13 A glimpse of combinatorial commutative algebra.

13.1 Simplicial complexes

In this chapter we will discuss a profound connection between commutative rings and some combinatorial properties of simplicial complexes. The deepest and most interesting results in this area require a background in algebraic topology and homological algebra beyond the scope of this book. However, we will be able to prove a highly nontrivial combinatorial result that relies on commutative algebra (i.e., the theory of commutative rings) in an essential way. This result is our Theorem 13.25, the characterization of f-vectors of shellable simplicial complexes. Of course we must first define these terms and then set up the necessary machinery.

Let $V = \{x_1, \ldots, x_n\}$ be a finite set, called a *vertex set*. An *abstract simplicial complex* on V, or just *simplicial complex* for short, is a collection Δ of subsets of V satisfying the following two conditions (of which the second is the significant one):

1. $\{x_i\} \in \Delta$ for $1 \leq i \leq n$
2. If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

An element F of Δ is called a *face*. A maximal face F, i.e., a face that is not contained in any larger face, is called a *facet*. The *dimension* of F is $\#F - 1$. In particular, the empty set $\emptyset$ is a face of dimension -1, unless $\Delta = \emptyset$ (see the remark below concerning empty simplicial complexes and empty faces). An i-dimensional face is called an *i-face*. Soon we will see the geometric reason for our definition of dimension.

13.1 Remark. There is a small subtlety about the definition of simplicial complex that can lead to confusion. Namely, one must distinguish between the empty simplicial complex $\Delta = \emptyset$ which has no faces whatsoever, and the simplicial complex $\Delta = \{\emptyset\}$ whose only face is the empty set $\emptyset$.

If Γ is any finite collection of finite sets, then $\langle\Gamma\rangle$ denotes the smallest simplicial complex containing the elements of Γ. Thus

$$\langle\Gamma\rangle = \{F \,:\, F \subseteq G \text{ for some } G \in \Gamma\}.$$

In presenting examples we will often abbreviate a set such as $\{1, 2, 3\}$ as simply 123. Thus for instance $\langle 123, 14, 24\rangle$ denotes the simplicial complex with faces

$$\emptyset, 1, 2, 3, 4, 12, 13, 23, 14, 24, 123.$$

It is worthwhile to understand simplicial complexes geometrically, though such understanding is not really germane to our main results here. Let us first review some basic definitions. A *convex set* in $\mathbb{R}^d$ is a subset S of $\mathbb{R}^d$ such that if $u, v \in S$, then the line segment joining u and v is also in S. Equivalently, $\lambda u + (1 - \lambda)v \in S$ for all real numbers $0 \leq \lambda \leq 1$. Clearly the intersection of convex sets is convex. The *convex hull* of any subset S of $\mathbb{R}^d$, denoted $\mathrm{conv}(S)$, is defined to be the intersection of all convex sets containing S. It is therefore the smallest convex set in $\mathbb{R}^d$ containing S.

A set $\{v_0, v_1, \ldots, v_j\} \subset \mathbb{R}^d$ is *affinely independent* if the following condition holds: if $\alpha_0, \alpha_1, \ldots, \alpha_j$ are real numbers for which $\sum \alpha_i v_i = 0$ and $\sum \alpha_i = 0$, then $\alpha_0 = \alpha_1 = \cdots = \alpha_j = 0$. Equivalently, define an *affine subspace* of $\mathbb{R}^d$ to be the translate of a linear subspace, i.e., a set

$$A = \{v \in \mathbb{R}^d \,:\, v \cdot y^{(1)} = \alpha_1, \ldots, y^{(k)} = \alpha_k\},$$

where $y^{(1)}, \ldots, y^{(k)} \in \mathbb{R}^d$ (with each $y^{(i)} \neq 0$) and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ are fixed, and where $v \cdot y$ denotes the usual dot product in $\mathbb{R}^d$ (with respect to some basis). The *dimension* of A is the dimension of the linear subspace $\{v \in \mathbb{R}^d \,:\, v \cdot y = 0\}$. The *affine span* of a subset S of $\mathbb{R}^d$, denoted $\mathrm{aff}(S)$, is the intersection of all affine subspaces containing S. It is easy to see that $\mathrm{aff}(S)$ is itself an affine subspace. It is then true that a set of $k+1$ points of $\mathbb{R}^d$ is affinely independent if and only if its affine span has dimension k, the maximum possible. In particular, the largest number of points of an affinely independent subset of $\mathbb{R}^d$ is $d+1$.

