018.)

36. (a) **Hint.** Apply the exponential specialization.


Problem 99 is a special case.

38. (a) Note that (b) is a restatement of (a) using the code \( C_\lambda \) of Exercise 7.59. We are also using part (b) of this exercise (in the reverse form of adding rather than removing a border strip). Hence it suffices to prove (b). Suppose that there are \( k \) integers \( j < b \) for which \( f(j) = 1 \). Then by definition of \( a \) and \( b \), the only possible values of \( j \) for which \( f(j) = 1 \) and \( f(j-n) = 0 \) are the \( k \) integers \( j < b \) for which \( f(j) = 0 \), together with the \( n \) integers \( b+1, b+2, \ldots, b+n \). The only ones which fail to have the desired property are those of the form \( j + n \) where \( j < b \) and \( f(j) = 1 \). Thus the number of integers \( j \) for which \( f(j) = 1 \) and \( f(j-n) = 0 \) is \( (n+k) - k = n \).

39. Let the successive maximum size border strips be \( D_1, \ldots, D_k \), so \( D_k \) is the “innermost” border strip. Let \( D_k \) correspond to the composition \( \alpha^1 \), i.e., \( D_k = B_{\alpha^1} \) in the notation of §7.23 of EC2. Then \( D_{k-1} \) can be decomposed into three (nonempty) border strips in a canonical way, where the middle border strip corresponds to \( \alpha^1 \). Let the two other border strips correspond to \( \alpha^2 \) and \( \alpha^3 \). Continuing this construction to \( D_{k-2} \), we decompose it into three border strips with (an isomorphic copy of) \( D_{k-1} \) in the middle, and border strips corresponding to \( \alpha^4 \).
and $\alpha^5$ at the ends. The middle $D_{k-1}$ decomposes into border strips corresponding to $\alpha^1, \alpha^2, \alpha^3$. At the end of this process, the skew shape is decomposed into $k$ copies of $\alpha^1$, $k-1$ copies of $\alpha^2$ and $\alpha^3$, $k-2$ copies of $\alpha^4$ and $\alpha^5$, etc. The process can be reversed given $\alpha^1, \ldots, \alpha^{2k-1}$.

**Example.** The figure below shows a skew shape of depth three. The innermost border strip has squares labelled 1. The next innermost has squares labelled 1* that duplicate 1, and two further pieces that are border strips, labelled 2 and 3. Finally the outermost border strip (the first one removed) duplicates 1 with 1’, 2 with 2’, 3 with 3’, and has 4 and 5 in addition.

![Skew shape diagram]

We obtain

$$B_k(x) = \sum_{\alpha^1, \ldots, \alpha^{2k-1}} x^{k|\alpha^1|+(k-1)(|\alpha^2|+|\alpha^3|)+(k-2)(|\alpha^4|+|\alpha^5|)+\cdots+(|\alpha^{2k-2}|+|\alpha^{2k-1}|)},$$

where $\alpha^1, \ldots, \alpha^{2k-1}$ are nonempty compositions. It follows easily that

$$B_k(x) = \frac{x^{k^2}}{(1-2x^k) \prod_{i=1}^{k-1} (1-2x^i)^2}.$$

40. (a) See Figure 7-5 (page 339), where $\nu = \lambda$ and $\rho = \emptyset$.

(b) Using (a) (extended to a product of finitely many Schur functions) and the fact that $h_\nu = s_{\nu_1}s_{\nu_2}\cdots$, we have for suitable $\rho, \sigma$ that

$$K_{\lambda/\mu, \nu} = \langle s_{\lambda/\mu}, h_\nu \rangle = \langle s_{\lambda}, s_{\mu}h_\nu \rangle = \langle s_{\lambda}, s_{\rho/\sigma} \rangle = c_{\lambda\alpha}^\rho.$$
41. (a) For $\lambda \vdash n$ let $\varphi^\lambda$ be an irreducible matrix representation of $\mathfrak{S}_n$ with character $\chi^\lambda$, such that $\varphi^\lambda(w)$ is a matrix of integers for each $w \in \mathfrak{S}_n$. (It is known that such representations exist. A good reference is B. Sagan, The Symmetric Group.) For each $\rho \vdash n$ let $\tilde{c}_\rho$ denote the sum (in the group algebra $\mathbb{Z}[\mathfrak{S}_n]$) of all elements in $\mathfrak{S}_n$ of cycle type $\rho$. Then $\tilde{c}_\rho$ commutes with each $w \in \mathfrak{S}_n$, so by a basic result in representation theory (Schur’s lemma) $\varphi(\tilde{c}_\rho)$ is a scalar multiple of the identity matrix, say

$$\varphi(\tilde{c}_\rho) = \omega^\lambda_\rho I_d,$$  \hspace{1cm} (14)

where $\omega^\lambda_\rho \in \mathbb{Z}$ and $d = n!/H_{\lambda}$, the degree of $\chi^\lambda$. Taking traces in equation (14) yields

$$\frac{n!}{z_\rho} \chi^\lambda_\rho = \frac{n!}{H_{\lambda}} \omega^\lambda_\rho.$$  

Therefore

$$\omega^\lambda_\rho = \frac{H_{\lambda}}{z_\rho} \chi^\lambda_\rho$$

is an integer for all $\lambda, \rho$. The desired result follows easily.

A $p$-integral formula for $H_{\lambda}s_{\lambda}$ was given by Hanlon, J. Comb. Theory Series A 47 (1988), 37–70 (Property 3 on page 63).

(b) Immediate from Problem 37, but this is cheating. For an elementary proof, let $f^\perp$ denote the adjoint to multiplication by $f$, i.e., $\langle f^\perp g, h \rangle = \langle g, fh \rangle$ for all $g, h$. It’s easy to check (by verifying it for $f = p_{\lambda}, g = p_{\mu},$ and $h = p_{\nu},$ and using linearity) that

$$f^\perp = f \left( \frac{\partial}{\partial p_1}, 2 \frac{\partial}{\partial p_2}, 3 \frac{\partial}{\partial p_3}, \ldots \right).$$

For a partition $\rho$, define the differential operator

$$D_\rho = \frac{\partial}{\partial p_{\rho_1}} \frac{\partial}{\partial p_{\rho_2}} \ldots .$$

Then

$$s_{\lambda/\mu} = s^\perp_{\mu} s_{\lambda}$$

$$= \left( \sum_\rho z^{-1}_\rho \chi_{\mu}(\rho) 1^{m_1(\rho)} 2^{m_2(\rho)} \ldots D_\rho \right) \frac{1}{H_{\lambda}} \sum_\sigma d_{\lambda\sigma} p_{\sigma}.$$
where \( d_{\lambda \sigma} \in \mathbb{Z} \) by (a). Carrying out the differentiation gives

\[
S_{\lambda/\mu} = \frac{1}{H_\lambda} \sum_{\sigma, \rho} d_{\lambda \sigma} z_\rho^{-1} \chi_\mu(\rho) 1^{m_1(\rho)} 2^{m_2(\rho)} \cdots (m_1(\sigma))^{m_1(\rho)} (m_2(\sigma))^{m_2(\rho)} \cdots p_{\sigma - \rho}.
\]

Now the falling factorial \((m_i(\sigma))_{m_i(\rho)}\) is divisible by \(m_i(\rho)!\). Hence

\[
z_\rho^{-1} 1^{m_1(\rho)} 2^{m_2(\rho)} \cdots (m_1(\sigma))^{m_1(\rho)} (m_2(\sigma))^{m_2(\rho)} \cdots \in \mathbb{Z},
\]

so we have

\[
S_{\lambda/\mu} = \frac{1}{H_\lambda} \sum g_{\lambda \sigma} p_{\sigma - \rho}
\]

for some integers \(g_{\lambda \sigma}\). Now take the coefficient of \(x_1 x_2 \cdots x_{n-k}\) on both sides. The coefficient of \(x_1 x_2 \cdots x_{n-k}\) in \(p_\nu\) is 0 unless \(\nu = (1^{n-k})\), in which case the coefficient is \((n-k)!\). Thus for some integer \(g\) we get

\[
f^{\lambda/\mu} = \frac{1}{H_\lambda} g \cdot (n-k)!,
\]

so

\[
g = \frac{H_\lambda f^{\lambda/\mu}}{(n-k)!} = \frac{(n)_k f^{\lambda/\mu}}{f^{\lambda}},
\]

completing the proof.

Is there a simpler proof?

Greta Panova has pointed out that the proof also follows immediately from Naruse’s formula (Problem 42) for the number of SYT of skew shape \(\lambda/\mu\). In fact (using notation from Problem 42),

\[
\frac{(n)_j}{f^{\lambda}} = \prod_{E \in (\lambda/\mu)} \prod_{u \in E} h(u).
\]

42. This formula is due to Hiroshi Naruse, 73rd Sém. Lothar. Combin., 2014; www.emis.de/journals/SLC/wpapers/s73vortrag/naruse.pdf. It has led to a lot of subsequent work. In particular, a \(q\)-analogue appears in a paper by A.H. Morales, I. Pak, and G. Panova, J. Combin. Theory, Ser. A 154 (2018), 26–49, arXiv:1512.08348. Morales et al. have written (at least) three sequels to this paper.

44. (a) This result is due to Bernstein and appears in Macdonald [7.96, Exam. I.5.29.b.5]. See M. Zabrocki, J. Alg. Comb. 13 (2000), 83–101, arXiv:math/9904084, for a host of related results.

(b) If we apply RSK to $w$, we get precisely those pairs $(P, Q)$ of SYT with at most $n - k$ columns, and for which the first row of $Q$ is $1, 2, \ldots, n - k$. Hence

$$f_k(n) = \sum_{\lambda \vdash k} f^{(n-k, \lambda)} f^\lambda.$$ By (a),

$$f_k(n) = [x_1 \cdots x_n y_1 \cdots y_k] \sum_{i \geq 0} (-1)^i h_{n-k+i}(x) e_i(x) \sum_{\lambda \vdash k} s_\lambda(x) s_\lambda(y).$$ By Exercise 7.27(c) and the Cauchy identity we get

$$f_k(n) = [x_1 \cdots x_n y_1 \cdots y_k] \sum_{i \geq 0} (-1)^i h_{k+i}(x) e_i(y) \prod (1 - x_i y_j)^{-1},$$

etc. This result is due to A. Garsia and A. Goupil, Electronic J. Combinatorics 16(2) (2009), R19.


45. Let $\ell(\mu) = \ell$. The first column of $\lambda$ consists of 1 and any $n - k$ of the numbers $2, 3, \ldots, \ell$. Hence

$$K_{(k,1^{n-k}), \mu} = \binom{\ell - 1}{n - k}.$$
46. There are various ways to describe $P$ and $Q$. One elegant description (though perhaps not the best for proving correctness) is to fill in each tableau in the order $(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \ldots$, always using the least number available that keeps the columns strictly increasing, and ignoring any positions that cannot be filled. For instance, if $m = 4$ and $n = 3$, then the numbers that go into $P$ are 111122223333. First let $P_{11} = 1$, then $P_{12} = 1$, then $P_{2,1} = 2$, etc. Note that no number is available for $P_{4,1}$, so we continue with $P_{4,1} = 2$, etc.

47. (a) Call an SYT $T$ shiftable if it becomes an SHSYT by pushing the $i$th row of $T$ $i - 1$ squares to the right, for all $i \geq 1$. The key fact is that for each $W$-equivalence class $X$, there is a unique shiftable SYT that is the insertion tableau of some $w \in X$. For further details, see D. Worley, Ph.D. thesis, M.I.T., 1984 (Theorem 6.2.2). Worley’s result has never been published.

♣ Has someone else published this result?

(b) One possible example of an “interesting way” is to fix $k \geq 1$ and define two permutations $u, v \in S_n$ to be $W_k$-equivalent if they have the same insertion tableau and the same set of their first $k$ elements (when written as words). Or perhaps the first $k$ elements of $u$ (in order) should be the reverse or inverse (after standardizing) of the first $k$ elements of $v$.

48. (a) Given $w_1w_2 \cdots w_n$, insert $w_1, w_2, \ldots, w_n$ successively into an insertion shifted tableau $P$ as follows. Use ordinary row insertion as long as a diagonal element (i.e., the first element of some row) is not bumped. As soon as a diagonal element is bumped, switch to column insertion. Define a recording shifted tableau $Q$ by inserting $1, 2, \ldots, n$ into $Q$ to keep the same shape as $P$ (just as in ordinary RSK). Moreover, if the new position in $P$ was obtained by a column-insertion (i.e., if sometime in the bumping process a diagonal element was bumped) then circle the corresponding element of $Q$. This sets up a bijection between $S_n$ and pairs $(P, Q)$ of shifted SYT of the same shape $\lambda \models n$, and with some subset of the $n - \ell(\lambda)$ nondiagonal elements of $Q$ circled, thereby proving (2). For instance, if $w = 2651743$ then $P$ is built up as follows:

```
  2  2  6  25  125  1257  1247  12367
  6  6  6  56  45
```
Moreover, $Q$ is given (with an element underlined instead of circled) by

\[
\begin{array}{ccccccc}
1 & 2 & 4 & 5 \underline{7} \\
3 & 6
\end{array}
\]


(b) Equation (3) is a formal consequence of the following facts, which are not difficult to verify. Let $V_n$ be the complex vector space with basis $\{\lambda : \lambda \vdash n\}$. Define linear transformations $U_n : V_n \rightarrow V_{n+1}$ and $D_n : V_n \rightarrow V_{n-1}$ by

\[
U_n(\lambda) = \sum_{\mu} \mu + \zeta \sum_\nu \nu
\]

\[
D_n(\lambda) = \sqrt{2} \sum_\sigma \sigma + \bar{\zeta} \sum_\tau \tau,
\]

where (i) $\lambda \subset \mu \vdash n + 1$ and $\ell(\mu) = \ell(\lambda)$, (ii) $\lambda \subset \nu \vdash n + 1$ and $\ell(\nu) = \ell(\lambda) + 1$, (iii) $\lambda \supset \sigma \vdash n - 1$ and $\ell(\sigma) = \ell(\lambda)$, and (iv) $\lambda \supset \tau \vdash n + 1$ and $\ell(\tau) = \ell(\lambda) - 1$. Let $I_n$ denote the identity operator on $V_n$. Then

\[
D_{n+1}U_n - U_{n-1}D_n = I_n
\]

\[
D_{n+1} \left( \sum_{\lambda \vdash n+1} \lambda \right) = U_{n-1} \left( \sum_{\lambda \vdash n-1} \lambda \right) + \bar{\zeta} \sum_{\lambda \vdash n} \lambda.
\]


This problem just scratches the surface of the theory of shifted shapes. Practically anything that can be done for ordinary shapes has a shifted analogue (including the connections with representation theory). See Problem 49 below for further examples.

