

## SUPPLEMENTARY EXERCISES (without solutions)

for Chapter 7 (symmetric functions) of

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1. [2] Find the number  $f(n)$  of pairs  $(\lambda, \mu)$  such that  $\lambda \vdash n$  and  $\mu$  covers  $\lambda$  in Young's lattice  $Y$ . Express your answer in terms of  $p(k)$ , the number of partitions of  $k$ , for certain values of  $k$ . Try to give a direct bijection, avoiding generating functions, recurrence relations, induction, etc.

2. [2] Let  $p_r(n)$  denote the number of partitions of  $n$  of rank  $r$ . Find the generating function

$$F_r(t) = \sum_{n \geq 0} p_r(n) t^n.$$

3. [1] Express the symmetric function  $p_1 e_\lambda$  in terms of elementary symmetric functions.
4. [2] Let

$$F_n(x) = (-x_1 + x_2 + x_3 + \cdots + x_n)(x_1 - x_2 + x_3 + \cdots + x_n) \\ \cdots (x_1 + x_2 + \cdots + x_{n-1} - x_n).$$

Show that

$$F_n(x) = \sum_{k=2}^n (-1)^k 2^k e_k e_1^{n-k} - e_1^n,$$

in the ring  $\Lambda_n$  of symmetric functions in  $n$  variables.

5. [2+] For what real numbers  $a$  is the symmetric formal power series  $F(x) = \prod_i (1 + ax_i + x_i^2)$   $e$ -positive, i.e., a nonnegative (infinite) linear combination of the  $e_\lambda$ 's?
6. [2+] Find all symmetric functions  $f \in \Lambda^n$  that are both  $e$ -positive and  $h$ -positive.
7. [1+] Find all  $f \in \Lambda^n$  for which  $\omega f = 2f$ .

8. [2] Let  $P(x)$  be a polynomial satisfying  $P(0) = 1$ . Express  $\omega \prod_i P(x_i)$  as an infinite product.
9. [2-] Let  $j, k \geq 1$ . Expand the monomial symmetric function  $m_{\langle kj \rangle}$  as a linear combination of power sums  $p_\lambda$ .
10. (a) [3-] Let  $\Lambda_{\mathbb{Z}}^n$  denote the (additive) abelian group with basis  $\{m_\lambda\}_{\lambda \vdash n}$ . Let  $\Pi_{\mathbb{Z}}^n$  denote the subgroup generated by  $\{p_\lambda\}_{\lambda \vdash n}$ . Thus by the Note after Corollary 7.7.2,

$$[\Lambda_{\mathbb{Z}}^n : \Pi_{\mathbb{Z}}^n] = \prod_{\mu \vdash n} d_\mu,$$

where  $d_\mu = \prod_{i \geq 1} m_i(\mu)!$ . Show that in fact

$$\Lambda_{\mathbb{Z}}^n / \Pi_{\mathbb{Z}}^n \cong \bigoplus_{\mu \vdash n} \mathbb{Z} / d_\mu \mathbb{Z}.$$

- (b) [2] (for readers familiar with Smith normal form) Let  $X_n$  denote the character table of  $\mathfrak{S}_n$ . Deduce from (a) that  $X_n$  has the same Smith normal form as the diagonal matrix with diagonal entries  $d_\mu$ ,  $\mu \vdash n$ .
11. [2-] Let  $\alpha \in \mathbb{R}$  (or consider  $\alpha$  to be an indeterminate). Expand the product  $\prod_i (1 + x_i)^\alpha$  as an (infinite) linear combination of the power sums  $p_\lambda$ .
12. [2] Let  $f(x, y) \in \Lambda(x) \otimes \Lambda(y)$ , i.e.,  $f(x, y)$  is symmetric with respect to  $x_1, x_2, \dots$  and separately with respect to  $y_1, y_2, \dots$ . Let  $\frac{\partial}{\partial p_k(x)} f(x, y)$  denote the partial derivative of  $f(x, y)$  with respect to  $p_k(x)$  when  $f(x, y)$  is written as a polynomial in the  $p_i(x)$ 's (regard the  $y_j$ 's as constants). Find a simple formula for

$$\frac{\partial}{\partial p_k(x)} \prod_{i,j} (1 - x_i y_j)^{-1}.$$

13. [2+] Fix  $n \geq 1$ . Find a simple formula for the number of pairs  $(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n$  such that  $uv = vu$ . Generalize to any finite group  $G$  instead of  $\mathfrak{S}_n$ .

14. [3] Fix  $n \geq 1$ , and let  $S$  be an  $n$ -element subset of  $\mathbb{P}$ . Show that the field  $\mathbb{Q}(p_1(x_1, \dots, x_n), p_2(x_1, \dots, x_n), \dots)$  of all rational symmetric functions over  $\mathbb{Q}$  in the variables  $x_1, \dots, x_n$  is generated by  $\{p_i(x_1, \dots, x_n) : i \in S\}$  if and only if  $\mathbb{P} - S$  is closed under addition.

15. (a) [2+] Show that the symmetric power series

$$T = \frac{\sum_{n \geq 0} h_{2n+1}}{\sum_{n \geq 0} h_{2n}}$$

is a power series in the *odd* power sums  $p_1, p_3, p_5, \dots$

- (b) [3-] Identify the coefficients when  $T$  is written as a power series in the power sums.

16. [5-] Define for  $n \geq 3$  the symmetric function (in  $n$  variables)

$$H_n = \prod (x_1 + \epsilon_2 x_2 + \dots + \epsilon_n x_n),$$

where the product ranges over all sequences  $(\epsilon_2, \dots, \epsilon_n) \in \{-1, 1\}^{n-1}$ . Show that when  $3H_n$  is expanded as a polynomial in the power sums  $p_j$ , the coefficients are integers. (This conjecture has been verified for  $n \leq 5$  but is probably false.)

17. (a) [2+] Let  $p$  be a prime, and define the symmetric function

$$F_p = F_p(x_1, \dots, x_{2p-1}) = \sum_{\substack{S \subseteq [2p-1] \\ \#S=p}} \left( \sum_{i \in S} x_i \right)^{p-1},$$

where the first sum ranges over all  $p$ -element subsets of  $1, 2, \dots, 2p - 1$ . Show that when  $F_p$  is written as a linear combination of monomials, every coefficient is divisible by  $p$ .

- (b) [2+] Deduce from (a) the Erdős-Ginzburg-Ziv theorem: given any  $(2p - 1)$ -element subset  $X$  of  $\mathbb{Z}$ , there is a  $p$ -element subset  $Y$  of  $X$  such that  $\sum_{i \in Y} i \equiv 0 \pmod{p}$ .

- (c) [2+] Show that when  $F_p$  is written as a linear combination of power sums  $p_\lambda$ , every coefficient is an integer divisible by  $p$ .

18. Given  $f \in \Lambda_{\mathbb{Q}}^n$  and  $k \in \mathbb{P}$ , let  $f(kx)$  denote the symmetric function  $f$  in  $k$  copies of each variable  $x_1, x_2, \dots$ . Thus for instance  $m_1(kx) = km_1(x)$ .

- (a) [2–] Let  $\{u_\lambda : \lambda \vdash n\}$  be a basis for  $\Lambda_{\mathbb{Q}}^n$ , and let

$$f(kx) = \sum_{\lambda \vdash n} c_\lambda(k) u_\lambda. \quad (1)$$

Show that  $c_\lambda(k)$  is a polynomial in  $k$  (with rational coefficients). This allows us to use equation (1) to *define*  $f(kx)$  for *any*  $k$  (in some extension field  $F$  of  $\mathbb{Q}$ , say).

- (b) [2–] For any  $j \in F$ , let  $g(x) = f(jx)$ . For any  $k \in F$  show that  $g(kx) = f(jkx)$ .
- (c) [2] Express  $f(-x)$  in terms of  $f(x)$  and  $\omega$ .
19. [2+] Evaluate the scalar product  $\langle h_{21^{n-2}}, h_{21^{n-2}} \rangle$ .
20. [2+] Fix  $n \geq 1$ . Find the dimension of the subspace of  $\Lambda_{\mathbb{Q}}^n$  spanned by  $\{h_\lambda + e_\lambda : \lambda \vdash n\}$ .
21. (a) [3–] Define a linear transformation  $\varphi : \Lambda_{\mathbb{Q}}^n \rightarrow \Lambda_{\mathbb{Q}}^n$  by  $\varphi(e_\lambda) = m_\lambda$ . Find the size of the largest block in the Jordan canonical form of  $\varphi$ .
- (b) [5–] Find the entire Jordan canonical form of  $\varphi$ .
- (c) [5–] Do the same for such linear transformations as  $e_\lambda \mapsto m_\lambda$ ,  $h_\lambda \mapsto m_\lambda$ ,  $p_\lambda \mapsto m_\lambda$ ,  $p_\lambda \mapsto h_\lambda$ .
22. Let  $\langle \cdot, \cdot \rangle$  denote the standard scalar product on  $\Lambda_{\mathbb{Q}}$ . Two linear transformations  $A, B : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$  are *adjoint* if  $\langle Af, g \rangle = \langle f, Bg \rangle$  for all  $f, g \in \Lambda_{\mathbb{Q}}$ .
- (a) [2] Find the adjoint to  $\omega + aI$ , where  $I$  denotes the identity transformation and  $a$  is a constant.
- (b) [2] Let  $\frac{\partial}{\partial p_i} f$  denote the partial derivative of  $f \in \Lambda$  with respect to  $p_i$  when  $f$  is written as a polynomial in  $p_1, p_2, \dots$ . Define the linear transformation  $M_j$  by  $M_j(f) = p_j f$ . Express the adjoint of  $M_j$  in terms of the operators  $\frac{\partial}{\partial p_i}$ .
23. [2] Let  $k \geq 2$ . Compute the Kostka number  $K_{(k,k,k),(k-1,k-1,1^{k+2})}$ .
24. (a) [2+] Let  $\lambda \vdash n$  and  $\lambda \subseteq \langle k^n \rangle$ . Give a bijective proof that

$$K_{\langle k^n \rangle / \lambda, \langle (k-1)^n \rangle} = f^\lambda.$$

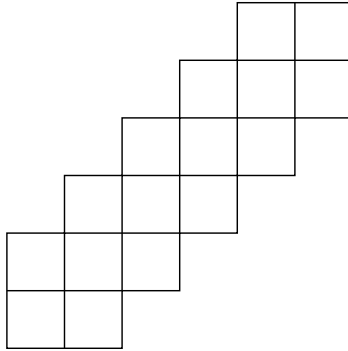
(b) [2] Deduce from (a) that  $K_{\langle k^n \rangle, \langle (k-1)^n, 1^n \rangle}$  is equal to the number of permutations in  $\mathfrak{S}_n$  with no increasing subsequence of length  $k + 1$ .