A *simplex* (plural *simplices*) σ in $\mathbb{R}^d$ is the convex hull of an affinely independent subset of $\mathbb{R}^d$. The *dimension* of a simplex σ is the dimension of its affine span. Equivalently, if σ is the convex hull of $j+1$ affinely independent points, then $\dim \sigma = j$. If S is affinely independent and $\sigma = \mathrm{conv}(S)$, then a *face* of σ is a set $\mathrm{conv}(T)$ for some $T \subseteq S$. In particular, taking $T = \emptyset$ shows that $\emptyset$ is a face of σ. A face τ of dimension zero (i.e., τ is a single point) is called a *vertex* of Γ. If $\dim \sigma = j$, then σ has $\binom{j+1}{i+1}$ i-dimensional faces. For instance, a zero-dimensional simplex is a point, a one-dimensional simplex is a line segment, a two-dimensional simplex is a triangle, a three-dimensional simplex is a tetrahedron, etc.

A (finite) *geometric simplicial complex* is a finite set Γ of simplices in $\mathbb{R}^d$ such that the following two conditions hold:

1. If $\sigma \in \Gamma$ and τ is a face of σ, then $\tau \in \Gamma$.
2. If $\sigma, \tau \in \Gamma$, then $\sigma \cap \tau$ is a common face (possibly empty) of σ and τ.

We sometimes identify Γ with the union $\bigcup_{\sigma \in \Gamma} \sigma$ of its simplices. In this situation Γ is just a subset of $\mathbb{R}^d$, but it is understood that it has been described as a union of certain simplices.

There is an obvious abstract simplicial complex Δ that we can associate with a geometric simplicial complex. Namely, the vertex set V of Δ consists of the vertices of Γ, and a set F of vertices of Δ is a face of Δ if F is the set of vertices of some simplex $\sigma \in \Gamma$. We then say that Γ (regarded as a union of its simplices) is a *geometric realization* of Δ, denoted $\Gamma = |\Delta|$. Note that if F is a face of Δ with $k+1$ vertices, then it corresponds to a k-dimensional simplex in Γ, explaining why we defined $\dim F = \#F - 1$.

Note. In some situations it is useful for Δ to have a unique (canonical) geometric realization. We can do this as follows. Suppose that Δ has n vertices $v_1, \ldots, v_n$. Let δ_i be the ith unit coordinate vector in $\mathbb{R}^n$. For each face $F = \{v_{i_1}, \ldots, v_{i_k}\} \in \Delta$, define the simplex $\sigma_F = \mathrm{conv}(\delta_{i_1}, \ldots, \delta_{i_k})$. The linear independence of the δ_i's guarantees that σ_F is indeed a simplex and that $\sigma_F \cap \sigma_G = \sigma_{F \cap G}$. Hence the set $\Gamma = \{\sigma_F : F \in \Delta\}$ is a geometric realization of Δ, so we could define Γ as *the* geometric realization of Δ (unique once we have labelled the vertices $v_1, \ldots, v_n$).

However, for our purposes we don't need this uniqueness.

13.2 Remark. (for those with some knowledge of topology) The geometric realization $|\Delta|$ is a topological space X (a topological subspace of some $\mathbb{R}^d$). We say that Δ is a *triangulation* of X.