49. (a) Analogous to the Bender-Knuth proof that $s_\lambda$ is a symmetric function.
Clearly we want a “shifted analogue” of RSK. See the thesis of Worley or paper of Sagan (Cor. 8.3) cited in #48 above.


50. See mathoverflow.net/questions/193459.

51. Given a skew SYT of shape $\lambda/2$, add 2 to each entry and fill in the two “missing” squares with 1,2 from left-to-right. This gives a bijection with SYT of shape $\lambda$ such that 2 appears in the first row. Thus the first sum is equal to the number of $w \in S_n$ such that under RSK the insertion tableau has a 2 in the first row. This will be the case if and only if 2 follows 1 in $w$ (i.e., $w^{-1}(2) > w^{-1}(1)$). There are $n!/2$ such permutations, so

$$\sum_{\lambda \vdash n} f^{\lambda/2} f^{\lambda} = n!/2. \quad (15)$$

By similar reasoning, the second sum is given by the number of $w \in S_n$ such that $w^{-1}(2) > w^{-1}(1)$ and $w(2) > w(1)$. If neither $w(1)$ nor $w(2)$ equals 1 or 2, then there are $\frac{1}{2}(n-2)!(n-2)!$ such permutations. We can never have $w(2) = 1$. If $w(1) = 1$ then there are $(n-1)!$ such permutations. Hence

$$\sum_{\lambda \vdash n} (f^{\lambda/2})^2 = \frac{1}{2} \binom{n-2}{2} (n-2)! + (n-1)!$$

$$= \frac{1}{4}(n^2 - n + 2)(n-2)!.$$ \hspace{1cm} (16)

**Alternative proof of (16).** Given a skew SYT of shape $\lambda/2$, add 1 to each entry and fill in the two missing squares with 1’s. This gives a bijection with SSYT of shape $\lambda$ and type $(2, 1^{n-2})$. The RSK algorithm shows that the number of pairs of such SSYT of the same shape is equal to the number of $N$-matrices with row and column sum vector $(2, 1^{n-2})$. 
This was enumerated in the solution to Problem 24. (Equation (15) can also be obtained in this way.)

52. (a) By Exercise 7.28(a), if \( w \in S_n \) is an involution and \( w \rightarrow^r (P, P) \), then the number of fixed points of \( w \) is the number of columns of \( P \) of odd length. Hence (after transposing \( P \)) we see that \( \sum_{\lambda \vdash n} o(\lambda) f^\lambda \) is the total number of fixed points of all involutions in \( S_n \). Since \( i \) is a fixed point in \( t_{n-1} \) involutions, the total number of fixed points in all involutions in \( S_n \) is \( nt_{n-1} \).

(b) By Corollary 7.15.9 we get

\[
d(\lambda) = \left< s_{\lambda/1}, \sum_{\mu \vdash n-1} s_{\mu} \right>.
\]

Hence

\[
\sum_{\lambda \vdash n} d(\lambda) f^\lambda = \left< \sum_{\lambda \vdash n} f^\lambda s_{\lambda}, \sum_{\lambda \vdash n} \left< s_{\lambda/1}, \sum_{\mu \vdash n-1} s_{\mu} \right> s_{\lambda} \right>
\]

\[
= \left< p_1^n, \sum_{\lambda \vdash n} s_{\lambda}, p_1 \sum_{\mu \vdash n-1} s_{\mu} \right> s_{\lambda}
\]

\[
= \sum_{\lambda} \left< s_{\lambda}, p_1 \sum_{\mu \vdash n-1} s_{\mu} \right> f^\lambda \quad \text{(since } \left< p_1^n, s_{\lambda} \right> = f^\lambda \text{)}
\]

\[
= \left< p_1^n, p_1 \sum_{\mu \vdash n-1} s_{\mu} \right>.
\]

Since the operation \( \frac{\partial}{\partial p_1} \) is adjoint to multiplication by \( p_1 \) (see the solution to Exercise 7.35(a)), we get

\[
\sum_{\lambda \vdash n} d(\lambda) f^\lambda = \left< np_1^{n-1}, \sum_{\mu \vdash n-1} s_{\mu} \right>
\]

\[
= n \sum_{\mu \vdash n-1} f^\mu
\]

\[
= nt_{n-1}.
\]
Both results of this exercise, with different proofs, are due to K. Carde, J. Loubert, A. Potechin, and A. Sanborn, 2008 REU report (Corollary 6.3), available at


Is it just a coincidence that the sums in (a) and (b) have the same value?

Alternative proof (D. Grinberg, 2021). Let

\[ d_n = \sum_{\lambda \vdash n} d(\lambda) f^\lambda. \]

Now,

\[ t_{n+1} = \sum_{\mu \vdash n+1} f^\mu = \sum_{\lambda \vdash n} (d(\lambda) + 1) f^\lambda, \]

(17)

since each \( f^\mu \) equals the sum of the \( f^\lambda \)'s over all \( \lambda \) that are covered by \( \mu \) in Young's lattice, and since each partition \( \lambda \) is covered by exactly \( d(\lambda) + 1 \) partitions. Equation (17) becomes

\[ t_{n+1} = \sum_{\lambda \vdash n} (d(\lambda) + 1) f^\lambda = \sum_{\lambda \vdash n} d(\lambda) f^\lambda + \sum_{\lambda \vdash n} f^\lambda = d_n + t_n, \]

so that \( d_n = t_{n+1} - t_n = nt_{n-1} \) by the standard recurrence for the numbers \( t_n \).

53. This is a minor reformulation of a result of K. Carde, J. Loubert, A. Potechin, and A. Sanborn, arXiv:0808:0928 (Theorem 1.1, second version), which in turn is a reformulation of a conjecture of G. Han.

54. 1, 2, \ldots, k will appear in the first row of ins(\( w \)) if and only if 1, 2, \ldots, k is a subsequence of \( w \) (written as a word) [why?]. The probability that 1, 2, \ldots, k is a subsequence of \( w \in \mathfrak{S}_n \) but 1, 2, \ldots, \( k+1 \) isn’t a subsequence is given by \( k/(k+1)! \) if \( k < n \), and by \( 1/n! \) if \( k = n \). Hence

\[ E_n = \sum_{k=1}^{n-1} k \cdot \frac{k}{(k+1)!} + \frac{n}{n!}. \]
Thus

\[
\lim_{n \to \infty} E_n = \sum_{k \geq 1} \frac{k^2}{(k+1)!} = \sum_{k \geq 1} \frac{k(k+1) - (k+1) + 1}{(k+1)!} = e - (e - 1) + (e - 2) = e - 1.
\]

**Alternative solution** (considerably more elegant). The probability that 1, 2, ..., \(i\) appear in the first row of \(\text{ins}(w)\) is \(1/i!\) for \(1 \leq i \leq n\). Hence

\[
E_n = \sum_{i=1}^{n} \frac{1}{i!}
\]

etc.

55. See [mathoverflow.net/questions/82353](http://mathoverflow.net/questions/82353). Taedong Yun (private communication, 16 April 2012) computed that the total number of permutations \(w \in \mathfrak{S}_n\) for which \(P(1, 3) = n\) is equal to \(\binom{2n-2}{n-3}\), from which it follows that

\[
v_{13} = \sum_{n \geq 1} \frac{1}{(n-1)!} \binom{2n-2}{n-3} = 5.090678729\ldots.
\]

This gives an alternative formula to the one in the MathOverflow reference above.

(e) The motivation for this conjecture comes from a paper of A. Berele and A. Regev, *Advances in Math.* 64 (1987), 118–175, with further elucidation at arXiv:1007.3833. Consider a standard Young tableau \(P\) with \(P(i, j) = n\). The positions occupied by 1, 2, ..., \(n - 1\) are contained in an \((i-1, j-1)\)-hook, using the terminology of the papers just cited. Conversely, when \(n\) is large compared to \(i\) and \(j\), we expect that “most” SYT \(T\) of size \(n - 1\) contained in an \((i-1, j-1)\)-hook will contain the squares \((i-1, j)\) and \((i, j-1)\), so we can extend \(T\) by adding the square \((i, j)\) and inserting an \(n\). By the result of Berele and Regev, for fixed \(i, j\) the number of \(w \in \mathfrak{S}_n\) whose shape under RSK is contained in
an \((i-1,j-1)\)-hook is roughly \((i+j)^{2n}\). (Their formula actually involves \((i+j-2)^{2n}\), but \((i+j)^{2n}\) is a sufficiently accurate estimate for our purposes.) Thus the expected value of \(P(i,j)\) is roughly \(\sum_n n \frac{(i+j)^{2n}}{n!}\), which is roughly \(e^{(i+j)^{2n}}\). Can this argument (or a better similar argument if the present one is flawed) be made precise?

56. See the answer by R. Stanley at MathOverflow, mathoverflow.net/questions/331579.


58. This result was proved for square shapes (but the proof is exactly the same for rectangles) by D. Romik, Advances in Applied Math. 37 (2006), 501–510; www.stat.berkeley.edu/~romik/paperfiles/extremal.pdf.

Perhaps the simplest proof is to consider the inverse map \((P,Q) \mapsto w\).

59. Put \(x_1 = \cdots = x_i = t\) and \(y_1 = \cdots = y_j = 1\) and all other \(x_r, y_s = 0\) in the Cauchy identity (Theorem 7.12.1). We get

\[
 f_n(i,j) = [n^n](1-t)^{-ij} \\
 = \binom{n+ij-1}{ij-1}.
\]
60. By setting \( x_i = y_i \) in (7.44) and comparing with (7.20) we get

\[
y_n = \sum_{\lambda \vdash n} z^{-1}_\lambda p^2_\lambda.
\]

Write \((\lambda, \lambda)\) for the partition \((\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots)\). By the orthogonality of the power sums (equation (7.34)) we get

\[
\langle y_n, y_n \rangle = \sum_{\lambda \vdash n} z^{-2}_\lambda z(\lambda, \lambda) = \sum_{\lambda=(1^{m_1} 2^{m_2} \ldots)} \frac{1^{2m_1} 2^{2m_2} \ldots (2m_1)! (2m_2)! \ldots}{1^{2m_1} 2^{2m_2} \ldots (m_1)!^2 (m_2)!^2 \ldots} \binom{2m_i}{m_i}.
\]

Thus

\[
\sum_{n \geq 0} \langle y_n, y_n \rangle x^n = \left( \sum_{m_1 \geq 0} \binom{2m_1}{m_1} t^{m_1} \right) \left( \sum_{m_2 \geq 0} \binom{2m_2}{m_2} t^{2m_2} \right) \ldots = \frac{1}{\sqrt{1 - 4x}} \frac{1}{\sqrt{1 - 4x^2}} \frac{1}{\sqrt{1 - 4x^3}} \ldots = P(x, 4)^{1/2}.
\]

61. See J. F. Willenbring, *J. Algebra* **242** (2001), 691–708. Willenbring’s proof is based on the representation theory of the orthogonal group. The formula given here is simpler than Willenbring’s (though of course equal to it). It can be obtained analogously to Problem 60 (though more complicated). To begin, set each \( x_i = y_i \) in (7.44) and use (7.20) to get the \( p \)-expansion of \( \sum s^2_\mu \). For the \( p \)-expansion of \( \sum s_{2\lambda} \) use Exercise 7.28. The equality of Willenbring’s formula with the one of this exercise can be proved by expanding their logarithms.

62. The symmetric function \( G \in \hat{\Lambda}_R \) has the desired property if and only if it has the form \( G = F(x_1) F(x_2) \ldots \), where \( F(x) \in R[[x]] \) and \( F(0) = 1 \). Equivalently (as is easy to see), \( \log G(x) \) has the form \( \sum_{k \geq 1} c_k p_k \), where \( c_k \in R \). (See Exercise 7.91 for more on such series.)
Proof. First assume that \( \log G = \sum_{k \geq 1} c_k p_k \). Then, as in Proposition 7.74, we have

\[
G = \sum_{\lambda} z_{\lambda}^{-1} \left( \prod_i c_{\lambda i} \right) p_{\lambda}.
\]

Let \( f = \sum_{\mu} a_{\mu} p_{\mu} \) and \( g = \sum_{\nu} b_{\nu} p_{\nu} \). For any partition \( \rho \) let \( c_\rho = \prod_i c_{\rho i} \).