25. [2–] How many SYT of shape  $(n^n)$  have main diagonal  $(1, 4, 9, 16, \dots, n^2)$ ?

26. [3] Let  $\delta_m = (m - 1, m - 2, \dots, 1)$ . Define skew shapes

$$\begin{aligned}\alpha_n &= (n, n, n - 1, n - 2, \dots, 2) / \delta_{n-1} \\ \beta_n &= (n, n, n, n - 1, n - 2, \dots, 2) / \delta_n \\ \gamma_n &= (n, n, n, n - 1, n - 2, \dots, 1) / \delta_n.\end{aligned}$$

For instance, the diagram below shows  $\alpha_6$ .



Show that

$$\begin{aligned}f^{\alpha_n} &= \frac{(3n - 2)! E_{2n-1}}{(2n - 1)! 2^{2n-2}} \\ f^{\beta_n} &= \frac{(3n - 1)! E_{2n-1}}{(2n - 1)! 2^{2n-1}} \\ f^{\gamma_n} &= \frac{(3n)! (2^{2n-1} - 1) E_{2n-1}}{(2n - 1)! 2^{2n-1} (2^{2n} - 1)},\end{aligned}$$

where  $E_{2n-1}$  denotes an Euler number.

27. [2–] For any partitions  $\lambda$  and  $\mu$ , express  $s_\lambda s_\mu$  as a skew Schur function.

28. [3–] Let  $\lambda/\mu$  be a skew shape. Let  $M_i$  be the set of all skew shapes obtained from  $\lambda/\mu$  by removing a vertical strip of size  $i$  from  $\mu$  (i.e., adding this strip to the inner boundary of  $\lambda/\mu$ ) and adding a horizontal

strip of size  $k - i$  to  $\lambda$  (i.e., adding this strip to the outer boundary of  $\lambda/\mu$ ). Show that

$$s_k s_{\lambda/\mu} = \sum_{i=0}^k (-1)^i \sum_{\rho \in M_i} s_{\rho}.$$

29. (a) [2+] Let  $\lambda$  be a partition and  $m \geq \lambda_1$ . Let  $\lambda \cup m$  denote the partition obtained by adding a part of length  $m$  to  $\lambda$ . Let  $e_i^\perp$  denote the linear operator on symmetric functions adjoint to multiplication by  $e_i$ . Show that

$$s_{\lambda \cup m} = \left( \sum_{i \geq 0} (-1)^i h_{m+i} e_i^\perp \right) s_{\lambda}.$$

- (b) [2+] Let  $1 \leq k \leq n/2$ . Let  $f_k(n)$  be the number of permutations  $w = a_1 \cdots a_n \in \mathfrak{S}_n$  such that  $a_1 < a_2 < \cdots < a_{n-k}$ , and the longest increasing subsequence of  $w$  has length exactly  $n - k$ . Show that

$$f_k(n) = \sum_{i=0}^k (-1)^i \binom{k}{i} (n)_{k-i}.$$

- (c) [3-] Is there a “nice” proof of (b) based on the Principle of Inclusion-Exclusion?

30. [2-] Let  $1 \leq k \leq n$  and  $\lambda = (k, 1^{n-k})$  (called a *hook shape*). For any  $\mu \vdash n$  find a simple formula for the Kostka number  $K_{\lambda\mu}$ .

31. [2+] Let  $A$  be the  $m \times n$  matrix of all 1's. If  $A \xrightarrow{\text{rsk}} (P, Q)$ , then describe (with proof) the SSYT's  $P$  and  $Q$ .

32. (a) [3] Let  $\lambda$  be a partition with distinct parts. A *shifted standard tableau* (SHSYT) of shape  $\lambda$  is defined just like an ordinary standard Young tableau of shape  $\lambda$ , except that each row is indented one space to the right from the row above. An example of an SHSYT of shape  $(5, 4, 2)$  is given by

|   |   |   |    |    |
|---|---|---|----|----|
| 1 | 2 | 3 | 5  | 9  |
|   | 4 | 6 | 8  | 11 |
|   |   | 7 | 10 |    |

Call two permutations  $u, v \in \mathfrak{S}_n$  *W-equivalent* if they belong to the same equivalence class of the transitive closure of the following relation: either (i) they have the same insertion tableau under the RSK-algorithm or (ii)  $u(1) = v(2)$  and  $u(2) = v(1)$ . For instance, the *W*-equivalence classes for  $n = 3$  are  $\{123, 213, 231, 321\}$  and  $\{312, 132\}$ . Show that the number of *W*-equivalence classes in  $\mathfrak{S}_n$  is equal to the number of SHSYT of size  $n$ .

(b) [5–] Can this be generalized in an interesting way?

33. Let  $\lambda$  be a partition of  $n$  with distinct parts, denoted  $\lambda \models n$ . Let  $g^\lambda$  denote the number of shifted SYT of shape  $\lambda$ , as defined in Problem 32.

(a) [3–] Prove by a suitable modification of RSK that

$$\sum_{\lambda \models n} 2^{n-\ell(\lambda)} (g^\lambda)^2 = n!. \quad (2)$$

(b) [3] The “shifted analogue” of Corollary 7.13.9 is the following curious result. Let  $\zeta = (1 + i)/\sqrt{2} = e^{2\pi i/8}$ . Let

$$u(n) = \sum_{\lambda \models n} \zeta^{\ell(\lambda)} 2^{(n-\ell(\lambda))/2} g^\lambda.$$

Show that

$$\sum_{n \geq 0} u(n) \frac{t^n}{n!} = e^{\zeta t + \frac{1}{2} t^2}. \quad (3)$$

34. (a) [2+] Let  $\lambda$  be a partition with distinct parts. A *strict shifted SSYT* (or S4YT) of shape  $\lambda$  is a way of filling the squares of the shifted diagram of  $\lambda$  with positive integers such that each row and column is weakly increasing, and every diagonal from the upper left to lower right is strictly increasing. A *component* of an S4YT is a maximal connected set of equal entries. For instance, the diagram

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 2 |
|   | 2 | 2 | 3 | 3 | 4 |
|   |   | 3 | 3 | 4 | 4 |
|   |   |   | 4 | 7 |   |

is an S4YT with seven components, where we have outlined each component. Given an S4YT  $T$ , let  $k(T)$  denote its number of components. Define

$$Q_\lambda(x) = \sum_T 2^{k(T)} x^T,$$

summed over all S4YT of shape  $\lambda$ . Show that  $Q_\lambda(x)$  is a symmetric function. For instance

$$Q_{31}(x) = 4m_{31} + 8m_{22} + 16m_{211} + 32m_{1111}.$$

(b) [3] Show that

$$\sum_\lambda 2^{-\ell(\lambda)} Q_\lambda(x) Q_\lambda(y) = \prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j},$$

where the sum is over all partitions of all  $n \geq 0$ .

(c) [3–] A *diagonal-strict plane partition* (DSPP) is a plane partition such that there are no  $2 \times 2$  squares of equal *positive* entries. If  $\pi$  is an DSPP, then define  $k(\pi)$  as before (ignoring 0 entries). Let  $|\pi|$  denote the sum of the parts of  $\pi$ . Use (b) to show that

$$\begin{aligned} \sum_\pi 2^{k(\pi)} q^{|\pi|} &= \prod_{j \geq 1} \left( \frac{1 + q^j}{1 - q^j} \right)^j \\ &= 1 + 2q + 6q^2 + 16q^3 + 38q^4 + 88q^5 + 196q^6 + \cdots, \end{aligned}$$

where  $\pi$  ranges over all DSPP.

35. [2+] Evaluate the sums

$$\sum_{\lambda \vdash n} f^{\lambda/2} f^\lambda \quad \text{and} \quad \sum_{\lambda \vdash n} (f^{\lambda/2})^2.$$

36. Let  $o(\lambda)$  denote the number of odd parts of the partition  $\lambda$ , and  $d(\lambda)$  the number of distinct parts. Let  $t_{n-1}$  denote the number of involutions in  $\mathfrak{S}_{n-1}$ .

(a) [2] Use the RSK algorithm to show that

$$\sum_{\lambda \vdash n} o(\lambda) f^\lambda = nt_{n-1}.$$

(b) [2+] Show that

$$\sum_{\lambda \vdash n} d(\lambda) f^\lambda = nt_{n-1}.$$

37. [3] With  $o(\lambda)$  as in Problem 36(a), show that

$$\sum_{\lambda \vdash n} f^\lambda \left( \frac{1+q}{1-q} \right)^{o(\lambda)} = \sum_{\lambda \vdash n} f^\lambda \prod_{u \in \lambda} \frac{1+q^{h(u)}}{1-q^{h(u)}},$$

where  $h(u)$  denotes the hook length of  $u$ . Equivalently,

$$\exp \left( \frac{1+q}{1-q} t + \frac{1}{2} t^2 \right) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\lambda \vdash n} f^\lambda \prod_{u \in \lambda} \frac{1+q^{h(u)}}{1-q^{h(u)}}.$$

38. [2+] Given an SYT  $T$ , let  $\sigma(T)$  be the largest integer  $k$  such that  $1, 2, \dots, k$  appear in the first row of  $T$ . Let  $E_n$  denote the expected value of  $\sigma(\text{ins}(w))$ , where  $w$  is a random (uniform) permutation in  $\mathfrak{S}_n$  and  $\text{ins}(w)$  denotes the insertion tableau of  $w$  under the RSK algorithm. Thus

$$E_n = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \sigma(\text{ins}(w)).$$

Find  $\lim_{n \rightarrow \infty} E_n$ .