13.3 Example. Let $\Delta = \langle 123, 234, 235, 36, 56, 57, 8\rangle$. A geometric realization of Δ is shown in Figure 1, projected from three dimensions. Note that since three triangles share the edge 23, any geometric realization in $\mathbb{R}^d$ requires $d \geq 3$. It is a result of Karl Menger, though irrelevant for us, that any d-dimensional simplicial complex can be realized in $\mathbb{R}^{2d+1}$, and that this result is best possible, i.e., the dimension $2d+1$ cannot in general be decreased. In fact, the simplicial complex whose facets are all the $(d+1)$-element subsets of a $(2d+3)$-element set cannot be realized in $\mathbb{R}^{2d}$. For example, when $d = 1$ we get that the complete graph K_5 cannot be embedded in the plane (without crossing edges), a famous result in graph theory known to Euler at least implicitly, since he showed in 1750 that $f_1 \leq 3f_0 - 6$ for any planar graph. The first person to realize explicitly that K_5 is not planar seems to be A. F. Möbius in 1840, who stated the result in the form of a puzzle.

13.4 Example. Let $V = \{1, \bar{1}, 2, \bar{2}, 3, \bar{3}\}$ and

$$\Delta = \langle 123, \bar{1}23, 1\bar{2}3, 12\bar{3}, \bar{1}\bar{2}3, \bar{1}2\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}\rangle.$$

Then the boundary of an octahedron is a geometric realization of Δ. See Figure 2. Thus we can also say, following Remark 13.2, that Δ is a triangulation of the 2-sphere (two-dimensional sphere).

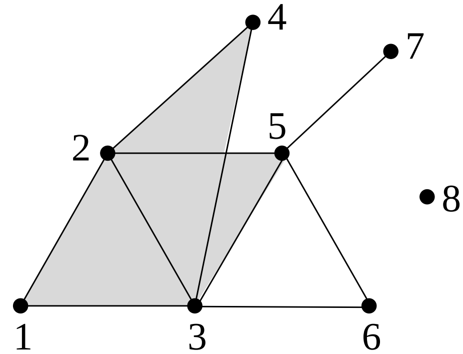

Figure 1: A geometric realization

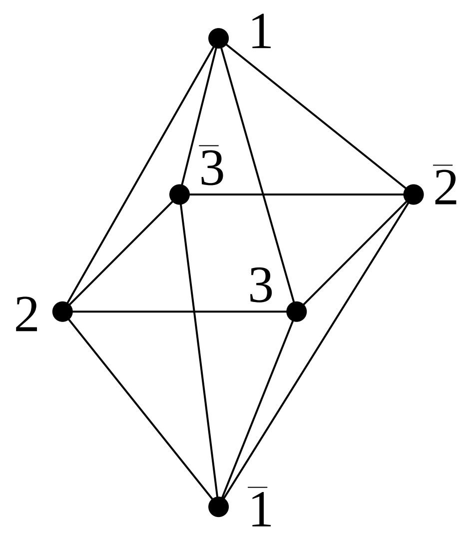

Figure 2: The boundary of an octahedron

We now come to the combinatorial information about simplicial complexes that is our primary interest in this chapter. For $i \geq -1$, let f_i be the number of i-dimensional faces of Δ. Thus $f_{-1} = 1$ unless $\Delta = \emptyset$, and $f_0 = \#V$, the number of vertices of Δ. If $\dim \Delta = d - 1$, then the vector

$$f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$$

is called the *f-vector* of Δ. Thus the simplicial complex Δ of Figure 1 has f-vector $(8, 10, 3)$, while that of Figure 2 has f-vector $(6, 12, 8)$.

An important general problem is to characterize the f-vector of various classes of simplicial complexes. The first class to come to mind is *all* simplicial complexes. In other words, what vectors $(f_0, f_1, \ldots, f_{d-1})$ of positive integers are f-vectors of $(d-1)$-dimensional simplicial complexes? Although this result is not directly related to the upcoming connection with commutative algebra, we will discuss it because of its general interest and its analogy to the upcoming Theorem 13.28.

We first make some strange-looking definitions and then explain their connection with f-vectors.