Then by the orthogonality of the power sums,

\[
\langle G, fg \rangle = \left\langle \sum_{\lambda} z_{\lambda}^{-1} c_{\lambda} p_{\lambda}, \sum_{\mu, \nu} a_{\mu} b_{\nu} p_{\mu} p_{\nu} \right\rangle = \sum_{\mu, \nu} a_{\mu} b_{\nu} c_{\mu} c_{\nu} = \left( \sum_{\mu} a_{\mu} c_{\mu} \right) \left( \sum_{\mu} b_{\mu} c_{\mu} \right) = \langle G, f \rangle \cdot \langle G, g \rangle.
\]

Conversely, let \( I = \sum_{\lambda} z_{\lambda}^{-1} d_{\lambda} p_{\lambda} \), and suppose that \( \langle I, fg \rangle = \langle I, f \rangle \cdot \langle I, g \rangle \) for all \( f, g \). In particular,

\[
d_{\mu \cup \nu} = \langle I, p_{\mu} p_{\nu} \rangle = d_{\mu} d_{\nu},
\]

so by iteration \( d_{\lambda} = \prod d_{\lambda i} \), completing the proof.

63. Since Schur functions \( s_\lambda(x_1, \ldots, x_n) \) are the polynomial characters of \( \text{GL}(n, \mathbb{C}) \) (Appendix 2), it follows by Weyl’s “unitary trick” that they are the irreducible characters of the unitary group \( U(n) \). Hence

\[
\langle V_n^{2k}, V_n^{2k} \rangle_n = \int_{u \in U(n)} V_n^{2k} \overline{V_n^{2k}} \, du = \int_{u \in U(n)} |V_n|^{4k} \, du.
\]

The Weyl integration formula allows us to express this integral as an integral over a torus. By the residue theorem, it is equal to

\[
\frac{1}{n!} \text{CT} \prod_{i,j=1}^{n} (1 - x_i x_j^{-1})^{2k+1},
\]

26

64. It is clear from Corollary 7.13.8 that

\[
\sum_{\lambda} s_{\lambda} = \sum_{\mu} L_{\mu} m_{\mu},
\]

where \(L_{\mu}\) is the number of symmetric \(N\)-matrices \((A_{ij})_{i,j\geq 1}\) with row\((A) = \) col\((A) = \mu\). Since \((h_{\lambda}, m_{\mu}) = \delta_{\lambda\mu}\), it follows that \(a_n\) is the number of \(n \times n\) symmetric \(N\)-matrices with every row and column sum equal to two. Hence by Example 5.2.7 we get

\[
F(t) = \frac{e^{t^2 + \frac{t}{1-t}}}{\sqrt{1-t}}.
\]

65. (a) It follows from Exercise 7.35(a) or from Problem 27(b) that \(\frac{\partial}{\partial p_1} s_{\lambda} = s_{\lambda/1}\). Hence from the case \(n = 1\) of Theorem 7.15.7 and Corollary 7.15.19, we have

\[
T_n = (U + D + DU)^n 1,
\]

where \(U = p_1\) (i.e., the operator given by multiplication by \(p_1\)) and \(D = \frac{\partial}{\partial p_1}\). (See Exercise 7.24 for similar reasoning.) From this it is clear that \(T_n = g_n(p_1)\), for some polynomial \(g_n \in \mathbb{Z}[t]\).

Now let \(y = \sum_{n \geq 0} g_n(t) \frac{x^n}{n!}\). Then from (18) we have

\[
(t + \frac{\partial}{\partial t} + \frac{\partial}{\partial t} t)y = \frac{\partial y}{\partial x}.
\]

a partial differential equation with the initial condition \(y(0) = 1\). There are various methods for solving this type of equation. For instance, let \(y = e^z\). Then

\[
t + z_t + 1 + tz_t = z_x.
\]
The general solution to the associated homogeneous equation $u_t + t u_t = u_x$ is $u(x,t) = F(\log(1 + t) + x)$ for any (smooth) $F$. A particular solution is seen by inspection (or by plugging in a linear polynomial with indeterminate coefficients) to be $-t$. Hence the general solution to (19) is given by

$$z(x,t) = -t + F(\log(1 + t) + x).$$

The initial condition $z(0,t) = 0$ shows that $F(\log(1 + t) + x) = -1 + \exp(\log(1 + t) + x)$. Hence

$$y = e^{-1-t+(1+0)e^x}. \tag{20}$$

Thus

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = y|_{t=0} = e^{ex-1} = \sum_{n \geq 0} B(n) \frac{x^n}{n!},$$


(b) We have by (20) that

$$\sum_{m,n \geq 0} \langle T_m, T_n \rangle \frac{x^m y^n}{m! n!} = \langle e^{-1-p_1+(1+p_1)e^x}, e^{-1-p_1+(1+p_1)e^y} \rangle$$
\[ = \exp(e^x + e^y - 2) \left\{ \exp(p_1(e^x - 1), \exp(p_1(e^y - 1)) \right. \]
\[ = \exp(e^x + e^y - 2) \left\{ \sum_{m \geq 0} p_1^m \frac{(e^x - 1)^m}{m!}, \sum_{n \geq 0} p_1^n \frac{(e^y - 1)^n}{n!} \right\} \]
\[ = \exp(e^x + e^y - 2) \sum_{n \geq 0} (e^x - 1)^n (e^y - 1)^n \frac{1}{n!} \]
\[ = \exp(e^x + e^y - 2 + (e^x - 1)(e^y - 1)) \]
\[ = \exp(e^{x+y} - 1) \]
\[ = \sum_{k \geq 0} B(k) \frac{(x + y)^k}{k!} \]
\[ = \sum_{m, n \geq 0} B(m + n) \frac{x^m y^n}{m! n!}, \]

whence \( \langle T_m, T_n \rangle = B(m + n) \).

The following much slicker argument was given by T. Lam. The result of part (a) may be restated as
\[ \langle T_n, 1 \rangle = \langle (U + D + DU)^n 1, 1 \rangle = B(n). \]

But \( U \) and \( D \) are adjoint with respect to \( \langle , \rangle \), so \( U + D + DU \) is self-adjoint. Hence
\[ \langle T_m, T_n \rangle = \langle (U + D + DU)^m 1, (U + D + DU)^n 1 \rangle \]
\[ = \langle (U + D + DU)^{m+n} 1, 1 \rangle \]
\[ = B(m + n). \]

Solutions to (a) and (b) can also be given using induction, working directly with the operators \( U \) and \( D \) and avoiding generating functions.

This problem is not as contrived as it may first appear. There is a semisimple algebra \( \mathcal{P}_n \), called the partition algebra, whose irreducible representations \( \sigma^\lambda \) are indexed by (certain) partitions \( \lambda \), such that \( \dim \sigma^\lambda = \langle T_n, s_\lambda \rangle \). See P. Martin, J. Knot Theory Ramifications 3 (1994), 51–82; P. Martin, J. Algebra 183 (1996), 319–358; P. Martin and G. Rollet, Compositio Math. 112 (1998), 237–254; P. Martin and D. Woodcock, J. Algebra 203 (1998), 91–124; C. Xi, Compositio Math. 119 (1999), 99–109; W. Doran and D. Wales, J. Algebra 231 (2000),
66. (a) It is easy to see that
\[ (g(p) + h(p) \frac{\partial}{\partial p}) F(x, p) = \frac{\partial}{\partial x} F(x, p), \]
with the initial condition \( F(0, p) = 1 \). It is then routine to check that the stated formula for \( F(x, p) \) does satisfy this differential equation and initial condition.

(b) We have \( L(t) = L^{(-1)}(t) = t \) and \( M(t) = \frac{1}{2} t^2 \). Hence by (a),
\[
F(x, p) = \exp \left[ -\frac{1}{2} p^2 + \frac{1}{2} (x + p)^2 \right]
= \exp \left( px + \frac{1}{2} x^2 \right).
\]

(c) We have [why?]
\[
\sum_{n \geq 0} \sum_{\lambda \in \text{Par}} g_\lambda(n) s_\lambda \frac{x^n}{n!} = F(x, p),
\]
where \( F(x, p) \) corresponds to \( g(t) = 1 + t \) and \( h(t) = 1 + t \). Hence \( L(t) = \log(1 + t) \), \( L^{(-1)}(t) = e^t - 1 \), and \( M(t) = t \), so
\[
L^{(-1)}(x + L(p)) = (1 + t)e^x - 1
\]
\[
F(x, p) = \exp \left[ -p + e^{x + \log(1+t)} - 1 \right]
= \exp(-1 - p + (1 + p)e^x).
\]

(d) We have \( g(t) = h(t) = 1/(1 - t) \), \( L(t) = t - \frac{1}{2} t^2 \), \( L^{(-1)}(t) = 1 - \sqrt{1 - 2t} \), and \( M(t) = t \). We get
\[
\sum_{n \geq 0} g_\delta(n) s_\lambda \frac{x^n}{n!} = F(x, p)
= \exp \left( -p + 1 - \sqrt{1 - 2 \left( x + p - \frac{1}{2} p^2 \right)} \right)
= \exp \left( 1 - p - \sqrt{(1-p)^2 - 2x} \right).
\]
(e) We have $g(t) = 1/(1-t)$, $h(t) = 1$, $L(t) = L^{(-1)}(t) = t$, and $M(t) = -\log(1-t)$. Hence

$$F(x,p) = \exp\left(\log(1-p) - \log(1-x-p)\right)$$

$$= \frac{1}{1 - \frac{x}{1-p}}$$

$$= \sum_{n \geq 0} \frac{x^n}{(1-p)^n}.$$ 

etc. Is there a nice bijective proof?

67. One first shows that $\text{ins}(w)$ can be obtained by row inserting $a_{n+1}$, then column inserting $a_n$, then row inserting $a_{n+2}$, then column inserting $a_{n-1}$, etc., ending with a row insertion of $a_{2n}$ followed by a column insertion of $a_1$. It can be shown that for each $1 \leq i \leq n$ the shape increases by the addition of one domino after inserting $a_{n+i}$ and $a_{n+1-i}$, and the proof follows. Alternatively, by Theorem A.1.2.10 we have $P = \text{evac}(P)$. Now consider the growth diagram for computing $\text{evac}(P)$ (Figure A1-13), or see Section 3 of R. Stanley, Promotion and evacuation, *Electronic J. Comb.* 15(2) (2008–2009). For a good overview of this topic, see M. Shimozono and D. E. White, *Electronic J. Combinatorics* 8(1) (2001), R21.

68. If $d \nmid n$ and $\lambda \vdash n$, then some $\lambda_j$ is not divisible by $d$. Hence

$$p_{\lambda_j}(1, \zeta, \ldots, \zeta^{d-1}) = \frac{1 - \zeta^{d\lambda_j}}{1 - \zeta^{\lambda_j}} = 0.$$ 

Thus $p_{\lambda}(1, \zeta, \ldots, \zeta^{d-1}) = 0$, and the proof follows (since the $p_{\lambda}$'s for $\lambda \vdash n$ are a basis for $\Lambda^n_Q$).

For stronger results about $s_{\lambda}(1, \zeta, \ldots, \zeta^{d-1})$, note that

$$\sum_{\lambda} s_{\lambda}(1, \zeta, \ldots, \zeta^{d-1}) s_{\lambda}(x) = \frac{1}{\prod_k (1 - x_k^d)} = \sum_{n \geq 0} h_n(x_1^d, x_2^d, \ldots)$$

and see Exercise 7.61.

69. See I. P. Goulden, *Canad. J. Math.* 42 (1990), 763–775 (Theorem 2.3(a)).
70. We have
\[ \sum_{u \in \lambda/\mu} h_{\lambda/\mu}(u) = \sum_{u \in \lambda/\mu} \sum_{v \in H(v)} 1. \]  \hspace{1cm} (21)

Given \( u, v \in \lambda/\mu \), it is easy to see that
\[ v \in H_{\lambda/\mu}(u) \iff u \in H_{(\lambda/\mu)^r}(v). \]

Hence reversing the order of summation in (21) gives \( \sum_{u \in \lambda/\mu} h_{(\lambda/\mu)^r}(u) \), as desired.

71. (a) This identity was conjectured by T. Amdeberhan on MathOverflow 312771 on October 13, 2018, with solutions by G. Zaimi and S. Hopkins.

72. (a) Let
\[ f(n) = \sum_{\lambda \vdash n} \eta_k(\lambda). \]

It is an easy consequence of Exercise 7.59(a,b) that the number of ways to add a border strip of size \( k \) to \( \lambda \) is \( k \) more than the number of border strips (or hooks) of \( \lambda \) of size \( k \). (In fact, this is a direct consequence of Exercise 7.59(e) and the observation that in \( Y^k \), an element \( \lambda \) is covered by \( k \) more elements than it covers. See Exercise 3.51(c) of EC1, second ed.) It follows that
\[ f(n + k) = kp(n) + f(n), \]
where \( p(n) \) denotes the number of partitions of \( n \). Let \( F(x) = \sum_{n \geq 0} f(n)x^n \) and \( P(x) = \sum_{n \geq 0} p(n)x^n \). We get
\[ x^{-k}F(x) = kP(x) + F(x), \]
whence
\[ F(x) = kx^kP(x)/(1 - x^k). \]
It is easy to see that
\[ \sum_{n \geq 0} \left( \sum_{\lambda \vdash n} m_k(\lambda) \right) x^n = \frac{x^kP(x)}{1 - x^k}, \]
and the proof follows. This result is due to R. Bacher and L. Manivel, Sém. Lotharingien de Combinatoire 47 (2002), B47d; arXiv.math/0108199.