39. [3] Show that as  $n \rightarrow \infty$ , for almost all (i.e., a  $(1 - o(1))$ -fraction) permutations  $w \in \mathfrak{S}_n$  the number of bumping operations performed in applying RSK to  $w$  is

$$(1 + o(1)) \frac{128}{27\pi^2} n^{3/2}.$$

Moreover, the number of comparison operations performed is

$$(1 + o(1)) \frac{64}{27\pi^2} n^{3/2} \log_2 n.$$

40. (a) [2+] Let  $n = pq$ ,  $w \in \mathfrak{S}_n$  and  $w \xrightarrow{\text{rsk}} (P, Q)$ . Suppose that the shape of  $P$  and  $Q$  is a  $p \times q$  rectangle. Show that when the RSK algorithm is applied to  $w$ , every bumping path is vertical (never moves strictly to the left).
- (b) [2] Let  $P = (a_{ij})$  and  $Q = (b_{ij})$  in (a). Deduce from (a) that  $w(b_{ij}) = a_{p+1-i,j}$ .
41. [2] Let  $i, j, n \geq 1$ . Evaluate the sum

$$f_n(i, j) = \sum_{\lambda \vdash n} s_\lambda(1^i) s_\lambda(1^j).$$

42. [3-] Let  $y_n = \sum_{\lambda \vdash n} s_\lambda^2$ . Find the generating function

$$F(q) = \sum_{n \geq 0} \langle y_n, y_n \rangle q^n.$$

Express your answer in terms of the generating function  $P(x, t) = \prod_{i \geq 1} (1 - tx^i)^{-1}$ .

43. [3] Let

$$f(n) = \left\langle \sum_{\mu \vdash n} s_\mu^2, \sum_{\lambda \vdash n} s_{2\lambda} \right\rangle,$$

where  $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$ . Thus

$$(f(0), f(1), \dots, f(10)) = (1, 1, 3, 5, 12, 20, 44, 76, 157, 281, 559).$$

Show that

$$\sum_{n \geq 0} f(n) q^n = \prod_{i \geq 1} \frac{1}{\sqrt{1 - 2q^i}} \cdot \prod_{j \geq 1} \frac{1}{(1 - q^{2j})^{2^{j-2}}}.$$

44. [3+] Let  $V_n = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ . Show that for  $k \geq 0$ ,

$$\langle V_n^{2k}, V_n^{2k} \rangle_n = \frac{((2k+1)n)!}{(2k+1)!^n n!}.$$

The notation  $\langle \cdot, \cdot \rangle_n$  indicates that the scalar product is taken in the ring  $\Lambda_n$ , i.e., the Schur functions  $s_\lambda(x_1, \dots, x_n)$  with  $\ell(\lambda) \leq n$  form an orthonormal basis.

45. [2+] Let

$$a_n = \left\langle h_2^n, \sum_{\lambda \vdash 2n} s_\lambda \right\rangle.$$

Find the generating function  $F(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$ . (A result in Chapter 5 may prove useful.)

46. (a) [3–] Find the number  $f(n)$  of ways to move from the empty partition  $\emptyset$  to  $\emptyset$  in  $n$  steps, where each step consists of either (i) adding a box, (ii) removing a box, or (iii) adding and then removing a box, always keeping the diagram of a partition (even in the middle of a step of type (iii)). For instance,  $f(3) = 5$ , corresponding to the five sequences

$$\begin{array}{cccc} \emptyset & (1, \emptyset) & (1, \emptyset) & (1, \emptyset) \\ \emptyset & (1, \emptyset) & 1 & \emptyset \\ \emptyset & 1 & (2, 1) & \emptyset \\ \emptyset & 1 & (11, 1) & \emptyset \\ \emptyset & 1 & \emptyset & (1, \emptyset) \end{array} .$$

Express your answer as a familiar combinatorial number and not, for instance, as a sum.

(b) [3–] Given a partition  $\lambda$ , let  $f_\lambda(n)$  be the same as in (a), except we move from  $\emptyset$  to  $\lambda$  in  $n$  steps. Define

$$T_n = \sum_{\lambda} f_\lambda(n) s_\lambda.$$

For instance,

$$T_3 = 5 + 10s_1 + 6s_2 + 6s_{11} + s_3 + 2s_{21} + s_{111}.$$

Find  $\langle T_m, T_n \rangle$ . As in (a), express your answer as a familiar combinatorial number.

47. (a) [2+] Let  $h(t) \in \mathbb{C}[[t]]$  with  $h(0) \neq 0$ , and let  $g(t) \in \mathbb{C}[[t]]$ . Write  $p = p_1 = \sum x_i$ . Let  $\Omega$  be the operator on  $\Lambda_{\mathbb{C}}$  defined by

$$\Omega = g(p) + h(p) \frac{\partial}{\partial p}.$$

Define

$$F(x, p) = \sum_{n \geq 0} \Omega^n(1) \frac{x^n}{n!}.$$

Show that

$$F(x, p) = \exp \left[ -M(p) + M(L^{(-1)}(x + L(p))) \right],$$

where

$$\begin{aligned} L(t) &= \int_0^t \frac{ds}{h(s)} \\ M(t) &= \int_0^t \frac{g(s)ds}{h(s)}, \end{aligned}$$

and where  $L^{(-1)}$  denotes the compositional inverse of  $L$ .

- (b) [1+] Let  $g(t) = t$  and  $h(t) = 1$ , so  $\langle \Omega^n(1), s_\lambda \rangle$  is the number of oscillating tableaux of shape  $\lambda$  and length  $n$ , as defined in Exercise 7.24(d). Show that

$$F(x, p) = \exp \left( px + \frac{1}{2}x^2 \right).$$

- (c) [2] Let  $f_\lambda(n)$  be the number of ways to move from the empty partition  $\emptyset$  to  $\lambda$  in  $n$  steps, where the steps are as in Problem 46. Use (a) to show that

$$\sum_{n \geq 0} \sum_{\lambda \in \text{Par}} f_\lambda(n) s_\lambda \frac{t^n}{n!} = \exp(-1 - p + (1 + p)e^x).$$

- (d) [2-] Let  $g_\lambda(n)$  be the number of ways to move from  $\emptyset$  to  $\lambda$  in  $n$  steps, where each step consists of adding one square at a time any number  $i$  of times (including  $i = 0$ ) to the current shape and then either stopping or deleting one square (always maintaining the shape of a partition). Show that

$$\sum_{n \geq 0} \sum_{\lambda \in \text{Par}} g_\lambda(n) s_\lambda \frac{x^n}{n!} = \exp \left( 1 - p - \sqrt{(1 - p)^2 - 2x} \right).$$

In particular,

$$\sum_{n \geq 0} g_\emptyset(n) \frac{x^n}{n!} = \exp \left( x + \sum_{k \geq 2} (2k - 3)!! \frac{x^k}{k!} \right).$$

- (e) [2–] Let  $j_\lambda(n)$  be the number of ways to move from  $\emptyset$  to  $\lambda \vdash k$  in  $n$  steps, where each step consists of adding one square at a time any number  $i$  of times (including  $i = 0$ ) to the current shape or else deleting one square (always maintaining the shape of a partition). Show that

$$j_\lambda(n) = n! \binom{n}{k} f^\lambda,$$

where as usual  $f^\lambda$  denotes the number of SYT of shape  $\lambda$ .

48. [3] Let  $w = a_1 a_2 \cdots a_{2n} \in \mathfrak{S}_{2n}$ . Suppose that  $a_i + a_{2n+1-i} = 2n + 1$  for all  $1 \leq i \leq n$ . Show that the shape of the insertion tableau  $\text{ins}(w)$  can be covered with  $n$  dominos.
49. [2–] Let  $d, n \geq 1$  and  $\zeta = e^{2\pi i/d}$ , a primitive  $d$ th root of unity. Let  $f \in \Lambda^n$ . Show that  $f(1, \zeta, \dots, \zeta^{d-1}) = 0$  unless  $d|n$ .
50. [3–] Let  $(n-3)/2 \leq m \leq n-1$ . Show that

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}} f^\lambda = t(n) - \sum_{\substack{i, j, l \geq 0 \\ 2i+j+2l=n-m-1}} \frac{(-1)^i \binom{n}{i+j} t(j)}{i! j!},$$

where  $t(j)$  denotes the number of involutions in  $\mathfrak{S}_j$ .

51. [2–] Let  $u$  be a square of the skew shape  $\lambda/\mu$ . We can define the *hook*  $H(u) = H_{\lambda/\mu}(u)$  just as for ordinary shapes, viz., the set of squares directly to the right of  $u$  and directly below  $u$ , counting  $u$  itself once. Similarly we can define the *hook length*  $h(u) = h_{\lambda/\mu}(u) := \#H(u)$ . Let  $(\lambda/\mu)^r$  denote  $\lambda/\mu$  rotated  $180^\circ$ , as in Exercise 7.56. Show that

$$\sum_{u \in \lambda/\mu} h_{\lambda/\mu}(u) = \sum_{u \in (\lambda/\mu)^r} h_{(\lambda/\mu)^r}(u).$$

52. (a) [3–] Let  $\eta_k(\lambda)$  be the number of hooks of length  $k$  of the partition  $\lambda$ . Show that

$$\sum_{\lambda \vdash n} \eta_k(\lambda) = k \sum_{\lambda \vdash n} m_k(\lambda).$$

As usual,  $m_k(\lambda)$  denotes the number of parts of  $\lambda$  equal to  $k$ . Note that Problem 1 is equivalent to the case  $k = 1$ . Is there a simple bijective proof similar to the solution to Problem 1?

- (b) [3−] Part (a) can be rephrased as follows. For  $u = (i, j) \in \lambda$ , let  $r(u) = \lambda_i$ , the length of the row in which  $u$  appears. Then the statistics  $h(u)$  and  $r(u)$  have the same distribution over all squares of all  $\lambda \vdash n$ , i.e.,

$$\sum_{\lambda \vdash n} \sum_{u \in \lambda} x^{h(u)} = \sum_{\lambda \vdash n} \sum_{u \in \lambda} x^{r(u)}.$$

Show in fact that  $h(u)$  and  $r(u)$  have a symmetric joint distribution, i.e., if

$$F(x, y) = \sum_{\lambda \vdash n} \sum_{u \in \lambda} x^{h(u)} y^{r(u)},$$

then  $F(x, y) = F(y, x)$ .

53. [3+] Given a partition  $\lambda$  and  $u \in \lambda$ , let  $a(u)$  and  $\ell(u)$  denote the arm and leg lengths of  $u$  as in Exercise 7.26. Define

$$\gamma(\lambda) = \#\{u \in \lambda : a(u) - \ell(u) = 0 \text{ or } 1\}.$$

Show that

$$\sum_{\lambda \vdash n} q^{\gamma(\lambda)} = \sum_{\lambda \vdash n} q^{\ell(\lambda)},$$

where  $\ell(\lambda)$  denotes the length (number of parts) of  $\lambda$ .