(b) This is a result of C. Bessenrodt and G. Han, Discrete Math. 309 (2009), 6070–6073. The proof follows easily from the following stronger result, which is proved by a combinatorial argument. Given \( u = (i, j) \in \lambda \), let \( a(u) = \lambda_i - i \) (the arm length of \( u \)) and
Let \( f_n(a, l, r) \) denote the number of ordered pairs \((\lambda, u)\) such that \( \lambda \vdash n \), \( u \in \lambda \), \( a(u) = a \), \( l(u) = l \), \( r(u) = r \). Then

\[
\sum_{n \geq 0} f_n(a, l, r)q^n = \frac{1}{\prod_{i \geq a+1}(1-q^i)} \left( \begin{array}{c} l + a \\ a \end{array} \right) \left( \begin{array}{c} r + a \\ a \end{array} \right) q^{(m+1)(l+1)+1}.
\]


74. Easy.

75. By Theorem 7.15.1 we have

\[
s_\lambda(x_1^2, \ldots, x_n^2) = \frac{a_2(\lambda+\delta)}{a_2\delta} \cdot \frac{a_\delta}{a_2\delta} = \frac{s_{2\lambda+\delta}(x_1, \ldots, x_n)}{s_\delta(x_1, \ldots, x_n)},
\]

whence

\[
s_\delta(x_1, \ldots, x_n)s_\lambda(x_1^2, \ldots, x_n^2) = s_{2\lambda+\delta}(x_1, \ldots, x_n).
\]

For some related results, see Exercise 7.30.

76. By the solution to Exercise 7.69(a) we have

\[
\left( \sum \limits_{n \geq 0} s_\lambda \right)^t = \exp \left( t \left( \sum \limits_{n \geq 1} \frac{1}{n} p_n + \sum \limits_{n \geq 1} \frac{1}{n} p_n^2 / 2 \right) \right).
\]

We can now apply the exponential formula as in the solution to Exercise 7.69(a).

77. Let \( N = 2n \) and \( \lambda = \delta_N = (N, N-1, \ldots, 1) \). Thus in Exercise 7.40, \( r = n \) and \( B(i, j) = B_{(2N+1-i-j, 1)} \), the “zigzag” border strip corresponding to the composition \((2^{N+1-i-j}, 1)\). The number \( f^B (i, j) \) of SYT of (skew)
shape \( B(i, j) \) is \( E_{|B(i,j)|} = E_{2N+3-2i-2j} \) (see Exercise 7.64(a)). Take the formula of Exercise 7.40, reflect the matrix through the antidiagonal (which doesn’t effect the determinant) and apply \( \exp \) (the exponential specialization) to obtain

\[
A_n = \frac{f^\delta_N}{|\delta_N|!}.
\]

Thus by the hook-length formula (Corollary 7.21.6) we get

\[
A_n = \frac{1}{\prod_{u \in \delta_N} h(u)} = \frac{1}{1^N 3^{N-1} 5^{N-2} \cdots (2N - 1)^1}.
\]

Exactly analogous reasoning for for \( N = 2n - 1 \) yields that

\[
B_n = \frac{1}{1^N 3^{N-1} 5^{N-2} \cdots (2N - 1)^1}.
\]

78. Let \( \tau = \tau_n \) be the “zigzag shape” of Exercise 7.64. By Corollary 7.23.8 we have \( f(n) = \langle \tau, \tau \rangle \). Let \( \chi = \text{ch}(\tau) \). Since \( \text{ch} \) is an isometry (Prop. 7.18.1) we have \( f(n) = \langle \chi, \chi \rangle \). By Exercise 7.64 we get for \( n \) odd that

\[
\langle \chi, \chi \rangle = \sum_{\mu \vdash n} z^{-1} \chi(\mu)^2
= \sum_{\mu} z^{-1} E_{2r}^2,
\]

where \( \mu \) ranges over all partitions of \( n \) with \( 2r + 1 \) odd parts and no even parts. The proof now follows from a routine application of Corollary 5.1.8. The proof for even \( n \) is similar, using a simple modification of Corollary 5.1.8.


80. This was a conjecture of R. Stanley, presented on June 25, 2003, at the 15th FPSAC meeting in Vadstena, Sweden, and available at

\[\text{math.mit.edu/~rstan/transparencies/fpsacproblem.pdf}\]
It was proved by M. Ishikawa, *Ramanujan J.* 16 (2008), 211–234; arXiv.math/0408204.

81. (a) By (7.66) we have

\[ s_\lambda(x, y) = \sum_{\mu \subseteq \lambda} s_\mu(x)s_{\lambda/\mu}(y). \]

Now apply \( \omega_y \) and use Theorem 7.15.6.

(b) It is clear from the definition (4) that

\[ p_n(x/y)_{x_1=t, y_1=-t} = p_n(x/y)_{x_1=y_1=0}. \]

Since \( \omega_y \) and the substitutions \( x_1 = t, y_1 = -t \) and \( x_1 = y_1 = 0 \) are homomorphisms (into suitable rings), it follows that \( g(x, y) \) satisfies (6) whenever \( g \in \text{im}(\omega_y) \).

Conversely, suppose that \( g(x, y) \in \Lambda(x) \otimes \Lambda(y) \) and that \( g(x, y) \) satisfies (6). Thus by hypothesis

\[ g((t, x), (-t, y)) = g(x, y), \tag{22} \]

where \( (t, x) = (t, x_1, x_2, \ldots) \) and similarly for \( (t, y) \). Define

\[ u_i = \frac{1}{2} \left( p_i(x) + (-1)^i p_i(y) \right) \]
\[ v_i = \frac{1}{2} \left( p_i(x) + (-1)^{i-1} p_i(y) \right). \]

It is clear that \( g(x, y) \) can be written uniquely as a polynomial \( P(u_1, u_2, \ldots; v_1, v_2, \ldots) \) in the \( u_i \)'s and \( v_i \)'s, and we want to show that no \( u_i \) appears. Now

\[ g((t, x), (-t, y)) = P(u_1 + t, u_2 + t^2, \ldots; v_1, v_2, \ldots). \tag{23} \]

By considering the coefficient of \( t \) in (23) and using (22), we see that \( u_1 \) does not appear in \( P \). Now by considering the coefficient of \( t^2 \) it follows that \( u_2 \) does not appear in \( P \), etc. This proof is due to John Stembridge (unpublished).

(c) See J. R. Stembridge, *J. Algebra* 95 (1985), 439–444 (Theorem 1).
(d) This is immediate from
\[ p_{2i}(x/x) = p_{2i}(x) + (-1)^{2i-1}p_{2i}(x) = 0. \]

(e) From (a) it is easy to see that
\[ [x^\alpha y^\beta]s\lambda(x/y) = \langle h^\alpha e^\beta, t^\lambda \rangle. \]
This corresponds to building up the shape \( \lambda \) by first adding horizontal strips of sizes \( \alpha_1, \alpha_2, \ldots \) and then vertical strips of sizes \( \beta_1, \beta_2, \ldots \). But multiplication of symmetric functions is commutative, so we could first add a horizontal strip \( H_1 \) of size \( \alpha_1 \), then a vertical strip \( V_1 \) of size \( \beta_1 \), then a horizontal strip \( H_2 \) of size \( \alpha_2 \), then a vertical strip \( V_2 \) of size \( \beta_2 \), etc. To get \( s\lambda(x/x) \), fill in each square of \( H_i \cup V_i \) with \( i \), getting an array \( T \), and sum all \( x^T \)'s. Now the array \( T \) is precisely a supertableau. How many times does a particular supertableau \( T \) get counted? For each component \( C \) of \( H_i \cup V_i \), there are exactly two ways to decompose \( C \) into a horizontal strip \( H \) followed by a vertical strip \( V \). (The last square in the top row can be part of \( H \) or \( V \), but there are no other choices.) Hence the number of times \( T \) gets counted is \( 2^{|(T)|} \), completing the proof.

(f) First note that \( s_{(nm)}(x_1, \ldots, x_m/y_1, \ldots, y_n) \) is a homogeneous polynomial of degree \( mn \). If \( \mu \subseteq (nm) \), then either \( \ell(\mu) = m \) or \( (nm)/\mu \) contains a row of length \( n \). In the former case \( s_{\mu}(x_2, x_3, \ldots, x_m) = 0 \) and in the latter \( s_{(nm)/\mu}(y_2, y_3, \ldots, y_n) = 0 \). Thus by (5) we have \( s_{(nm)}(0, x_2, \ldots, x_m/0, y_2, \ldots, y_n) = 0 \), so by (6) we have
\[ s_{(nm)}(x_1, x_2, \ldots, x_m/y_1, y_2, \ldots, y_n)|_{x_1=-y_1} = 0. \]
Hence \( s_{(nm)}(x_1, x_2, \ldots, x_m/y_1, y_2, \ldots, y_n) \) is divisible by \( x_1+y_1 \). By symmetry, \( s_{(nm)}(x_1, x_2, \ldots, x_m/y_1, y_2, \ldots, y_n) \) is divisible by \( x_i + y_j \) for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). This accounts for \( mn \) factors, so \( s_{(nm)}(x_1, x_2, \ldots, x_m/y_1, y_2, \ldots, y_n) = c \Pi(x_i + y_j) \) for some constant \( c \). It is easy to see that \( c = 1 \), e.g., by considering the coefficient of \( x_1^n x_2^n \cdots x_m^n \), completing the proof. This result is due to D. E. Littlewood and can be found in [7.88, XVIII on page 115].

**Bijective proof.** Immediate from Exercise 7.42 (with the typo \( s_{\tilde{\lambda}}(y) \) replaced with \( s_{\tilde{\lambda}}(y) \)).
(g) This result is due to A. Berele and A. Regev, *Advances in Math.* 64 (1987), 118–175. A bijective proof was given by J. B. Remmel, *Linear and Multilinear Algebra* 28 (1990), 119–154. For another proof, see [7.96, Exam. I.3.23(4)].


83. (a) We have $\langle s^k \rho_\lambda, \rho_\mu \rangle = \langle \rho_\lambda, s^k \rho_\mu \rangle$. The proof now follows from $s_k = \sum_{\mu \vdash k} z_\mu^{-1} \rho_\mu$ and the orthogonality relation $\langle \rho_\lambda, \rho_\nu \rangle = \delta_\lambda \nu z_\lambda$.

(b) See mathoverflow.net/questions/376494 (answer by R. Stanley).

84. Sketch of proof. Write $\vartheta_n$ for $\vartheta$ specialized to $t = n$, where $n \in \mathbb{P}$. Note that

$$\vartheta_n(p_k)(x_1, \ldots, x_n) = p_k(x_1 + 1, \ldots, x_n + 1).$$

Hence

$$\vartheta_n(s_\lambda)(x_1, \ldots, x_n) = s_\lambda(x_1 + 1, \ldots, x_n + 1).$$

Now

$$s_\lambda(x_1 + 1, \ldots, x_n + 1) = \frac{a_{\lambda + \delta}(x_1 + 1, \ldots, x_n + 1)}{a_\delta(x_1 + 1, \ldots, x_n + 1)} = \frac{a_{\lambda + \delta}(x_1 + 1, \ldots, x_n + 1)}{a_\delta(x_1, \ldots, x_n)}.$$

By expanding the entries of $a_{\lambda + \delta}(x_1 + 1, \ldots, x_n + 1)$ and using the multilinearity of the determinant we get (see I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Example I.3.10 on page 47)

$$s_\lambda(x_1 + 1, \ldots, x_n + 1) = \sum_{\mu \subseteq \lambda} d_{\lambda \mu} s_\mu, \quad (24)$$

where

$$d_{\lambda \mu} = \det \left( \begin{array}{c} \lambda_i + n - i \\ \mu_j + n - j \end{array} \right)_{1 \leq i, j \leq n}.$$

We can factor out factorials from the numerators of the row entries and denominators of the column entries of the above determinant. These factorials altogether yield $\prod_{u \in \lambda/\mu} (n + c(u))$. What remains is exactly
the determinant for $f^{\lambda/\mu}/|\lambda/\mu|!$ given by Corollary 7.16.3, and we obtain

$$\vartheta_n(s_\lambda)(x_1, \ldots, x_n) = \sum_{\mu \subseteq \lambda} \frac{f^{\lambda/\mu}}{|\mu/\mu|!} \left( \prod_{u \in \lambda/\mu} (n + c(u)) \right) s_\mu(x_1, \ldots, x_n).$$

We can set $x_{i+1} = \cdots = x_n = 0$ to obtain the above equation in $i$ variables. In other words,

$$\vartheta_n(s_\lambda)(x_1, \ldots, x_i) = \sum_{\mu \subseteq \lambda} \frac{f^{\lambda/\mu}}{|\mu/\mu|!} \left( \prod_{u \in \lambda/\mu} (n + c(u)) \right) s_\mu(x_1, \ldots, x_i)$$

for all $n \geq i$. Both sides are polynomials in $n$, so they are equal as polynomials, and we can replace $n$ with the indeterminate $t$. Now let $i \to \infty$.

Is there a more conceptual proof that doesn’t involve the evaluation of a determinant?


86. Part (a) follows from Section 4.2 of

www.mat.univie.ac.at/˜slc/wpapers/FPSAC2018/

50-Thiel-Williams.html

by M. Thiel and N. Williams. The key to doing the entire exercise is to first show that for any symmetric function $f$, we have

$$[z^r] f(z + 1, 1, \cdots, 1, 0, 0, \ldots) = \frac{1}{n \cdot r!} \psi_r f,$$

38
where \([z^r]g\) denotes the coefficient of \(z^r\) in \(g\) (when expanded as a power series in \(z\)). This exercise is based on work of X. Li, L. Mu, and R. Stanley, in preparation.

87. This is the special case of Exercise 7.47(j) where \(P\) is both \((3 + 1)\)-free and \((2 + 2)\)-free. It was shown by M. Guay-Paquet, arXiv:1306.2400, that this special case actually implies the full conjecture.

For the case \(I = \{(1, 2), (2, 3), \ldots, (n - 1, n)\}\) (i.e., no two consecutive elements of \(i_1i_2 \cdots i_n\) are equal), see Exercise 7.47(k). Ben Joseph claimed to have a proof based on the involution principle for the case \(I = \{(1, 2, 3), (2, 3, 4), \ldots, (n - 2, n - 1, n)\}\), but this was never written up. A different proof is due to S. Dahlberg, Sém. Lotharingien de Combinatoire 82B (2019), Article #59, 12 pp.

David Gebhard and Bruce Sagan, J. Algebraic Comb. 13 (2001), 227–255, have generalized Exercise 7.47(k) to the case \(I = \{(1, 2, \ldots, k_1), (k_1, k_1 + 1, \ldots, k_2), (k_2, k_2 + 1, \ldots, k_3), \ldots, (k_r, k_r + 1, \ldots, n)\}\). For further work on chromatic symmetric functions and their quasisymmetric generalization, see the web page

https://www.math.upenn.edu/~peal/polynomials/chromaticQuasisymmetric.htm

of Per Alexandersson.

88. (a) We have

\[
a(m, n) = \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\mu f^\nu f^\lambda c_{\mu\nu}^\lambda \\
= \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\mu f^\nu f^\lambda \langle s_\mu s_\nu, s_\lambda \rangle \\
= \langle \left( \sum_{\mu \vdash m} f^\mu s_\mu \right) \left( \sum_{\nu \vdash n-m} f^\nu s_\nu \right), \sum_{\lambda \vdash n} f^\lambda s_\lambda \rangle \\
= \langle p_1^m p_{n-m}^\nu, p_1^n \rangle \\
= n!.
\]
Similarly we obtain

\[ b(m, n) = \left\langle p_1^n, \sum_{\lambda \vdash n} s_{\lambda} \right\rangle = t(n), \]

since e.g. if \( f \in \Lambda^n \) then \( \left\langle p_1^n, f \right\rangle = [x_1 x_2 \cdots x_n]f \), while \( \sum_{\lambda \vdash n} f^\lambda = t(n) \) by Corollary 7.13.9. In the same way we obtain

\[ c(m, n) = \left\langle \left( \sum_{\mu \vdash m} s_{\mu} \right) p_1^{n-m}, p_1^n \right\rangle. \]

An elegant way to proceed is to use the fact (Problem 27(b)) that multiplication by \( p_1 \) is adjoint to \( \frac{\partial}{\partial p_1} \). Hence

\[ c(m, n) = \left\langle \sum_{\mu \vdash m} s_{\mu}, \frac{\partial^{n-m}}{\partial_{n-m} p_1} p_1^n \right\rangle = \left\langle \sum_{\mu \vdash m} s_{\mu}, (n)_{n-m} p_1^{n-m} \right\rangle = (n)_{n-m} t(m). \]

Finally,

\[ d(m, n) = \left\langle \left( \sum_{\mu \vdash m} s_{\mu} \right) \left( \sum_{\nu \vdash n-m} s_{\nu} \right), p_1^n \right\rangle = \left[ x_1 x_2 \cdots x_n \right] \left( \sum_{\mu \vdash m} s_{\mu} \right) \left( \sum_{\nu \vdash n-m} s_{\nu} \right) = \binom{n}{m} t(m) t(n - m). \]

(b) See C. Ikenmeyer, *J. Alg. Combinatorics* 44 (2016), 1–29 (Theorem 1.1); *arXiv* 1209.1521.
(c) We have (pointed out by B. Tenner)

\[ e(m, n) = \left\langle \left( \sum_{\mu \vdash m} s_\mu \right) \left( \sum_{\nu \vdash n-m} f_\nu s_\nu \right), \sum_{\lambda \vdash n} s_\lambda \right\rangle \]

\[ = \left\langle \left( \sum_{\mu \vdash m} s_\mu \right) p_1^{n-m}, \sum_{\lambda \vdash n} s_\lambda \right\rangle \]

\[ = \sum_{\mu \vdash m} \sum_{\lambda \vdash n} \langle p_1^{n-m} s_\mu, s_\lambda \rangle \]

\[ = \sum_{\mu \vdash m} \sum_{\lambda \vdash n} \langle p_1^{n-m}, s_\lambda/\mu \rangle \]

\[ = \sum_{\mu \vdash m} \sum_{\lambda \vdash n} f^{\lambda/\mu} . \]

Now use Exercise 7.27(a).

89. (a) Let

\[ Y_n = \left( \sum_{\mu \vdash n} s_\mu(x)s_\mu(y) \right)^2 . \]

It is easy to see that the coefficient of \( q^n \) in the left-hand side of (7) is just \( \langle Y_n, Y_n \rangle \) (scalar product in the ring \( \Lambda(x) \otimes \Lambda(y) \)). Expanding \( Y_n \) in terms of power sums yields

\[ \langle Y_n, Y_n \rangle = \sum_{|\mu|+|\nu|=n} z_{\mu}^{-1} z_{\nu}^{-1} z_{\mu,\nu}, \]

from which it is easy to complete the proof. The original proof of Jeb Willenbring (private communication of 5 June 2003) is based on representation theory. See P. E. Harris and J. F. Willenbring, in *Symmetry: representation theory and its applications*, Birkhäuser, New York, 2014, pp. 305–326.

(e) Follows easily from Exercise 7.71(c) and letting \( n \to \infty \) in part (d) of the present exercise.

90. Conjectured by Stijn Lievens, and Neli Stiolova, and proved by Ron King in July, 2007, in a manuscript entitled “Notes on certain sums of Schur functions.”

♣ Can this manuscript be accessed online? Is there another reference?
91. This conjecture is due to S. Sundaram. It has been checked for even $n \leq 20$.


94. This result is closely related to the saturation conjecture (Problem 93 above) and to the problem of characterizing the possible eigenvalues of hermitian matrices $A, B, A + B$. For a nice survey of this area see W. Fulton, *Bull. Amer. Math. Soc.* 37 (2000), 209–249; math.AG/9908012.

95. (a) This result follows from the theory of Hall polynomials, not discussed in EC2. See (4.3) on page 188 of Macdonald, *Symmetric Functions and Hall Polynomials*, second ed. To complete the proof, one needs to show that $g^\lambda_{\mu\nu}(p) \neq 0$ (when $p$ is prime). This fact follows from a result of F. M. Maley, *J. Algebra* 184 (1996), 363–371, that $g^\lambda_{\mu\nu}(t)$ has nonnegative coefficients when expanded in powers of $t − 1$.

(b) This result (if true) would give an algebraic strengthening of the Saturation Conjecture (Exercise 93 above). For a somewhat more general context, see mathoverflow.net/questions/212368.

96. (a) One method is to note that by the Murnaghan-Nakayama rule we have $\chi^\lambda((n)) \neq 0$ (if and) only if $\lambda$ is a hook. But the only hooks $\lambda$ with a border strip of size $n − 1$ (otherwise $\chi^\lambda(n − 1, 1) = 0$) are $(n)$ and $(1^n)$. Since $\chi^n(\mu) = \varepsilon_{\mu} x^n(\mu) = 1 \neq 0$ for all $\mu \vdash n$, it follows that $\lambda = (n)$ or $\lambda = (1^n)$.

97. Let $\mu_1$ be the size of the largest border strip $B_1$ of $\lambda$. Thus $\mu_1 = h(1,1) = \lambda_1 + \lambda_1' - 1$, the hook length of square $(1,1)$. Let $\mu_2$ be
the size of the largest border strip $B_2$ after we remove $B_1$ from $\lambda$. Thus $\mu_1 = h(2,2) = \lambda_2 + \lambda'_2 - 3$. Continue in this way to obtain $\mu = (\mu_1, \mu_2, \ldots)$. The number of parts of $\mu$ is $\text{rank}(\lambda)$. Since each $B_i$ is unique, there is only one border strip tableau of type $\mu$. Thus by the Murnaghan-Nakayama rule $\chi^\lambda(\mu) = \pm 1$.

98. Given $w \in S_n$, let

$$\text{sq}(w) = \#\{u \in S_n : u^2 = w\},$$

the number of square roots of $w$. Let $\varphi : \Lambda_Q \to Q$ be the linear transformation (not a ring homomorphism) defined by $\varphi(s_\lambda) = 1$. Thus $\varphi^{\otimes k}$ (the $k$th tensor power of $\varphi$) is a linear transformation from $\Lambda_Q(x^{(1)}) \otimes \cdots \otimes \Lambda_Q(x^{(k)})$ to $Q$ defined by

$$\varphi(s_{\lambda_1}(x^{(1)}) \cdots s_{\lambda_k}(x^{(k)})) = 1.$$

By the solution to Exercise 7.69(b) we have $\varphi(p_\lambda) = \text{sq}(w)$, where $\rho(w) = \lambda$. Now apply $\varphi^{\otimes k}$ to both sides of (7.186).

99. Equivalent to equation (4.7) of J. B. Lewis, V. Reiner, and D. Stanton, *J. Alg. Combinatorics* 40(3) (2014), 663–691; arXiv:1308.1468, where the result is attributed to Kerov and to Garsia-Haiman. Several persons subsequently found elegant combinatorial proofs, the first being X. Chen （陈小美）.


101. Let $\lambda$ be a $p$-core with $\lambda_1 = k$. Remove from $\lambda$ all rows such that the first square $(i,1)$ of the row satisfies $h(i,1) \equiv h(1,1) \pmod{p}$, where $h(i,j)$ is the hook length of the square $(i,j)$. One can check that this sets up a bijection with $(p - 1)$-cores $\mu$ with $\mu_1 \leq k$. Thus if $f_p(k)$ denotes the number of $p$-cores with largest part $k$, then

$$f_p(k) = f_{p-1}(1) + f_{p-1}(2) + \cdots + f_{p-1}(k),$$

from which the proof follows easily. This result was explained by M. Vazirani in a conversation at MSRI on March 24, 2008.

103. (a) We have 
\[(\chi^{\lambda_1} \cdots \chi^{\lambda_k}, \chi^{(n)}) = (\chi^{\lambda_1} \cdots \chi^{\lambda_{k-1}}, \chi^{\lambda_k}).\]
Thus [why?]
\[u_k(n) = \sum_{\lambda_1, \ldots, \lambda_{k-1}} \langle \chi^{\lambda_1} \cdots \chi^{\lambda_{k-1}}, \chi^{\lambda_1} \cdots \chi^{\lambda_{k-1}} \rangle = \sum_{\lambda_1, \ldots, \lambda_{k-1}} \langle (\chi^{\lambda_1})^2 \cdots (\chi^{\lambda_{k-1}})^2, \chi^{(n)} \rangle = \langle \left( \sum_{\mu \vdash n} (\chi^{\mu})^2 \right)^{k-1}, \chi^{(n)} \rangle.\]
By Exercise 7.82(a),
\[\text{ch} \sum_{\mu \vdash n} (\chi^{\mu})^2 = \sum_{\mu \vdash n} p_\mu.\]
Hence [why?]
\[u_k(n) = \left\langle \left( \sum_{\mu \vdash n} p_\mu \right)^{*(k-1)}, h_n \right\rangle,\]
where \(\ast(k-1)\) denotes the \((k-1)\)th power with respect to \(\ast\). Now since
\[p_\lambda \ast p_\mu = z_\mu p_\mu \delta_{\lambda\mu},\]
and \(h_n = \sum_{\mu \vdash n} z_\mu^{-1} p_\mu\), finally we get [why?]
\[u_k(n) = \sum_{\mu \vdash n} (z_\mu)^{k-1}.\]
Thanks to Sam Hopkins for asking about the value of \(u_2(n)\).
Some references are J. B. Geloun and S. Ramgoolan, arXiv:1307.6490 (equation (18)), OEIS A110143, MO 41337, and MO 162428, where MO = MathOverflow.
(b) We now have [why?]
\[
v_k(n) = \sum_{\lambda_1, \ldots, \lambda_k} \langle \chi_{\lambda_1}^{\lambda_1} \cdots \chi_{\lambda_k}^{\lambda_k}, \chi^{(n)} \rangle
\]
\[
= \left\langle \left( \sum_{\lambda} s_\lambda \right)^* h_n, h_n \right\rangle.
\]
Now by Exercise 7.69(a) we have
\[
\sum_{\lambda \vdash n} s_\lambda = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} p_{\rho(w^2)} = \sum_{\mu \vdash n} z_\mu^{-1} \text{sq}(w_\mu),
\]
where \( w_\mu \) is a permutation in \( \mathfrak{S}_n \) of cycle type \( \mu \). Hence [why?]
\[
v_k(n) = \sum_{\mu} z_\mu^{-1} \text{sq}(w_\mu)^k
\]
\[
= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{sq}(w)^k.
\]
Note the special case \( k = 2 \):
\[
\frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{sq}(w)^2 = p(n).
\]
(For a different generalization, see Problem 98.)
For any finite group \( G \) of order \( d \) with \( k \) conjugacy classes we have
\[
\# \{(u, v) \in G \times G : uv = vu\} = kd.
\]
Hence
\[
\# \{(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n : uv = vu\} = \# \{(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n : u^2 = v^2\}.
\]
(In fact, this is true for any finite group \( G \) all of whose complex representations are equivalent to real representations.) Is there a combinatorial proof?
(c) Let $k = 3$ in equation (10). The number of terms in the two sums is around $e^{c\sqrt{n}}$. The largest term on the right-hand side is $n!$, coming from $\mu = (1^n)$. Hence the largest term on the left-hand side is $n!$, up to a factor of the order $e^{c\sqrt{n}}$. From this observation and Stirling’s formula for $n!$ the proof follows easily. For further information, including the partitions $\lambda, \mu, \nu$ achieving the maximum, see I. Pak, G. Panova, and D. Yeliussizov, *J. Combinatorial Theory Ser. A* 165 (2019), 44–77; arXiv:1804.04693.