54. [2] Let  $\delta = (n-1, n-2, \dots, 0)$  as usual, and let  $\lambda \in \text{Par}$  with  $n \geq \ell(\lambda)$ . Find the Schur function expansion of the product

$$s_\delta(x_1, \dots, x_n) s_\lambda(x_1^2, \dots, x_n^2).$$

55. [2] Let  $E_k$  denote an Euler number (the number of alternating permutations of  $1, 2, \dots, k$ ). Evaluate the determinants

$$A_n = \left| \frac{E_{2i+2j-1}}{(2i+2j-1)!} \right|_{i,j=1}^n$$

and

$$B_n = \left| \frac{E_{2i+2j-3}}{(2i+2j-3)!} \right|_{i,j=1}^n.$$

**HINT.** Use Exercise 7.40.

56. [2+] Let  $f(n)$  be the number of permutations  $w \in \mathfrak{S}_n$  such that both  $w$  and  $w^{-1}$  are alternating. Let

$$\begin{aligned} L(x) &= \frac{1}{2} \log \frac{1+x}{1-x} \\ &= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots. \end{aligned}$$

Use Corollary 7.23.8 and Exercise 7.64 to show that

$$\begin{aligned} \sum_{k \geq 0} f(2k+1)x^{2k+1} &= \sum_{k \geq 0} E_{2k+1}^2 \frac{L(x)^{2k+1}}{(2k+1)!} \\ \sum_{k \geq 0} f(2k)x^{2k} &= \frac{1}{\sqrt{1-x^2}} \sum_{k \geq 0} E_{2k}^2 \frac{L(x)^{2k}}{(2k)!}, \end{aligned}$$

where  $E_n$  denotes an Euler number.

57. [3-] Let  $a(n)$  denote the number of alternating involutions in  $\mathfrak{S}_n$ , i.e., the number of involutions in  $\mathfrak{S}_n$  that are alternating permutations in the sense of the last two paragraphs of Section 3.16. Let  $E_m$  denote an Euler number. Use Problem 89 below and Exercise 7.64 to show that

$$\begin{aligned} \sum_{k \geq 0} a(2k+1)x^{2k+1} &= \sum_{i,j \geq 0} \frac{E_{2i+2j+1}}{(2i+1)! j! 4^j} (\tan^{-1} x)^{2i+1} \left( \log \frac{1+x^2}{1-x^2} \right)^j \\ \sum_{k \geq 0} a(2k)x^{2k} &= \frac{1}{\sqrt[4]{1-x^4}} \sum_{i,j \geq 0} \frac{E_{2i+2j}}{(2i)! j! 4^j} (\tan^{-1} x)^{2i} \left( \log \frac{1+x^2}{1-x^2} \right)^j. \end{aligned}$$

58. [3+] Let  $\lambda \vdash n$ , and let  $a, b, c, d$  be (commuting) indeterminates. Define

$$w(\lambda) = a^{\sum \lceil \lambda_{2i-1}/2 \rceil} b^{\sum \lfloor \lambda_{2i-1}/2 \rfloor} c^{\sum \lceil \lambda_{2i}/2 \rceil} d^{\sum \lfloor \lambda_{2i}/2 \rfloor}.$$

For instance, if  $\lambda = (5, 4, 4, 3, 2)$  then  $w(\lambda)$  is the product of the entries below in the diagram of  $\lambda$ :

$$\begin{array}{c} a b a b a \\ c d c d \\ a b a b \quad . \\ c d c \\ a b \end{array}$$

Let  $y = \sum_{\lambda} w(\lambda) s_{\lambda}$ , where  $\lambda$  ranges over all partitions. Show that

$$\log(y) - \sum_{n \geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n \geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \dots]].$$

Note that if we set  $a = qt$ ,  $b = q^{-1}t$ ,  $c = qt^{-1}$ ,  $d = q^{-1}t^{-1}$  and then set  $q = t = 0$ , then  $y$  becomes  $\sum_{\lambda} s_{\lambda}$ , where  $\lambda$  ranges over all partitions such that each  $\lambda_i$  and  $\lambda'_i$  is even.

59. Let  $\omega_y : \Lambda(x, y) \rightarrow \Lambda(x) \otimes \Lambda(y)$  be the algebra homomorphism defined by

$$\omega_y p_n(x, y) = p_n(x) + (-1)^{n-1} p_n(y). \quad (4)$$

Equivalently,  $\omega_y$  is the automorphism  $\omega$  acting on the  $y$ -variables only. Write  $\omega_y f(x, y) = f(x/y)$ . In particular,  $s_{\lambda}(x/y)$  is called a *super Schur function*. Let

$$\Sigma = \text{im}(\omega_y) = \{f(x/y) : f \in \Lambda\},$$

a subalgebra of  $\Lambda(x) \otimes \Lambda(y)$ .

- (a) [2–] Show that

$$s_{\lambda}(x/y) = \sum_{\mu \subseteq \lambda} s_{\mu}(x) s_{\lambda/\mu}(y). \quad (5)$$

- (b) [3–] Let  $g(x, y) \in \Lambda(x) \otimes \Lambda(y)$ , and let  $t$  be an indeterminate. Show that  $g \in \Sigma$  if and only if

$$g(x, y)|_{x_1=t, y_1=-t} = g(x, y)|_{x_1=y_1=0}. \quad (6)$$

- (c) [3] Prove the following “finite analogue” of (b). Let  $g \in \Lambda(x_1, \dots, x_m) \otimes \Lambda(y_1, \dots, y_n)$ . Then

$$g|_{x_1=t, y_1=-t} = g(x, y)|_{x_1=y_1=0}$$

if and only if  $g$  is a polynomial in the “variables”  $p_i(x_1, \dots, x_m) + (-1)^{i-1} p_i(y_1, \dots, y_n)$ ,  $i \geq 1$ .

- (d) [2–] Show that for any  $f \in \Lambda$ ,  $f(x/x)$  is a polynomial in the odd power sums  $p_1, p_3, p_5, \dots$ .

- (e) [3–] Define a *supertableau* of shape  $\lambda$  to be an array  $T$  of positive integers of shape  $\lambda$  such that (i) the rows and columns are weakly increasing, and (ii) the diagonals from the upper left to lower right are strictly increasing (equivalently, there is no  $2 \times 2$  square of equal entries). A maximal rookwise-connected subset of equal entries is called a *component* of  $T$ . Let  $c(T)$  denote the number of components. For instance, if  $T$  is given by:

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 2 | 3 |
| 1 | 2 | 2 | 2 | 3 | 4 |   |
| 1 | 2 | 3 | 3 | 3 | 4 |   |
| 2 | 2 | 4 | 4 |   |   |   |
| 3 | 4 | 4 |   |   |   |   |

then  $T$  has one component of 1's, two components of 2's, three components of 3's, and two components of 4's, so  $c(T) = 8$ . Show that

$$s_\lambda(x/x) = \sum_T 2^{c(T)} x^T,$$

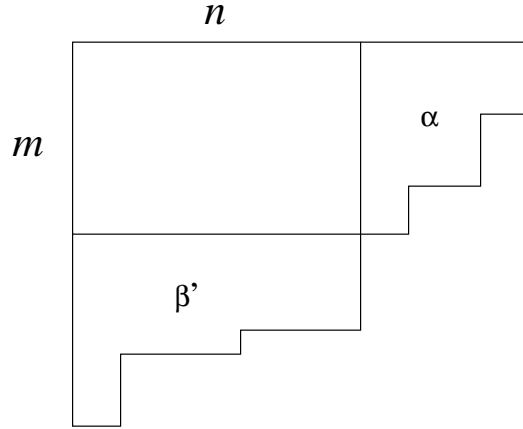
where  $T$  ranges over all supertableaux of shape  $\lambda$  and  $x^T$  has its usual meaning.

- (f) [2] Let  $(n^m)$  denote an  $m \times n$  rectangular shape. Show in two different ways that

$$s_{(n^m)}(x_1, \dots, x_m / y_1, \dots, y_n) = \prod_{i=1}^m \prod_{j=1}^n (x_i + y_j).$$

The first proof (easy) should use Exercises 7.41 and 7.42. The second proof should use (b) above (in the easy “only if” direction) but no RSK, Cauchy identity, etc.

- (g) [3–] More generally, let  $\alpha, \beta$  be partitions with  $\ell(\alpha) \leq m$  and  $\ell(\beta) \leq n$ . Let  $[m, n, \alpha, \beta]$  denote the partition obtained by adjoining  $\alpha$  to the right of  $(n^m)$  and  $\beta'$  below  $(n^m)$ , as illustrated below.



Show that

$$s_{[m,n,\alpha,\beta]}(x_1, \dots, x_m / y_1, \dots, y_n) = s_\alpha(x_1, \dots, x_m) s_\beta(y_1, \dots, y_n) \cdot \prod_{i=1}^m \prod_{j=1}^n (x_i + y_j).$$

60. (a) [3] Define a grading on the ring  $\Lambda$  of symmetric functions by setting  $\deg(p_i) = 1$  for all  $i \geq 1$ . Thus  $\deg(p_\lambda) = \ell(\lambda)$ . Let  $\hat{s}_\lambda$  denote the terms of least degree appearing in the expansion of  $s_\lambda$  in terms of power sums. (It is an easy consequence of Exercise 7.52 and the Murnaghan-Nakayama rule that this least degree is equal to  $\text{rank}(\lambda)$ .) For instance,

$$s_{221} = \frac{1}{24}p_1^5 - \frac{1}{12}p_2p_1^3 - \frac{1}{6}p_3p_1^2 + \frac{1}{8}p_2^2p_1 + \frac{1}{4}p_4p_1 - \frac{1}{6}p_3p_2,$$

so

$$\hat{s}_{221} = \frac{1}{4}p_4p_1 - \frac{1}{6}p_3p_2.$$

Let  $V_n$  denote the subspace of  $\Lambda_{\mathbb{Q}}$  spanned by all  $\hat{s}_\lambda$  such that  $\lambda \vdash n$ . Show that a basis for  $V_n$  is given by  $\{\hat{s}_\lambda : \text{rank}(\lambda) = \ell(\lambda)\}$ .