104. (a) See R. Stanley, *Advances in Applied Math.* 30 (2003), 283–294; arXiv.math/0106115 (Theorem 2.2). The proof consists simply of applying the exponential specialization $e^x$ (EC2, pp. 304–306) to the identity of Exercise 7.27(e). Since this identity has a bijective proof via a skew version of RSK, its exponential specialization is therefore just a special case of this bijective proof. A somewhat different bijective proof was given by Aaron D. Jaggard, *Electronic J. Combinatorics* 12 (2005), R14 (Theorem 3.2); arXiv.math/0107130.

(b) See R. Stanley, *ibid.* (Theorem 2.1).

105. (b) See D. White, *Advances in Math.* 50, 160–186, though there may be earlier references.

(c) Easy consequence of (b), the isomorphism $\phi$, and the Murnaghan-Nakayama rule.


107. (a) We may assume $q \in \mathbb{P}$. In Exercise 7.70, let $k = 3$ and $x^{(3)} = 1^q$. Take the scalar product of both sides with $p_n(x^{(1)}) p_n(x^{(2)})$. The right-hand side becomes after some simplification

$$\text{RHS} = n \sum_{\rho(w) = (n)} q^{\rho(w, 1, 2, \ldots, n)}.$$
The left-hand side becomes (using Corollary 7.21.4)

\[
\text{LHS} = \sum_{\lambda \vdash n} \chi^{\lambda}(n)^2 \prod_{u \in \lambda} (q + c(u)).
\]

Now e.g. by the Murnaghan-Nakayama rule we have

\[
\chi^{\lambda}(n) = \begin{cases} 
(-1)^{n-i}, & \text{if } \lambda = (i, 1^{n-i}) \\
0, & \text{otherwise}
\end{cases}
\]

From this we easily get

\[
P_n(q) = \frac{1}{n} \sum_{i=1}^{n} (q + i - 1)_n = \frac{1}{n(n+1)} ((q + n)_{n+1} - (q)_{n+1}).
\]

(b) If \( P_n(z) = 0 \) then

\[|(z + 1)(z + 2) \cdots (z + n)|^2 = |(z - 1)(z - 2) \cdots (z - n)|^2. \quad (25)\]

Let \( z = a + bi \) where \( a, b \in \mathbb{R} \) and \( a > 0 \). Then for \( j > 0 \) we have

\[|(z + j)|^2 - |(z - j)|^2 = 4aj > 0.\]

Hence the left-hand side of (25) is greater than the right. The reverse inequality holds if \( a < 0 \). Hence if (25) holds then \( a = 0 \), and the proof follows.


(d) (sketch) As in (a) we obtain

\[
P_\lambda(q) = \sum_{i=1}^{n} \chi^{(n-i,1^i)}(\mu) (-1)^{i-1} (q + n - i)_n.
\]

Now if we apply the specialization (homomorphism) \( \psi(p_n) = 1 - (-t)^n \), then

\[
\psi(s_\lambda) = \begin{cases} 
t^k(1 + t), & \lambda = (n - k, 1^k), \ 0 \leq k \leq n - 1 \\
0, & \text{otherwise}
\end{cases}
\]

47
Hence from $p_{\mu} = \sum_{\lambda} \chi_{\lambda}(\mu)s_{\lambda}$ we get

$$ \prod \frac{1 - (-t)^{\mu_i}}{1 + t} = \sum_{k=0}^{n-1} \chi^{(n-k,1^k)}(\mu)t^k, $$

so

$$ P_{\lambda}(q) = \prod \frac{1 - (-t)^{\mu_i}}{1 + t} \bigg|_{t^k \rightarrow (-1)^k(q+n-k-1)}.$$

It is now not hard to complete the proof using Lemma 9.3 of A. Postnikov and R. Stanley, *J. Combinatorial Theory (A)* 91 (2000), 544–597.


108. (a) Let $\alpha = (\alpha_1, \ldots, \alpha_k)$. Define

$$ m \circ \alpha = (\alpha_1, [\alpha_2, \ldots, \alpha_{k-1}, \alpha_k + \alpha_1]^{m-1}, \alpha_2, \ldots, \alpha_k), $$

where $[\gamma]^m$ denotes the concatenation of $\gamma$ with itself $m$ times. For instance, $3 \circ (2, 1, 3, 2) = (2, 1, 3, 4, 1, 3, 4, 1, 3, 2)$. Define

$$ (\beta_1, \ldots, \beta_h) \circ \alpha = (\beta_1 \circ \alpha, \ldots, \beta_h \circ \alpha). $$

For instance, $(1, 3, 2) \circ (2, 1) = (2, 1, 2, 3, 1, 2, 3, 1)$. Given a composition $\alpha$, there is unique way to write it as $\beta^1 \circ \beta^2 \circ \cdots \circ \beta^j$, where each $\beta^i$ is irreducible with respect to $\circ$ (i.e., $\beta \neq \gamma \circ \delta$, where $\gamma$ and $\delta$ are compositions of integers greater than one). If $\beta = (\beta_1, \ldots, \beta_h)$, let $\bar{\beta} = (\beta_h, \ldots, \beta_1)$. L. Billera and S. van Willigenburg, *Advances in Math.* 204 (2006), 204–240, arXiv.math/0405434, have shown that the compositions equivalent to $\alpha$ are exactly those of the form $(\beta^1)' \circ (\beta^2)' \circ \cdots \circ (\beta^j)'$, where each $($ and $)$ are compositions of integers greater than one. For instance, since $12132 = 12 \circ 12$, the compositions equivalent to $12132$ are $12132, 12 \circ 21 = 21231, 21 \circ 12 = 13212,$ and $21 \circ 21 = 23121$.


112. (a,b) See B. Joseph, Ph.D. thesis, M.I.T, 2001 (Chapter 3), available at dspace.mit.edu/handle/1721.1/8225. This proof was never published.

(c) Immediate from (a) and the definition of the scalar product $\langle f_n, \text{sgn} \rangle$. This result is attributed to Paul D. Hanna, 9 March 2013, in OEIS A033917.

113. (a) We have

$$\sum_{\lambda \vdash n} s_{\lambda}(x)s_{\lambda}(y) = \sum_{\lambda \vdash n} z_{\lambda}^{-1}p_{\lambda}(x)p_{\lambda}(y).$$

Set $y_i = q^i - 1$ and multiply by $(1 - q)(1 - q^2) \cdots (1 - q^n)$. By Proposition 7.19.11 we obtain

$$\sum_{\lambda \vdash n} \left( \sum_{sh(T) = \lambda} q^{\text{maj}(T)} \right) s_{\lambda} = \sum_{\lambda \vdash n} z_{\lambda}^{-1} (1 - q) \cdots (1 - q^n) (1 - q^{\lambda_1}) \cdots (1 - q^{\lambda_{\ell(\lambda)}}) p_{\lambda},$$

where $\ell = \ell(\lambda)$. Now set $q = -1$. The left-hand side becomes $R_n$. The term indexed by $\lambda$ on the right-hand side vanishes unless $\lambda = (2^m)$ or $\lambda = (2^m, 1)$. Moreover, $z_{(2^m)} = z_{(2^m, 1)} = 2^m m!$, and

$$\lim_{q \to -1} \frac{(1 - q)(1 - q^2) \cdots (1 - q^{2m})}{(1 - q^2)^m} = \lim_{q \to -1} \frac{(1 - q)(1 - q^2) \cdots (1 - q^{2m+1})}{(1 - q)(1 - q^2)^m} = 2^m m!.$$

Hence

$$R_n = \begin{cases} 
  p_2^m, & n = 2m \\
  p_1 p_2^m, & n = 2m + 1. 
\end{cases}$$
(b) For definiteness let \( n = 2m \); the case \( n = 2m + 1 \) is essentially the same. By (a) we need to characterize all \( \lambda \vdash n \) for which \( \langle p_2^m, s_\lambda \rangle \neq 0 \). By Corollary 7.17.4, if \( \langle p_2^m, s_\lambda \rangle \neq 0 \) then there exists a border strip tableau of shape \( \lambda \) and type \( (2^m) \). Moreover, a simple argument shows that all border strip tableaux of shape \( \lambda \) and type \( (2^m) \) have the same parity of horizontal (and hence of vertical) dominos. Thus there is no cancellation of signs in the computation of \( \chi^\lambda(2^m) = \langle p_2^m, s_\lambda \rangle \), so \( \langle p_2^m, s_\lambda \rangle \neq 0 \) if and only if there exists a border strip tableau of shape \( \lambda \) and type \( (2^m) \). (This fact also follows from \#105.) Such a border strip tableau defines a covering of the shape of \( \lambda \) with \( m \) (disjoint) dominos. Moreover, it’s easy to see that the dominos of any covering of \( \lambda \) with \( m \) dominos can be ordered so that they define a border strip tableau, and the proof follows. (A crucial point in extending the argument to odd \( n \) is that a border strip tableau of type \( (2, 2, \ldots, 2, 1) \) always has the same square \((1, 1)\) as the border strip of size 1. This fact breaks down for skew shapes.)


(c) For the first statement, see Prop. 5.3 of the previous reference, which in fact is valid for more general labelled posets than Schur labelled skew shapes.

For the second statement, let \( \lambda/\mu = 43/2 \). Then we can place \( [5/2] = 2 \) disjoint dominos on the diagram of \( 43/2 \), but \( E(43/2) = 5 \) and \( O(43/2) = 4 \).

(d) Similar to (a)–(c); details omitted.

114. (a) For the case of rectangular shapes, see D. E. White, *J. Combinatorial Theory (A)* 95 (2001), 1–38.


115. (a) For any homogeneous symmetric function \( Y \) of degree \( n \), \( \langle Y, p_1^n \rangle \) is equal to \( [x_1 x_2 \cdots x_n] Y \) (the coefficient of \( x_1 x_2 \cdots x_n \) in \( Y \)). For any \( n \)-vertex graph \( G \), \( [x_1 x_2 \cdots x_n] X_G \) is equal to the number of
proper colorings of $G$ using the colors $1, 2, \ldots, n$ once each. Hence $\langle X_G, p_1^n \rangle = n!$, so the proof follows since $F_n$ is a sum of $C_n X_G$'s.

(b) For any $n$-vertex graph $G$ with $X_G = \sum_{\lambda \vdash n} c_\lambda e_\lambda$, Exercise 7.47(g) asserts that $\sum_{\ell(\lambda) = k}^{\lambda \vdash n} c_\lambda$ is the number of acyclic orientations of $G$ with $k$ sinks. Hence $c_{(1^n)}$ is the total number of acyclic orientations of $n$-vertex unit interval graphs (always up to isomorphism) with $n$ sinks. The only such graph is the one with no edges, and it has one acyclic orientation. Hence $c_{(1^n)} = 1$.

(c) Since $(2, 1^{n-2})$ is the only partition of $n$ with $n-1$ parts, we want the total number of acyclic orientations with $n-1$ sinks of $n$-vertex unit interval graphs. Such a graph can have exactly one edge (with two acyclic orientations having $n-1$ sinks), or have two incident edges (with one acyclic orientation having $n-1$ sinks). There are $n - 1$ $n$-vertex unit interval graphs with exactly one edge, and $n - 2$ with two incident edges. Hence

$$c_{(2, 1^{n-2})} = 2(n - 1) + (n - 2) = 3n - 4.$$ 

(d) Let $G$ be any $n$-vertex graph and $v$ any vertex of $G$. Greene and Zaslavsky (1983) showed that the number of acyclic orientations of $G$ having $v$ as the only sink is equal to $[q] \chi_G(q)$. Using the notation of (e) below, it follows that

$$c_{(n)} = n \sum_{(a_1, \ldots, a_n) \in D_n} (d_2 - 1)(d_3 - 1) \cdots (d_n - 1).$$

The set of sequences $(d_2 - 1, \ldots, d_n - 1)$ with no term equal to 0 is just the set $D_{n-1}$, so the proof follows from (e).

(e) Let the unit interval graph $G$ have vertices $1, 2, \ldots, n$ in the order of the unit intervals that define it. (The unit intervals are ordered by the order of their left endpoints.) Let vertex $i$ be adjacent to $d_i$ vertices $j < i$ (so $d_1 = 0$). It is easy to see that

$$\chi_G(q) = \prod_{i=1}^{n} (q - d_i).$$

It is well-known that $(-1)^n \chi_G(-1) = ao(G)$, the number of acyclic orientations of $G$. Hence $ao(G) = \prod (d_i + 1)$. It is also easy to see
that these sequences \((d_1, \ldots, d_n) \in \mathbb{P}^n\) are characterized by \(d_1 = 1\) and \(d_{i+1} \leq 1 + d_i\). Let \(\mathcal{D}_n\) denote the set of all such sequences. (We have \(|\mathcal{D}_n| = C_n\) by Exercise 80 in R. Stanley, Catalan Numbers.) We need to show that

\[
\prod_{(d_1, \ldots, d_n) \in \mathcal{D}_n} d_i = (2n - 1)!!.
\]

Let \(e_i = 2i - d_i\). It is not hard to see that the number of complete matchings \(M\) on \([2n]\) such that \(e_1, \ldots, e_n\) are the smaller vertices of the edges of \(M\) is equal to \(d_1 \cdots d_n\). Moreover, all possible such “smaller vertex” sets \(\{e_1, \ldots, e_n\}\) are obtained in this way, and the proof follows.

116. (b) This result is equivalent to a result of I. Schur, Math. Z. 1 (1918), 184–207, as may be seen by expanding each \(p_{\rho(w)}\) in the definition of \(\text{sf det}(A)\) in terms of Schur functions and taking the coefficient of \(s_\lambda\).

(d) Similarly, this result is equivalent to a result of J. Stembridge, Bull. London Math. Soc. 23 (1991), 422–428. For a generalization of (b) and (c), see B. Kostant, J. Amer. Math. Soc. 8 (1995), 181–186.

(e) Similarly, this assertion is equivalent to Stembridge, Canad. J. Math. 44 (1992), 1079–1099 (Conjecture 2.1).

(f) M. Guay-Paquet, arXiv:1306.2400, has reduced Exercise 7.47(j) to the case where \(P\) is both \((3 + 1)\)-free and \((2 + 2)\)-free, i.e., a semiorder (solution to Exercise 6.19(ddd)). It follows from the sentence preceding Conjecture 5.1 of R. Stanley, Advances in Math. 111 (1995), 166–194, that Exercise 7.47(j) is equivalent to the \(h\)-positivity of \(\text{sf det}(A)\), where if \(G\) has \(n\) vertices then \(A\) is the \(n \times n\) \((0, 1)\)-matrix with the 0’s occupying a certain rotated Young diagram justified into the lower left-hand corner and lying below the main diagonal. Such a matrix is totally nonnegative, so Exercise 7.47(j) is a special case of (d). For another combinatorial application of the total nonnegativity of \(A\), see the final paragraph of R. Stanley, J. Combinatorial Theory (A) 74 (1996), 169–172.


119. By Corollary 7.23.6 we need to expand $L_\lambda$ in terms of the quasisymmetric functions $L_\alpha$. (The dual use of $L$ is an unfortunate notational lapse.) This was done by I. Gessel and C. Reutenauer, J. Combinatorial Theory (A) 64 (1993), 189–215 (Theorem 2.1).

120. Hint. By Exercise 7.89(f), for fixed $k$ we have

$$\sum_{r \geq 0} L_{(kr)} = \exp \sum_{n \geq 1} \frac{1}{nk} \sum_{d|k} \mu(d)p^{k/d}. $$

Use the multiplicative property $L_\lambda = L_{(1^{m_1})}L_{(2^{m_2})} \cdots$ of Exercise 7.89(f).

121. Let $\sigma$ be an orbit of the action of $\mathfrak{S}_k$ on $C_k$. Let $G_\sigma(p_1, p_2, \ldots)$ be the Frobenius characteristic symmetric function of the action of $\mathfrak{S}_k$ on $\sigma$. Thus $\sigma$ corresponds to an unlabelled structure $\sigma$. It follows from Theorem A2.8 on page 449 that the action of $\mathfrak{S}_{kn}$ on $n$ disjoint copies of $\sigma$ is given by $h_n[G_\sigma]$. Now it follows directly from the definition of plethysm that if $f(x_1, x_2, \ldots) \in \Lambda$, then

$$p_n[f] = f[p_n] = f(x_1^n, x_2^n, \ldots).$$

53
Hence by equation (28),

\[ F_0 := \sum_{n \geq 0} h_n[G_0]t^{kn} \]

\[ = \exp \sum_{n \geq 1} \frac{1}{n} p_n[G_0]t^{kn} \]

\[ = \exp \sum_{n \geq 1} \frac{1}{n} G_0(p_n, p_{2n}, p_{3n}, \ldots) t^{kn}. \]

It follows from Proposition 7.18.2 that

\[ F_n = \prod_{m \geq 1} \prod_{\sigma} F_{\sigma}, \]

where \( \sigma \) ranges over all orbits of the action of \( \mathfrak{S}_m \) on \( \mathcal{C}_m \), and the proof follows.

See equation (20)c on page 46 of F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial Species and Tree-Like Structures*.

122. (a) Suppose that \( f(n) = \beta(S, T) \). By Corollary 7.23.8 we have \( \beta(S, T) = \langle s_{B_S}, s_{B_T} \rangle \). Now

\[ \langle s_{B_S}, s_{B_S} \rangle - 2\langle s_{B_S}, s_{B_T} \rangle + \langle s_{B_T}, s_{B_T} \rangle = \langle s_{B_S} - s_{B_T}, s_{B_S} - s_{B_T} \rangle \geq 0. \]

Hence \( \frac{1}{2}(\beta(S, S) + \beta(T, T)) \geq \beta(S, T) \). Therefore either \( \beta(S, S) \geq \beta(S, T) \) or \( \beta(T, T) \geq \beta(S, T) \), and the proof follows.

(b) This problem was raised by Ira Gessel (private communication, 2007).

123. (a) This result can be deduced from #119, using the fact that a multi-set permutation (of a totally ordered set) and its standardization have the same descent set, together with [why?]

\[ h_\alpha = \sum_{\beta \subseteq \alpha} s_{S_\beta}. \]

(c) The set of all parking functions is a union of sets $\mathcal{S}_M$ of all permutations of a multiset $M$. The proof follows from (a) and the definition of $\text{PF}_n$.

(d) We $L_n = L(n) = \frac{1}{n} \sum d(n) \mu(d) p_d^n/d$ (equation (7.191)) and

$$\text{PF}_n = \sum_{\lambda \vdash n} (n + 1)^{\ell(\lambda)-1} z_\lambda^{-1} p_\lambda$$

(by applying $\omega$ to the first formula of Exercise 7.48(f)). The proof follows from the orthogonality relation $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda \mu}$ (Proposition 7.9.3).


(e) By Problem 123(b) above and Exercise 7.48(f), we have that the two series

$$F(t) = t + \text{PF}_1 t^2 + \text{PF}_2 t^3 + \cdots,$$
$$G(t) = t - e_1 t^2 + e_2 t^3 - \cdots$$

are compositional inverses. Equating coefficients of $t^{n+1}$ on both sides of the identity $t = F(G(t))$ yields an expansion of $\text{PF}_n$ as a polynomial in the symmetric functions $(-1)^n e_n$. Doing the same for $t = G(F(t))$ yields an expansion of $(-1)^n e_n$ as a polynomial in the symmetric functions $\text{PF}_n$ with exactly the same coefficients. A simple homogeneity argument shows the same is true using $e_n$ instead of $(-1)^n e_n$. Since both $e_\lambda$ and $\text{PF}_\lambda$ are multiplicative bases, it follows that $R_n = R_n^{-1}$.

(f) Immediate from $R_n^2 = I$.

(h) Follows from applying $\omega$ to Problem 6 and using Problem 123(b) and part (f) of the present exercise.

(i) It is easy to see that the $e$-expansion of $\text{PF}_n$ has the form $\text{PF}_n = C_n e_1^n + \cdots$. The proof follows from the definition $\text{PF}_\lambda = \text{PF}_{\lambda_1} \text{PF}_{\lambda_2} \cdots$.

(j) Generalizing (i) above, it is not hard to show that

$$\text{PF}_n = C_n e_1^n - \frac{n - 1}{2} C_n e_2 e_1^{n-2} + \cdots.$$
Thus if $\lambda \vdash n$, then the coefficient of $e_2 e_1^{n-2}$ in $PF_\lambda$ is

$$-\frac{1}{2} \sum_{i=1}^{\ell(\lambda)} (\lambda_i - 1) \cdot \prod C_{\lambda_i} = -\frac{1}{2} (n - \ell(\lambda)) \prod C_{\lambda_i}. $$


126. **Answer:** the column vector $[\chi^\lambda(\nu)]_{\lambda \vdash n}$. Once the answer is guessed, the verification is straightforward.

127. (a) There are many approaches. We give an elementary argument. The character $\chi$ of the defining representation is given by $\chi(w) = \#\text{Fix}(w)$, the number of fixed points of $w$. Thus

$$s_n + s_{n-1,1} = \sum_{\lambda \vdash n} z_\lambda^{-1} m_1(\lambda) p_\lambda,$$

where $m_1(\lambda)$ is the number of parts of $\lambda$ equal to one. Hence by definition of internal product,

$$(s_n + s_{n-1,1})^k = \sum_{\lambda \vdash n} z_\lambda^{-1} m_1(\lambda)^k p_\lambda.$$

Now $s_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda$, so

$$\sum_{\lambda \vdash n} z_\lambda^{-1} m_1(\lambda)^k p_\lambda = \left( p_1 \frac{\partial}{\partial p_1} \right)^k s_n. $$

It is formal consequence of the commutation relation

$$p_1 \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_1} p_1 = 1$$

(where the 1 on the right-hand side denotes the identity operator) that

$$\left( p_1 \frac{\partial}{\partial p_1} \right)^k = \sum_{i=1}^{k} S(k, i) p_1^i \frac{\partial^i}{\partial p_1^i}. \quad (27)$$

(See Exercise 3.209.) Since $\frac{\partial}{\partial p_1} s_n = s_{n-i}$, the proof follows.
A rather similar argument, but formulated in terms of representation theory, is given by Dan Petersen at MathOverflow 284054.


(b) The value of the character of this $\mathfrak{S}_n$-action on a permutation of cycle type $\lambda$ is $\binom{m_2}{2} + m_2$. Hence

$$ (s_n + s_{n-1,1} + s_{n-2,2})^k = \left( \frac{1}{2} p_1^2 \frac{\partial^2}{\partial p_1^2} + p_2 \frac{\partial}{\partial p_2} \right)^k s_n. $$

Thus in analogy with equation (27) we need to find the coefficients $c_{ij}$ so that

$$ \left( \frac{1}{2} p_1^2 \frac{\partial^2}{\partial p_1^2} + p_2 \frac{\partial}{\partial p_2} \right)^k = \sum_{i,j} c_{ij} p_1^i p_2^j \frac{\partial^i}{\partial p_1^i} \frac{\partial^j}{\partial p_2^j}. $$

We can then use $\frac{\partial^i s_n}{\partial p_1^i} = s_{n-i}$ (as before) and $\frac{\partial^i s_n}{\partial p_2^i} = 2^{-j} s_{n-2j}$.

128. Open.


130. Immediate from Theorem 7.19.7 and Corollary 7.23.6 (setting $f = s_\lambda$).

131. (a) Let $\alpha = co(S)$ and $\bar{B}_\alpha$ the rotation of $B_\alpha$ by $180^\circ$ as in Exercise 7.56. Hence by this exercise we have $s_{B_\alpha} = s_{\bar{B}_\alpha}$. Given an SSYT of shape $\bar{B}_\alpha$ such as

\[
\begin{array}{ccc}
2 & 2 & 3 \\
3 & 3 \\
4 \\
2 & 6
\end{array}
\]

for $n = 8$ and $S = \{2, 3, 5\}$, simply read the entries from left-to-right and bottom-to-top. For example above we get the sequence
26433223. This sets up a bijection between sequences \( u = u_1 \cdots u_n \) with descent set \( S \) and terms \( x_{u_1} \cdots x_{u_n} \) of \( s_{B_u} \).

(b) Straightforward consequence of EC1, second ed., Exercise 4.40.

132. (a) Let \( \lambda \vdash n \). The largest \( \alpha \in \text{Comp}(n) \) in lexicographic order such that \([L_\alpha]s_\lambda \neq 0\) is given by \( \alpha = \lambda \). It follows that \([\text{why?}]\) \( f = \sum_{\mu \leq L_\lambda} a_\mu s_\mu \), where \( a_\mu \in \mathbb{Z} \) and \( \leq_L \) denotes lex order. Similarly the smallest \( \beta \in \text{Comp}(n) \) in lex order such that \([L_\beta]s_\lambda \neq 0\) is defined by \([n-1] - S_\beta = S_\lambda'\). Thus \( f = \sum_{\nu \geq L_\lambda} b_\mu s_\mu \) where \( b_\mu \in \mathbb{Z} \). Hence \( f_\lambda = cs_\lambda \) for some \( c \in \mathbb{Q} \). Since the \( s_\lambda \)'s form an integral basis, we must have \( c = 0 \) or \( 1 \). This result and proof are due to L. Billera.