- (b) [2] Deduce from (a) that  $\dim V_n$  is the number of  $\mu \vdash n$  whose parts differ by at least 2. (By the Rogers-Ramanujan identities, this is also the number of  $\mu \vdash n$  whose parts are  $\equiv \pm 1 \pmod{5}$ .)
- (c) [3] Define the *augmented monomial symmetric function*  $\tilde{m}_\lambda = r_1!r_2! \cdots m_\lambda$ , where  $\lambda = \langle 1^{r_1}, 2^{r_2}, \dots \rangle$ . Let  $t_\lambda$  denote the result

of substituting  $ip_i$  for  $p_i$  in the expansion of  $\hat{s}_\lambda$  in terms of power sums. Suppose that  $t_\lambda = \sum_\mu a_{\lambda\mu} p_\mu$ . Show that

$$t_\lambda = \sum_\mu a_{\lambda\mu} \tilde{m}_\mu.$$

- (d) [5–] Let  $W_n$  denote the space of all  $f \in \Lambda_{\mathbb{Q}}^n$  such that if  $f = \sum_\mu a_{\lambda\mu} p_\mu$ , then  $f = \sum_\mu a_{\lambda\mu} \tilde{m}_\mu$ . Find  $\dim W_n$ . Does  $W_n$  have a nice basis?
- (e) [5–] Let  $\varphi_k(s_\lambda)$  denote the terms of the least  $k$  degrees (that is, of degrees  $\text{rank}(\lambda)$ ,  $\text{rank}(\lambda) + 1, \dots, \text{rank}(\lambda) + k - 1$ ) appearing in the expansion of  $s_\lambda$  in terms of power sums, so in particular  $\hat{s}_\lambda = \varphi_1(s_\lambda)$ . Let  $V_n^{(k)}$  denote the subspace of  $\Lambda_{\mathbb{Q}}^n$  spanned by all  $\varphi_k(s_\lambda)$ . Show that a basis for  $V_n^{(2)}$  is given by  $\{\hat{s}_\lambda : \text{rank}(\lambda) \geq \ell(\lambda) - 1\}$ .
- (f) [5–] Find a basis and/or the dimension of  $V_n^{(k)}$  for  $k \geq 3$ . NOTE. It is *false* that a basis for  $V_n^{(3)}$  is given by  $\{\hat{s}_\lambda : \text{rank}(\lambda) \geq \ell(\lambda) - 2\}$ .
61. (a) [3] Let  $t$  be an indeterminate. Let  $\vartheta : \Lambda \rightarrow \Lambda[t]$  be the specialization (homomorphism) defined by

$$\vartheta(p_k) = t + \sum_{i=1}^k \binom{k}{i} p_i.$$

Show that

$$\vartheta(s_\lambda) = \sum_{\mu \subseteq \lambda} \frac{f^{\lambda/\mu}}{|\lambda/\mu|!} \left( \prod_{u \in \lambda/\mu} (t + c(u)) \right) s_\mu,$$

where  $c(u)$  denotes the content of the square  $u$ .

62. [3–] Define a  $\mathbb{Q}$ -linear transformation  $\varphi : \Lambda_{\mathbb{Q}} \rightarrow \mathbb{Q}[t]$  by

$$\varphi(s_\lambda) = \frac{\prod_{i=1}^n (t + \lambda_i + n - i)}{H_\lambda},$$

where  $\lambda = (\lambda_1, \dots, \lambda_n) \vdash n$  and  $H_\lambda$  denotes the product of the hook lengths of  $\lambda$ . Show that for any  $\mu \vdash n$  with  $\ell(\mu) = \ell$  and  $m_1(\mu) = m$  (the number of parts of  $\mu$  equal to 1), we have

$$\varphi(p_\mu) = (-1)^{n-\ell} \sum_{i=0}^m \binom{m}{i} t(t+1) \cdots (t+i-1).$$

63. [5] Let  $\mathcal{I}$  be a collection of subintervals  $\{i, i+1, \dots, i+j\}$  of  $[n]$ . (Without loss of generality we may assume that  $\mathcal{I}$  is an *antichain*, i.e., if  $I, J \in \mathcal{I}$  and  $I \subseteq J$ , then  $I = J$ .) Define

$$f_{\mathcal{I}}(x) = \sum_{i_1 i_2 \cdots i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where  $i_1 i_2 \cdots i_n$  ranges over all  $n$ -tuples of positive integers such that if  $j, k \in I \in \mathcal{I}$  and  $j \neq k$ , then  $x_{i_j} \neq x_{i_k}$ . Thus  $f_{\mathcal{I}} \in \Lambda$ . For instance, if  $\mathcal{I} = \emptyset$  then  $f_{\mathcal{I}} = e_1^n$ , and if  $\mathcal{I} = \{[n]\}$  then  $f_{\mathcal{I}} = n! e_n$ . Show that  $f_{\mathcal{I}}$  is  $e$ -positive.

64. (a) [2+] Fix integers  $1 \leq m \leq n$ . Find simple formulas for the four sums

$$\begin{aligned} a(m, n) &= \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\mu f^\nu f^\lambda c_{\mu\nu}^\lambda \\ b(m, n) &= \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\mu f^\nu c_{\mu\nu}^\lambda \\ c(m, n) &= \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\nu f^\lambda c_{\mu\nu}^\lambda \\ d(m, n) &= \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\lambda c_{\mu\nu}^\lambda, \end{aligned}$$

where  $c_{\mu\nu}^\lambda$  denotes a Littlewood-Richardson coefficient. Some of the formulas may involve the number  $t(k)$  of involutions in  $\mathfrak{S}_k$  for certain  $k$ .

- (b) [2+] Let

$$e(m, n) = \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} f^\nu c_{\mu\nu}^\lambda.$$

Show that

$$\sum_{m \geq 0} \sum_{k \geq 0} e(m, m+k) x^m \frac{y^k}{k!} = P(x) \exp\left(\frac{y}{1-x} + \frac{y^2}{2(1-x^2)}\right),$$

where  $P(x) = \prod_{i \geq 1} (1-x^i)^{-1}$ .

- (c) [5-] Do something similar for

$$f(m, n) = \sum_{\mu \vdash m} \sum_{\nu \vdash n-m} \sum_{\lambda \vdash n} c_{\mu\nu}^\lambda.$$

65. [3–] Show that

$$\sum_{\mu, \nu, \lambda} (c_{\mu\nu}^\lambda)^2 q^{|\lambda|} = \frac{1}{\prod_{i \geq 1} (1 - 2q^i)}. \quad (7)$$

66. [3–] Let  $k \geq 1$  and

$$B_k(x) = \sum_{\ell(\lambda) \leq k} s_\lambda(x),$$

as in Exercise 7.16(a). Show that

$$B_k(x) = \frac{\sum_{\mu} (-1)^{c_{\mu}} s_{\mu}(x)}{\prod_i (1 - x_i) \cdot \prod_{i < j} (1 - x_i x_j)},$$

where  $\mu$  ranges over all partitions whose Frobenius notation has the form

$$\mu = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 + k & a_2 + k & \cdots & a_r + k \end{pmatrix},$$

and where  $c_{\mu} = (|\mu| - rk + r)/2$ .

67. [3+] Let  $\lambda, \mu, \nu$  be partitions and  $n \in \mathbb{P}$ . Show that  $c_{\mu\nu}^\lambda \neq 0$  if and only if  $c_{n\mu, n\nu}^{n\lambda} \neq 0$ .

68. [4–] Let  $\lambda, \mu, \nu$  be partitions of length at most  $n$ . If  $A$  is an  $n \times n$  hermitian matrix with eigenvalues  $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ , then write  $\text{spec}(A) = (\alpha_1, \dots, \alpha_n)$ . Show that the following two conditions are equivalent:

- There exist  $n \times n$  hermitian matrices  $A, B, C$  such that  $A = B + C$ ,  $\text{spec}(A) = \lambda$ ,  $\text{spec}(B) = \mu$ , and  $\text{spec}(C) = \nu$ .
- $c_{\mu\nu}^\lambda \neq 0$ , where  $c_{\mu\nu}^\lambda$  denotes a Littlewood-Richardson coefficient.

69. (a) [2–] Find all partitions  $\lambda \vdash n$  such that  $\chi^\lambda(\mu) \neq 0$  for all  $\mu \vdash n$ .

(b) [5–] Find all partitions  $\mu \vdash n$  such that  $\chi^\lambda(\mu) \neq 0$  for all  $\lambda \vdash n$ .

70. [2+] Given  $\lambda \vdash n$ , let  $H_\lambda$  denote the product of the hook lengths of  $\lambda$ , so  $H_\lambda = n!/f^\lambda$ . Show that for  $k \in \mathbb{N}$ ,

$$\sum_{\lambda \vdash n} H_\lambda^{k-2} = \frac{1}{n!} \#\{(w_1, w_2, \dots, w_k) \in \mathfrak{S}_n^k : w_1^2 w_2^2 \cdots w_k^2 = 1\}.$$

**HINT.** Use Exercises 7.69(b) (or more precisely, its solution) and 7.70.

71. (a) [2+] Show that

$$\sum_{n \geq 0} \sum_{\lambda \vdash n} (f^\lambda)^2 \prod_{u \in \lambda} (t + c_u^2) \cdot \frac{x^n}{n!^2} = (1 - x)^{-t},$$

where  $c_u$  denotes the content of the square  $u$  in the diagram of  $\lambda$ .

(b) [3+] Show that

$$\sum_{n \geq 0} \sum_{\lambda \vdash n} (f^\lambda)^2 \prod_{u \in \lambda} (t + h_u^2) \cdot \frac{x^n}{n!^2} = \prod_{i \geq 1} (1 - x^i)^{-1-t},$$

where  $h_u$  denotes the hook length of the square  $u$  in the diagram of  $\lambda$ .

(c) [3] Show that for any  $r \geq 0$  we have

$$\frac{1}{n!} \sum_{\lambda \vdash n} (f^\lambda)^2 \sum_{u \in \lambda} \prod_{i=0}^{r-1} (c_u^2 - i^2) = \frac{(2r)!}{(r+1)!^2} (n)_{r+1}. \quad (8)$$

(d) [2] Deduce from equation (8) that

$$\frac{1}{n!} \sum_{\lambda \vdash n} (f^\lambda)^2 \sum_{u \in \lambda} c_u^{2k} = \sum_{j=1}^k T(k, j) \frac{(2j)!}{(j+1)!^2} (n)_{j+1},$$

where  $T(k, j)$  is a *central factorial number* (EC2, Exercise 5.8).

(e) [3] Show that for any  $r \geq 0$  we have

$$\frac{1}{n!} \sum_{\lambda \vdash n} (f^\lambda)^2 \sum_{u \in \lambda} \prod_{i=1}^r (h_u^2 - i^2) = \frac{1}{2(r+1)^2} \binom{2r}{r} \binom{2r+2}{r+1} (n)_{r+1}. \quad (9)$$

(f) [2] Deduce from (e) that

$$\frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \sum_{u \in \lambda} h_u^{2k} = \sum_{j=1}^{k+1} T(k+1, j) \frac{1}{2j^2} \binom{2(j-1)}{j-1} \binom{2j}{j} (n)_j,$$

where  $T(k+1, j)$  is as in (d).

(g) [3] Let  $F = F(x) \in \Lambda_{\mathbb{Q}}$  be a symmetric function. Define

$$\Phi_n(F) = \frac{1}{n!} \sum_{\lambda \vdash n} (f^\lambda)^2 F(h_u^2 : u \in \lambda).$$

Here  $F(h_u^2 : u \in \lambda)$  denotes substituting the quantities  $h_u^2$ , where  $u$  is a square of the diagram of  $\lambda$ , for  $n$  of the variables of  $F$ , and setting all other variables equal to 0. Show that  $\Phi_n(F)$  is a polynomial function of  $n$ .

(h) [3] Let  $G(x; y)$  be a power series of bounded degree (say over  $\mathbb{Q}$ ) that is symmetric separately in the  $x$  variables and  $y$  variables. Let

$$\Psi_n(G) = \frac{1}{n!} \sum_{\lambda \vdash n} (f^\lambda)^2 G(c_u : u \in \lambda; \lambda_i - i : 1 \leq i \leq n).$$

Show that  $\Psi_n(G)$  is a polynomial function of  $n$ .

72. Let  $k \geq 1$  and  $p \geq 2$ . Show that the number of  $p$ -cores (as defined in Exercise 7.59(d)) with largest part  $k$  is  $\binom{k+p-2}{p-2}$ .

73. Fix a partition  $\mu \vdash k$ , and define  $N(n; \mu) = \sum_{\lambda \vdash n} f^{\lambda/\mu}$ . Let  $t(j)$  denote the number of involutions in  $\mathfrak{S}_j$ .

(a) [2+] Show that for all  $n, k \geq 0$  we have

$$N(n+k; \mu) = \sum_{j=0}^k \binom{n}{j} \left( \sum_{\nu \vdash k-j} f^{\mu/\nu} \right) t(n-j).$$

(b) [3-] Let  $\tilde{\nu}$  be the partition obtained from  $\nu$  by replacing each even part  $2i$  with  $i, i$ . Equivalently, if  $w$  is a permutation of cycle type  $\nu$ , then  $w^2$  has cycle type  $\tilde{\nu}$ . Show that for  $n \geq k$ ,

$$N(n; \mu) = \sum_{j=0}^k \frac{t(n-j)}{(k-j)!} \sum_{\substack{\nu \vdash j \\ m_1(\nu)=m_2(\nu)=0}} z_\nu^{-1} \chi^\mu(\tilde{\nu}, 1^{k-j}).$$

For instance,

$$N(n; 32) = N(n; 221) = \frac{1}{24} (t(n) - 4t(n-3) + 6t(n-4)).$$

74. (a) [3–] Fix a partition  $\mu \vdash k$ . Given  $\lambda \vdash n \geq k$ , define

$$\widehat{\chi}^\lambda(\mu, 1^{n-k}) = \frac{(n)_k \chi^\lambda(\mu, 1^{n-k})}{\chi^\lambda(1^n)}.$$

Let  $p \times q$  denote the partition with  $p$  parts equal to  $q$ . Fix a partition  $w_\mu \in \mathfrak{S}_k$  of cycle type  $\mu$ , and let  $\kappa(w)$  denote the number of cycles of the permutation  $w \in \mathfrak{S}_k$ . Show that

$$\widehat{\chi}^{p \times q}(\mu, 1^{pq-k}) = (-1)^k \sum_{uv=w_\mu} p^{\kappa(u)} (-q)^{\kappa(v)},$$

where the sum ranges over all  $k!$  pairs  $(u, v) \in \mathfrak{S}_k \times \mathfrak{S}_k$  satisfying  $uv = w_\mu$ . **HINT.** Use the Murnaghan-Nakayama rule and Exercise 7.70.

- (b) [3+] [to be inserted]

75. (a) [2+] Let  $\kappa(w)$  denote the number of cycles of  $w \in \mathfrak{S}_n$ . Show that

$$P_n(q) := \sum_w q^{\kappa(w(1,2,\dots,n))} = \frac{1}{n(n+1)} ((q+n)_{n+1} - (q)_{n+1}).$$

where  $w$  ranges over all  $(n-1)!$   $n$ -cycles in  $\mathfrak{S}_n$  and  $w(1, 2, \dots, n)$  denotes the product of  $w$  with the  $n$ -cycle  $(1, 2, \dots, n)$ . For instance,

$$\begin{aligned} \sum_{\rho(w)=(3)} q^{\kappa(w(1,2,3))} &= \frac{1}{12} ((q+3)_4 - (q)_4) \\ &= q^3 + q. \end{aligned}$$

**HINT.** Use Exercise 7.70.

- (b) [2+] Show that all the zeros of  $P_n(q)$  have real part 0.  
(c) [5–] It follows from (a) that

$$P_n(q) = \frac{1}{\binom{n+1}{2}} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} c(n+1, n-2i) q^{n-2i},$$

where  $c(n+1, n-2i)$  denotes the number of permutations  $w \in \mathfrak{S}_{n+1}$  with  $n-2i$  cycles. Is there a bijective proof? (In fact, it isn't so obvious that  $c(n+1, n-2i)$  is divisible by  $\binom{n+1}{2}$ . J. Burns has proved the stronger result that if  $\lambda \vdash n+1$  and  $\varepsilon_\lambda = -1$ , then  $(n+1)!/z_\lambda$  is divisible by  $\binom{n+1}{2}$ .)

(d) [3] Generalize (b) as follows. Fix  $\lambda \vdash n$ . Define

$$P_\lambda(q) = \sum_{\rho(w)=\lambda} q^{\kappa(w(1,2,\dots,n))},$$

where  $w$  ranges over all permutations in  $\mathfrak{S}_n$  of cycle type  $\lambda$ . Show that all the zeros of  $P_\lambda(q)$  have real part 0.

76. [3] Define two compositions  $\alpha$  and  $\beta$  of  $n$  to be *equivalent* if  $s_{B_\alpha} = s_{B_\beta}$  (as defined in §7.23). Describe the equivalence classes of this equivalence relation, showing in particular that the cardinality of each equivalence class is a power of two.

NOTE. A “trivial” equivalence is given by

$$(\alpha_1, \alpha_2, \dots, \alpha_k) \sim (\alpha_k, \dots, \alpha_2, \alpha_1).$$

It is surprising that an equivalence class can have more than two elements, e.g.,  $\{(1, 2, 1, 3, 2), (2, 3, 1, 2, 1), (2, 1, 2, 3, 1), (1, 3, 2, 1, 2)\}$ .

77. [3] Define the *rank* of a skew shape  $\lambda/\mu$  to be the minimal number of border strips in a border strip tableau of shape  $\lambda/\mu$ . It is easy to see that when  $\mu = \emptyset$  this definition agrees with that on page 289. Let  $|\lambda/\mu| = n$ , and let  $\nu$  be a partition of  $n$  satisfying  $\ell(\nu) = \text{rank}(\lambda/\mu)$ . Show that  $\chi^{\lambda/\mu}(\nu)$  is divisible by  $m_1(\nu)! m_2(\nu)! \cdots$ . (Incidentally, note that by the definition (7.75) of  $\chi^{\lambda/\mu}(\nu)$  we have  $\chi^{\lambda/\mu}(\nu) = 0$  if  $\ell(\nu) < \text{rank}(\lambda/\mu)$ .)
78. Let  $\lambda/\mu$  be a skew shape, identified with its Young diagram  $\{(i, j) : \mu_i < j \leq \lambda_i\}$ . We regard the points  $(i, j)$  of the Young diagram as squares. An *outside top corner* of  $\lambda/\mu$  is a square  $(i, j) \in \lambda/\mu$  such that  $(i-1, j), (i, j-1) \notin \lambda/\mu$ . An *outside diagonal* of  $\lambda/\mu$  consists of all squares  $(i+p, j+p) \in \lambda/\mu$  for which  $(i, j)$  is a fixed outside top corner. Similarly an *inside top corner* of  $\lambda/\mu$  is a square  $(i, j) \in \lambda/\mu$  such that  $(i-1, j), (i, j-1) \in \lambda/\mu$  but  $(i-1, j-1) \notin \lambda/\mu$ . An *inside diagonal* of  $\lambda/\mu$  consists of all squares  $(i+p, j+p) \in \lambda/\mu$  for which  $(i, j)$  is a fixed inside top corner. If  $\mu = \emptyset$ , then  $\lambda/\mu$  has one outside diagonal (the main diagonal) and no inside diagonals. Figure 1 shows the skew shape 8874/411, with outside diagonal squares marked by + and inside diagonal squares by -. Let  $d^+(\lambda/\mu)$  (respectively,  $d^-(\lambda/\mu)$ ) denote the

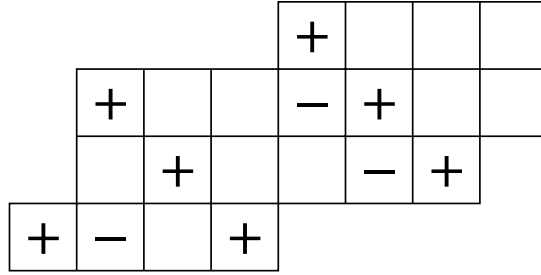


Figure 1: Outside and inside diagonals of the skew shape 8874/411

total number of outside diagonal squares (respectively, inside diagonal squares) of  $\lambda/\mu$ .