(b) The smallest example is
\[
s_{31} + s_{211} - s_{22} = L_{31} + L_{13} + L_{211} + L_{112}.
\]
It is an open (and probably very difficult) problem to find all \( L \)-positive symmetric functions \( u \) such that if \( u = f + g \), where \( f, g \in \Lambda \) and \( f, g \) are \( L \)-positive, then \( f = 0 \) or \( g = 0 \).

133. Since \( h_n = \frac{1}{n!} \sum_{w \in S_n} p_{\rho(w)} \) and \( e_n = \frac{1}{n!} \sum_{w \in S_n} \varepsilon_w p_{\rho(w)} \), we get
\[
h_n + e_n = \frac{2}{n!} \sum_{w \in A_n} p_{\rho(w)}.
\]
Hence \( Z_{\mathfrak{A}_n} = h_n + e_n = s_n + s_{1^n} \).

134. We have \( c_\mu = \langle \text{ch}(\chi), h_\mu \rangle \). Now \( h_\mu \) is the Frobenius characteristic of the induction \( 1_\mu \) of the trivial representation \( 1_{S_\mu} \) from \( S_\mu \) to \( S_n \). Hence \( \langle \text{ch}(\chi), h_\mu \rangle = \langle \chi, 1_\mu \rangle \). By Frobenius reciprocity this is \( \langle \chi |_\mu, 1_{S_\mu} \rangle \), as desired. This result for permutation representations is due to V. Dotsenko, arXiv:0802.1340.

135. (a) This follows from the fact that for \( \lambda \vdash n \), \( \text{ex}(s_\lambda)|_{t=1} = f^\lambda/n! > 0 \), while
\[
\text{ex}(p_\lambda)|_{t=1} = \begin{cases} 1, & \lambda = (1^n) \\ 0, & \text{otherwise} \end{cases}.
\]
Here \( \text{ex} \) denotes the exponential specialization (EC2, pp. 304–306).
136. (a) Clearly \( \mathfrak{S}_n \) acts transitively on the set \( \mathfrak{S}_n/G \) of left cosets of \( G \). (In fact, for any finite group \( K \), the transitive permutation representations of \( K \) correspond to the actions of \( K \) on left cosets of subgroups \( G \).) On the other hand, by Burnside’s lemma (Lemma 7.24.5) and the fact that \( Z_G = \chi_{\mathfrak{S}_n} \) (equation (7.119)), we have that the number of orbits is \( \langle Z_G, s_n \rangle \). Thus \( \langle Z_G, s_n \rangle = 1 \).

If \( Z_G = \sum a_{\lambda} h_{\lambda} \), then \( a_{\lambda} \) is an integral basis. Moreover, if \( \lambda \vdash n \) then \( \langle h_{\lambda}, s_n \rangle = K_{n, \lambda} = 1 \). Hence if \( Z_G = \sum a_{\lambda} h_{\lambda} \) and \( Z_G \) is \( h \)-positive, then

\[
1 = \langle Z_G, s_n \rangle = \sum a_{\lambda},
\]

where each \( a_{\lambda} \in \mathbb{N} \). It follows that \( Z_G = h_{\lambda} \) for some \( \lambda \vdash n \).

(b) Assume that \( Z_G = h_{\lambda} = Z_{\mathfrak{S}_{\lambda}} \), and let \( \ell = \ell(\lambda) \). The number of orbits of \( G \) itself acting on \( 1, 2, \ldots, n \) is by Burnside’s lemma the average number of fixed points of elements of \( G \) and hence is determined by \( Z_G \). Since the action on left cosets of \( \mathfrak{S}_{\lambda} \) has \( \ell \) orbits, the same is true for \( G \). Now \( \mathfrak{S}_{\lambda} \) contains an element of cycle type \( \lambda \), so \( G \) also contains such an element \( w \). Since \( w \) has exactly \( \ell \) cycles and \( G \) has \( \ell \) orbits, it follows that \( G \subseteq H \), where \( H \) is a subgroup of \( \mathfrak{S}_n \) conjugate to \( \mathfrak{S}_{\lambda} \). But \([p_1^n]Z_G = 1/\#G\), so \( \#G = \#H \). Thus \( G = H \), as desired.

137. The identity is equivalent to

\[
1 + \left( \sum_{n \geq 1} \tilde{Z}_{\lambda_n} x^n \right) \left( \sum_{n \geq 0} n!h_n x^n \right) = \sum_{n \geq 0} n!h_n x^n.
\]

Equating coefficients of \( x^n \) gives

\[
\sum_{k=1}^{n} \tilde{Z}_{\lambda_k} (n-k)! h_{n-k} = n! h_n.
\]

For every permutation \( w = a_1 \cdots a_n \in \mathfrak{S}_n \), there is a unique \( 1 \leq k \leq n \) for which \( u := a_1 \cdots a_k \in \mathfrak{S}_k \). For fixed \( k \), the cycle indicator of all such \( u \) is \( \tilde{Z}_{\lambda_k} \). For any such \( u \), the cycle indicator of the remaining terms \( a_{k+1} \cdots a_n \) is \( (n-k)! h_{n-k} \) by Exercise 7.111(a). Hence for fixed \( k \), the cycle indicator of all such \( w \) is \( \tilde{Z}_{\lambda_k} (n-k)! h_{n-k} \), and the proof follows
by summing on $k$. This proof is completely analogous to the sketched proof in the solution to Exercise 1.128(a) of EC1, second ed.

138. (a,b) This is due to Brendan Pawlowski. See MathOverflow #254782.

(c) Take $n = 4$ and the three partitions to be 12-3-4, 13-2-4, and 14-2-3. Take the three characters to be trivial. We get

$$f = p_1^4 + 3p_2^2p_2 + 3p_1p_3 + p_4 = 8s_4 + 5s_31 - s_22 + s_211.$$


140. (a) Suppose that the cycle $c$ of $w_1 \cdots w_k$ containing 1 intersects $A_j$ in $i_j$ elements, so the length of this cycle is $i_1 + \cdots + i_k - k + 1$. The cycle $c$ will have the form

$$c = (1, u_1, u_2, \ldots, u_k),$$

where $u_j$ is a sequence of $i_j - 1$ distinct elements of $A_j - \{1\}$. Hence there are $\prod_{j=1}^k (a_j - 1)_{i_j - 1}$ choices for $c$. The elements of $A_j$ not in $c$ can be any permutation of size $a_j - i_j$. The generating function by cycle type of these permutations is

$$\sum_{w \in S_{a_j - i_j}} p_{\rho(w)} = (a_j - i_j)!h_{a_j - i_j}.$$

Thus the terms of the symmetric function (12) for fixed $a_1, \ldots, a_k, i_1, \ldots, i_k$ yield

$$\left(\prod_{j=1}^k (a_j - 1)_{i_j - 1}(a_j - i_j)\right) \cdot p_{i_1 + \cdots + i_k - k + 1}x_1^{i_1} \cdots x_k^{i_k} h_{a_1 - i_1}(x_1) \cdots h_{a_k - i_k}(x_k).$$

Note that $(a_j - 1)_{i_j - 1}(a_j - i_j)! = (a_j - 1)!$. Divide by $\prod (a_j - 1)!$ and sum on $i_1, \ldots, i_k \geq 1$ and $a_1, \ldots, a_k \geq 0$ to complete the proof.

(b) This result is due to Miriam Farber in 2015. The $s$-expansion uses the Schur function expansion of $h_i h_j$ and the Murnaghan-Nakayama rule. The $h$-expansion uses the $s$-expansion and the Jacobi-Trudi identity. Are there more conceptual proofs?
(c) $G_{2,2,2} = 8s_4 + 5s_{31} - s_{22} + s_{211}$
(d) This conjecture is due to M. Farber.
(e) Farber has a few more sporadic results and conjectures.

141. Answer. Suppose that $G \subseteq S_S$. Suppose our set of colors is $X \cup \bar{X}$, where $X = \{c_1, c_2, \ldots \}$ and $\bar{X} = \{\bar{c}_1, \bar{c}_2, \ldots \}$. Let $f : S \to X \cup \bar{X}$ be a coloring of $S$. Define

$$H_f = \{w \in G : w \cdot f = f\},$$

the subgroup of $G$ fixing the coloring $f$, and let $\tilde{H}_f$ be the restriction of $H_f$ to $f^{-1}(\bar{X})$. Call $f$ a $G$-super coloring if $\tilde{H}_f$ does not contain an odd permutation. For instance, let $S = \{1, 2, 3, 4\}$ and let $G$ be generated by the 4-cycle $(1, 2, 3, 4)$. Write $c_1 = a$ and let $f$ be the coloring $\bar{a}a\bar{a}a$ of 1234. Then $H_f = \{(1)(2)(3)(4), (1, 3)(2, 4)\}$ and $\tilde{H}_f = \{(2)(4), (2, 4)\} \not\subseteq A_2$ (although $H_f \subseteq A_4$). Hence $f$ is not a $G$-super coloring.

Theorem. The coefficient of $x^\alpha y^\beta$ in $Z_G(x/y)$ is equal to the number of $G$-orbits of $G$-super colorings $f$ such that $\alpha_i = \#f^{-1}(c_i)$ and $\beta_i = \#f^{-1}(\bar{c}_i)$ for all $i$.


143. Let $F(t) = \exp f$ and $G(t) = \exp g$. We claim that $\exp f[g] = F(G(t))$. Since the maps $\exp$ and $f \mapsto f[g]$ are homomorphisms, it suffices to take $f = p_n$, in which case the computation is straightforward.

144. There are many approaches. One way is to note that $h_2 = \frac{1}{2}(p_1^2 + p_2)$. Hence

$$h_2[h_n] = \frac{1}{2}(h_n^2 + h_n(x_1^2, x_2^2, \ldots)).$$
Now

\[
\sum_{n \geq 0} h_n(x_1^2, x_2^2, \ldots) = \frac{1}{\prod(1 - x_i^2)}
\]

\[
= \frac{1}{\prod(1 - x_i)(1 + x_i)}
\]

\[
= \left( \sum_{j \geq 0} h_j \right) \left( \sum_{k \geq 0} (-1)^k h_k \right),
\]

whence

\[
h_2[h_n] = \frac{1}{2} \left( h_n^2 + \sum_{k=0}^{2n} (-1)^k h_k h_{2n-k} \right).
\]

Expand \(h_k h_{2n-k}\) into Schur functions by Theorem 7.17.5 (Pieri’s rule) and collect terms to get

\[
h_2[h_n] = \sum_{k=0}^{\lfloor n/2 \rfloor} s(2n-2k, 2k).
\]

This result has been extended e.g. to \(h_3[h_n]\) (see for instance S. P. O. Plunkett, *Canad. J. Math.* 24 (1972), 541–552), but becomes increasingly unmanageable for \(h_4[h_n], h_5[h_n],\) etc.

145. Since \(e_1^2 = \sum_{i,j} x_i x_j\) and

\[
\sum_{n \geq 0} e_n = \prod(1 + x_i),
\]

we have

\[
\sum_{n \geq 0} e_n[e_1^2] = \prod_{i,j}(1 + x_i x_j).
\]

Setting \(y_i = x_i\) in the dual Cauchy identity gives

\[
\prod_{i,j}(1 + x_i x_j) = \sum_{\lambda} s_{\lambda}(x)s_{\lambda'}(x).
\]

Taking the degree \(2n\) part gives

\[
e_n[e_1^2] = \sum_{\lambda \vdash n} s_{\lambda}(x)s_{\lambda'}(x).
\]
146. Follows from the fact that the operation $g \mapsto g[f]$ is a (continuous) ring homomorphism and the formula

$$
\sum_{n\geq 0} h_n t^n = \exp \sum_{n\geq 1} \frac{p_n}{n}.
$$

(28)

If $f$ and $g$ are sums of monomials, the proof also follows from the fact that when $M$ is a multiset of monomials and $r = \sum_{u \in M} u$, then

$$
\sum_{n\geq 0} h_n [r] = \frac{1}{\prod_{u \in M} (1 - u)}.
$$


Note that the exponential specialization of equation (13) is equivalent to $e^{a+b} = e^a e^b$.

147. (a) Answer. Let $F(x) = \sum a_n x^n$ and let $F(x)^{(-1)} = \sum b_n x^n$ be its compositional inverse. Then the plethystic inverse of $f$ is $\sum b_n p^n_1$ (easy to prove e.g. from the fact that $p^n_1 [p^n_1] = p^{mn}_1$).

(b) Answer. Define $\delta : \mathbb{P} \to \mathbb{Z}$ by $\delta(n) = \delta_1 n$ (the Kronecker delta). Note that $\delta$ is the identity for $\ast$, i.e., $F \ast \delta = \delta \ast F = F$ for all $F$. Since $a_1 \neq 0$, the function $a_n$ possesses a unique Dirichlet inverse $b_n$, i.e., $a * b = \delta$. (For instance, if $a_n = 1$ for all $n$ then $b_n = \mu(n)$, the number-theoretic Möbius function.) Then the plethystic inverse of $f$ is $\sum_{n\geq 1} b_n p_n$.


149. Let $E_{ij}$ be the matrix in Mat$(n, \mathbb{C})$ with a 1 in the $(i,j)$-position and 0’s elsewhere, as on the middle of page 443. If $A = \text{diag}(\theta_1, \ldots, \theta_n)$ then $AE_{ij} = \theta_i E_{ij}$. Hence the eigenvalues of $A$ acting on Mat$(n, \mathbb{C})$ are just $\theta_1, \ldots, \theta_n$, $n$ times each, so the character of the action is just $n(x_1 + \cdots + x_n) = n s_1(x_1, \ldots, x_n)$.