Generalizing the code  $C_\lambda$  of Exercise 7.59, define the *code*  $C_{\lambda/\mu}$  of  $\lambda/\mu$  to be the two-line array whose top line is  $C_\lambda$  and whose bottom line is  $C_\mu$ , where the indexing is “in phase.” For instance,

$$C_{8874/411} = \begin{array}{cccccccccccccccc} \cdots & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots \end{array}.$$

[3–, for the first four] Show that the following numbers are equal:

- The rank of  $\lambda/\mu$ , as defined in Exercise 77 above.
- $d^+(\lambda/\mu) - d^-(\lambda/\mu)$
- The number of rows in the Jacobi-Trudi matrix for  $\lambda/\mu$  (i.e., the matrix of Theorem 7.16.1) which don’t contain a 1.
- The number of columns of  $C_{\lambda/\mu}$  equal to  $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$  (or to  $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$ ).
- [3+] The largest power of  $t$  dividing the polynomial  $s_{\lambda/\mu}(1^t)$ .

79. [2+] Let  $\kappa(w)$  denote the number of cycles of  $w \in \mathfrak{S}_n$ . Regard  $\kappa$  as a class function on  $\mathfrak{S}_n$ . Let  $\lambda \vdash n$ . Show that

$$\langle \kappa, \chi^\lambda \rangle = \begin{cases} \sum_{i=1}^n \frac{1}{i}, & \text{if } \lambda = (n) \\ (-1)^{n-p-q} \frac{p-q+1}{(n-q+1)(n-p)}, & \text{if } \lambda = (p, q, 1^{n-p-q}), q > 0 \\ 0, & \text{otherwise.} \end{cases}$$

80. (a) [3] Define a class function  $f_n : \mathfrak{S}_n \rightarrow \mathbb{Z}$  by

$$f_n(w) = n!(\kappa(w) + 1)^{\kappa(w)-1},$$

where  $\kappa(w)$  denotes the number of cycles of  $w$ . Show that  $f_n$  is a character of  $\mathfrak{S}_n$ .

- (b) [3-] Let  $F(x) = x x^{x^{\cdot^{\cdot^{\cdot}}}}$ , so  $F(x)^{\langle -1 \rangle} = x^{1/x}$ . Let the Taylor series expansion of  $F(x)$  about  $x = 1$  be given by

$$\begin{aligned} F(x) &= \sum_{n \geq 0} a_n \frac{(x-1)^n}{n!} \\ &= 1 + u + 2\frac{u^2}{2!} + 9\frac{u^3}{3!} + 56\frac{u^4}{4!} + 480\frac{u^5}{5!} + 5094\frac{u^6}{6!} + \cdots, \end{aligned}$$

where  $u = x - 1$ . Show that  $\langle f_n, \text{sgn} \rangle = a_n$ , where  $\text{sgn}$  denotes the sign character of  $\mathfrak{S}_n$ . In particular, by (a) it follows that  $a_n \geq 0$ .

81. Let  $E(\lambda)$  (respectively,  $O(\lambda)$ ) be the number of SYT of shape  $\lambda$  whose major index is even (respectively, odd).

- (a) [2+] Express the symmetric function

$$R_n = \sum_{\lambda \vdash n} (E(\lambda) - O(\lambda)) s_\lambda$$

in terms of the power sum symmetric functions.

- (b) [2+] Deduce from (a) that if  $\lambda \vdash n$ , then  $E(\lambda) = O(\lambda)$  if and only if one cannot place  $\lfloor n/2 \rfloor$  disjoint dominos (i.e., two squares with an edge in common) on the diagram of  $\lambda$ .
- (c) [2+] Show that (b) continues to hold for skew shapes  $\lambda/\mu$  when  $|\lambda/\mu|$  is even, but that the “only if” part can fail when  $|\lambda/\mu|$  is odd.
- (d) [2+] Let  $p$  be prime. Generalize (a)–(c) to the case  $A_0(\lambda) = A_1(\lambda) = \cdots = A_{p-1}(\lambda)$ , where  $A_i(\lambda)$  denotes the number of SYT  $T$  of shape  $\lambda$  satisfying  $\text{maj}(T) \equiv i \pmod{p}$ .

82. (a) [5] A problem superficially similar to 81(b) is the following. We can regard an SYT of shape  $\lambda$  (or more generally, a linear extension of a finite poset  $P$ ) as a permutation of the squares of  $\lambda$  (or the elements of  $P$ ), where we fix some particular SYT  $T$  to correspond to the identity permutation. Define an *even* SYT to be one which, regarded as a permutation, is an even permutation, and similarly *odd* SYT. For which  $\lambda$  is the number of even SYT the same as the number of odd SYT? (It's easy to see that the answer does not depend on the choice of the "identity SYT"  $T$ .) This problem has been solved for rectangular shapes by a difficult argument (rating [3] or even [3+]).
- (b) [3] Given an SYT  $T$  with  $n$  squares, let  $w_T$  be the permutation of  $[n]$  obtained by reading the elements of  $T$  in the usual reading order (left-to-right, top-to-bottom). Write  $\text{sgn}(T) = \text{sgn}(w_T)$ , i.e.,  $\text{sgn}(T) = 1$  if  $w_T$  is an even permutation, and  $\text{sgn}(T) = -1$  if  $w_T$  is an odd permutation. Show that

$$\sum_T \text{sgn}(T) = 2^{\lfloor n/2 \rfloor},$$

where  $T$  ranges over all SYT with  $n$  squares.

83. (a) [3−] Let  $g_\lambda = \sum_\pi x_1^{c_1(\pi)} x_2^{c_2(\pi)} \cdots$ , where  $\pi$  ranges over all reverse plane partitions of shape  $\lambda$ , and  $c_i(\pi)$  is the number of columns of  $\pi$  that contain the part  $i$ . Show that  $g_\lambda$  is an (inhomogeneous) symmetric function whose highest degree part is  $s_\lambda$ .
- (b) [3] Define an *elegant* SSYT of skew shape  $\lambda/\mu$  to be an SSYT of shape  $\lambda/\mu$  for which the numbers in row  $i$  lie in the interval  $[1, i - 1]$ . In particular, there are no elegant SSYT of shape  $\lambda/\mu$  if the first row of  $\lambda/\mu$  is nonempty. Let  $f_\lambda^\mu$  be the number of elegant SSYT of shape  $\lambda/\mu$ . Show that

$$g_\lambda = \sum_{\mu \subseteq \lambda} f_\lambda^\mu s_\mu.$$

In particular,  $g_\lambda$  is Schur positive.

*Example.* Let  $\lambda = (2, 1)$ . Then there is one elegant SSYT of the empty shape  $(2, 1)/(2, 1)$  and one elegant SSYT of shape  $(2, 1)/(2)$ . Hence  $g_{2,1} = s_{2,1} + s_2$ .

- (c) [3] For  $k \geq 0$  and  $n \geq 1$ , let  $g_n^{(k)} = \sum_{j=0}^n \binom{k-1+j}{j} h_{n-j}$ . For instance,  $g_n^{(0)} = h_n$  and  $g_n^{(1)} = h_n + g_{n-1} + \cdots + h_1 + 1$ . Set  $g_0^{(k)} = 1$  and  $g_{-n}^{(k)} = 0$  for  $n > 0$ . Show that if  $\lambda = (\lambda_1, \dots, \lambda_m)$ , then

$$g_\lambda = \det \left( g_{\lambda_i - i + j}^{(i-1)} \right)_{i,j=1}^m.$$

84. (a) [2] A *set-valued tableau* of shape  $\lambda/\mu$  is a filling of the diagram of  $\lambda/\mu$  with nonempty finite subsets of  $\mathbb{P}$  such that if each subset is replaced by one of its elements, then an SSYT always results. If  $T$  is a set-valued tableau, then let  $x^T = x_1^{c_1(T)} x_2^{c_2(T)} \cdots$ , where  $c_i(T)$  is the number of boxes of  $T$  containing  $i$ . Set  $|T| = \sum c_i(T)$ , the total number of elements appearing in all the boxes. Define

$$G_{\lambda/\mu}(x) = \sum_T (-1)^{|T| - |\lambda/\mu|} x^T,$$

where  $T$  ranges over all set-valued tableaux of shape  $\lambda/\mu$ . For instance,

$$G_{1^n} = e_n - n e_{n+1} + \binom{n+1}{2} e_{n+2} - \binom{n+2}{3} e_{n+3} + \cdots.$$

Show that  $G_{\lambda/\mu}$  is a symmetric formal power series (i.e., an element of the completion  $\hat{\Lambda}$  of the ring  $\Lambda$  of symmetric functions) whose least degree part is  $s_{\lambda/\mu}$ .

- (b) [3] Let  $f_\lambda^\mu$  have the meaning of the previous problem. Show that

$$s_\mu = \sum_{\lambda \supseteq \mu} f_\lambda^\mu G_\lambda.$$

For instance,

$$s_{1^n} = e_n = G_n + n G_{n+1} + \binom{n+1}{2} G_{n+2} + \binom{n+2}{3} G_{n+3} + \cdots.$$

- (c) [2] Deduce from (a) and (b) that  $\langle g_\lambda, G_\mu \rangle = \delta_{\lambda\mu}$ , where  $g_\lambda$  has the meaning of the previous problem.

- (d) [3] For  $k \geq 0$  and  $n \geq 1$ , let  $G_n^{(k)} = \sum_{i,j \geq 0} (-1)^j \binom{k+i-2}{i} s_{(n+i,1^j)}$ . For instance,  $G_n^{(1)} = s_n - s_{(n,1)} + s_{(n,1,1)} - \cdots$ . Set  $G_0^{(k)} = 1$  and  $G_{-n}^{(k)} = 0$  for  $n > 0$ . Show that if  $\lambda = (\lambda_1, \dots, \lambda_m)$ , then

$$G_\lambda = \det \left( G_{\lambda_i - i + j}^{(m-i+1)} \right)_{i,j=1}^m.$$

85. [3] Let  $L_\lambda$  be the symmetric function of Exercise 7.89(f), where  $\lambda \vdash n$ . Let  $\alpha \in \text{Comp}(n)$ , and let  $B_\alpha$  be the corresponding border strip (as defined on page 383). Show that

$$\langle L_\lambda, s_{B_\alpha} \rangle = \#\{w \in \mathfrak{S}_n : \rho(w) = \lambda, D(w) = S_\alpha\},$$

where  $D(w)$  denotes the descent set of  $w$ .

86. (a) [2+] Fix  $n \geq 1$ . Given  $S, T \subseteq [n-1]$ , let

$$\beta(S, T) = \#\{w \in \mathfrak{S}_n : D(w) = S, D(w^{-1}) = T\}.$$

Let  $f(n) = \max_{S, T \subseteq [n-1]} \beta(S, T)$ . Show that there is some  $S \subseteq [n-1]$  for which  $f(n) = \beta(S, S)$ .

- (b) [5-] Show that  $f(n) = \beta(S, S)$ , where  $S = \{1, 3, 5, \dots\} \cap [n-1]$ .

87. [3-] Let  $y := \sum_\lambda s_\lambda$ . Show that

$$y * y = \exp \left( \sum_{n \geq 1} \frac{p_{2n-1}}{(2n-1)(1-p_{2n-1})} \right) \cdot \left( \prod_{n \geq 1} (1-p_n^2) \right)^{-1/2},$$

where  $*$  denotes internal product.

88. [5-] Let  $|\lambda/\mu| = n$  and

$$\begin{aligned} f^{\lambda/\mu}(q) &= (1-q)(1-q^2) \cdots (1-q^n) s_{\lambda/\mu}(1, q, q^2, \dots) \\ &= \sum_T q^{\text{maj}(T)}, \end{aligned}$$

where  $T$  ranges over all skew SYT of shape  $\lambda/\mu$ . (See Proposition 7.19.11.)

We can regard  $f^{\lambda/\mu}(q)$  as the “natural”  $q$ -analogue of  $f^{\lambda/\mu}$ . Investigate when  $f^{\lambda/\mu}(q)$  has unimodal coefficients. This isn’t always the case (e.g.,  $\lambda = (2, 2), \mu = \emptyset$ ) but it does seem to be unimodal in certain cases, such as when  $\mu = \emptyset$  and  $\lambda$  is an arithmetic progression ending with 1.

89. [2–] We follow the notation of Sections 7.19 and 7.23. Let  $\alpha \in \text{Comp}(n)$  and  $\lambda \vdash n$ . Show that  $\langle s_{B_\alpha}, s_\lambda \rangle$  is equal to the number of SYT of shape  $\lambda$  and descent set  $S_\alpha$ .

90. (a) [2–] For a sequence  $u = u_1 \cdots u_n$  of positive integers, define the *descent set*  $D(u)$  in analogy to permutations, i.e.,

$$D(u) = \{i : u_i > u_{i+1}\} \subseteq [n-1].$$

Given  $S \subseteq [n-1]$ , define

$$f_S = \sum x_{u_1} \cdots x_{u_n},$$

where  $u_1 \cdots u_n$  ranges over all sequences  $u$  of positive integers of length  $n$  satisfying  $D(u) = S$ . Show that  $f_S = s_{B_{\text{co}(S)}}$ , using the notation of Sections 7.19 and 7.23.

(b) [2+] Let  $\mathcal{S}_k$  denote the set of all finite sequences  $u_1 u_2 \cdots u_n$  of positive integers containing no strictly decreasing factor of length  $k$ , i.e., we never have  $u_i > u_{i+1} > \cdots > u_{i+k-1}$ . Show that

$$\begin{aligned} & \sum_{u_1 \cdots u_n \in \mathcal{S}_k} x_{u_1} \cdots x_{u_n} \\ &= \frac{1}{1 - e_1 + e_k - e_{k+1} + e_{2k} - e_{2k+1} + e_{3k} - e_{3k+1} + \cdots}. \end{aligned}$$

91. (a) [3–] Let  $L_\alpha$  be as in (7.89). Suppose that  $s_\lambda = f + g$ , where  $f, g \in \Lambda$  and  $f, g$  are  $L$ -positive. Show that  $f = 0$  or  $g = 0$ .

(b) [2+] Give an example of an  $L$ -positive symmetric function that isn't  $s$ -positive.

92. [2–] Let  $\mathfrak{A}_n$  denote the alternating group of degree  $n$  (regarded as a subgroup of  $\mathfrak{S}_n$ ). Express the cycle index  $Z_{\mathfrak{A}_n}$  as a linear combination of Schur functions.

93. [2+] Let  $\chi$  be a character of  $\mathfrak{S}_n$ . Let  $\text{ch}(\chi) = \sum_{\mu \vdash n} c_\mu m_\mu$ . Show that

$$c_\mu = \langle \chi|_\mu, 1_{\mathfrak{S}_\mu} \rangle,$$

the multiplicity of the trivial character  $1_{\mathfrak{S}_\mu}$  of the Young subgroup  $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots$  in the restriction  $\chi|_\mu$  of  $\chi$  to  $\mathfrak{S}_\mu$ . In particular, if  $\chi$  is a permutation representation then  $c_\mu$  is the number of orbits of  $\mathfrak{S}_\mu$ .

94. (a) [1+] Let  $X$  be a nonempty subset of  $\mathfrak{S}_n$ . Suppose that the cycle indicator  $Z_X$  is  $s$ -positive. Show that  $X$  contains the identity element of  $\mathfrak{S}_n$ .
- (b) [5] What can be said about subsets  $X$  of  $\mathfrak{S}_n$  for which  $Z_X$  is  $s$ -positive or  $h$ -positive? (See equation (7.120), Exercise 7.111(c,d), and Problem 95 below for some information.)
95. Let  $G$  be a subgroup of  $\mathfrak{S}_n$  for which the cycle indicator  $Z_G$  is  $h$ -positive.
- (a) [2+] Show that  $Z_G = h_\lambda$  for some  $\lambda \vdash n$ .
- (b) [3–] Show in fact that  $G$  is conjugate to the Young subgroup  $\mathfrak{S}_\lambda$ .
96. [2] Let  $\mathfrak{I}_n$  denote the set of all indecomposable permutations in  $\mathfrak{S}_n$ , as defined in Exercise 1.32(a). Let  $\tilde{Z}_{\mathfrak{I}_n}$  denote the augmented cycle indicator of  $\mathfrak{I}_n$ , as defined in Definition 7.24.1. Show that

$$\sum_{n \geq 1} \tilde{Z}_{\mathfrak{I}_n} x^n = 1 - \frac{1}{\sum_{n \geq 0} n! h_n x^n},$$

a direct generalization of Exercise 1.32(a).

97. [3–] Give a super-analogue of Theorem 7.24.4 (Pólya's theorem). More precisely, when  $Z_G(x/y)$  is expanded as a linear combination of the  $m_\lambda(x)m_\mu(y)$ 's, give a combinatorial interpretation of the coefficients.
98. (a) [2+] Let  $T$  be an SYT of shape  $\lambda \vdash n$ . We can regard the tableau  $\text{evac}(T)$  (as defined in Appendix 1) as a permutation of the entries  $1, 2, \dots, n$  of  $T$ . Show that this permutation is even if and only if the integer  $\binom{n}{2} + (\mathcal{O}(\lambda) - \mathcal{O}(\lambda'))/2$  is even, where  $\mathcal{O}(\mu)$  denotes the number of odd parts of the partition  $\mu$ . (Note that this condition depends only on the shape  $\lambda$  of  $T$ .)
- (b) [3] Let  $e(n)$  denote the number of permutations  $\lambda \vdash n$  for which  $\text{evac}(T)$  is an even permutation of  $T$ , for some (or every) SYT  $T$  of shape  $\lambda$ . Let  $p(n)$  denote the total number of partitions of  $n$ . Show that  $e(n) = (p(n) + (-1)^{\binom{n}{2}} f(n))/2$ , where

$$\sum_{n \geq 0} f(n)x^n = \prod_{i \geq 1} \frac{1 + x^{2i-1}}{(1 - x^{4i})(1 + x^{4i-2})^2}.$$

99. [2] Express  $\text{ex } f[g]$  in terms of  $\text{ex } f$  and  $\text{ex } g$ , where  $\text{ex}$  denotes the exponential specialization and  $f[g]$  denotes plethysm.
100. [2+] Expand the plethysm  $h_2[h_n]$  in terms of Schur functions.
101. The *plethystic inverse* of  $f \in \Lambda$  is a symmetric function  $g \in \Lambda$  satisfying  $f[g] = g[f] = p_1$  (the identity element of the operation of plethysm). (See Exercise 7.88(d).) It is easy to see that if  $g$  exists, then it is unique. Moreover,  $g$  exists if and only if  $f$  has constant term 0 and  $[p_1]f \neq 0$ .
- (a) [2] Describe the plethystic inverse of  $f = \sum_{n \geq 1} a_n p_1^n$ , where  $a_1 \neq 0$ , in terms of “familiar” objects.
- (b) [2] Let  $f = \sum_{n \geq 1} a_n p_n$ , where  $a_1 \neq 0$ . Describe the plethystic inverse of  $f$  in terms of Dirichlet convolution. The *Dirichlet convolution*  $f * g$  of two functions  $a, b : \mathbb{P} \rightarrow \mathbb{C}$  is defined by

$$a * b(n) = \sum_{d|n} f(d)g(n/d).$$

102. [2-] The group  $\text{GL}(n, \mathbb{C})$  acts on the space  $\text{Mat}(n, \mathbb{C})$  of  $n \times n$  complex matrices by left multiplication. Express the character of this action as a linear combination of irreducible characters.

## **CHRONOLOGY OF NEW PROBLEMS (beginning 4/13/02)**

Some items prior to November 22, 2007, may be missing from this list.

- 67. April 13, 2002
- 68. April 13, 2002
- 98. May 5, 2002
- 65. June 8, 2003
- 10. June 10, 2003
- 79. October 6, 2003
- 80. October 6, 2003
- 58. October 10, 2003
- 60. October 10, 2003
- 87. October 10, 2003
- 76. October 13, 2003
- 4. July 3, 2004
- 78. August 17, 2004
- 64. (b) January 1, 2005
- 31. February 13, 2005
- 40. April 16, 2005
- 39. April 17, 2005
- 47. December 13, 2005
- 56. December 13, 2005
- 96. December 31, 2005

- 57. January 3, 2006
- 90. August 2, 2006
- 17. October 22, 2006
- 66. August 7, 2007
- 26. September 4, 2007
- 83. September 29, 2007
- 84. September 29, 2007
- 24. November 22, 2007
- 93. February 13, 2008
- 83(c). March 14, 2008
- 84(d). March 14, 2008
- 72. March 25, 2008
- 86. April 26, 2008
- 71. June 29, 2008
- 3. July 11, 2008
- 71. (expanded) July 15, 2008
- 62. July 15, 2008
- 36. October 9, 2008
- 37. February 20, 2009
- 34. February 20, 2009
- 28. March 22, 2009
- 29. March 22, 2009
- 52(b). April 4, 2009
- 14. November 17, 2009