13.5 Proposition. Given positive integers n and j, there exist unique integers

$$n_j > n_{j-1} > \cdots > n_1 \geq 0$$

such that

$$n = \binom{n_j}{j} + \binom{n_{j-1}}{j-1} + \cdots + \binom{n_1}{1}. \tag{1}$$

Proof. The proof is based on the following simple combinatorial identity. Let $1 \leq i \leq m$. Then

$$\binom{m}{i} + \binom{m-1}{i-1} + \cdots + \binom{m-i+1}{1} + 1 = \binom{m+1}{i}. \qquad (2)$$

This identity can easily be proved by induction on i, for instance. It also has a simple combinatorial interpretation. Namely, the right-hand side is the number of i-element subsets S of the set $[m+1] = \{1, 2, \ldots, m+1\}$. The number of such subsets for which the least missing element is $s+1$ is equal to $\binom{m-s}{i-s}$. Summing over all $0 \leq s \leq i$ completes the proof of equation (2).

We now prove the proposition by induction on j. For $j = 1$ we have $n = \binom{n}{1}$, while $n \neq \binom{m}{1}$ for $m \neq n$. Hence the proposition is true for $j = 1$.

Assume the proposition for $j-1$. Given n, j, define m_j to be the largest integer for which $n \geq \binom{m_j}{j}$. Hence if the proposition is true for n and j, then $n_j \leq m_j$. But by (2),

$$\binom{m_j - 1}{j} + \binom{m_j - 2}{j-1} + \cdots + \binom{m_j - j}{1} = \binom{m_j}{j} - 1 < n.$$

Since the above sum is the largest possible number of the form (1) beginning with $\binom{m_j - 1}{j}$, we have $n_j \geq m_j$. Hence $n_j = m_j$. By induction there is a unique way to write

$$n - \binom{n_j}{j} = \binom{n_{j-1}}{j-1} + \binom{n_{j-2}}{j-2} + \cdots + \binom{n_1}{1},$$

where $n_{j-1} > n_{j-2} > \cdots > n_1 \geq 0$. Thus we need only check that $n_{j-1} < n_j$. If on the contrary $n_{j-1} \geq n_j$ then

$$\binom{n_j}{j} + \binom{n_{j-1}}{j-1} \geq \binom{n_j}{j} + \binom{n_j}{j-1} = \binom{n_j + 1}{j},$$

contradicting the maximality of n_j and completing the proof. $\square$

The representation of n in the form (1) is called the *j-binomial expansion* of n. Given this formula, define

$$n^{(j)} = \binom{n_j}{j+1} + \binom{n_{j-1}}{j} + \cdots + \binom{n_1}{2}.$$

In other words, add 1 to the bottom of all the binomial coefficients in the j-binomial expansion of n. For instance, $51 = \binom{7}{4} + \binom{5}{3} + \binom{4}{2} + \binom{0}{1}$, so $51^{(4)} = \binom{7}{5} + \binom{5}{4} + \binom{4}{3} + \binom{0}{2} = 30$. For notational simplicity we sometimes suppress the binomial coefficients equal to 0, e.g., $51 = \binom{7}{4} + \binom{5}{3} + \binom{4}{2}$. Note that a binomial coefficient $\binom{n_i}{i} = 0$ in the j-binomial expansion of n if and only if $n_i = i - 1$, in which case $n_r = r - 1$ for all $1 \le r \le i$.

We can now state a famous theorem of Schützenberger and Kruskal-Katona, often called the *Kruskal-Katona theorem.*

13.6 Theorem. A vector $(f_0, f_1, \ldots, f_{d-1}) \in \mathbb{P}^d$ is the f-vector of a ($(d-1)$-dimensional) simplicial complex if and only if

$$f_{i+1} \le f_i^{(i+1)}, \quad 0 \le i \le d-2. \tag{3}$$

As an example, the fact that $51^{(4)} = 30$ means that in any simplicial complex with $f_3 = 51$ we must have $f_4 \le 30$, and that this result is best possible. Theorem 13.6 says qualitatively the intuitively clear result that given f_i, the number f_{i+1} cannot be too big. However, the precise quantitative result given by this theorem is by no means intuitively obvious. Let us try to provide some intuition and at the same time convey some idea of the proof.

Let $\alpha = (a_1, \ldots, a_j)$ and $\beta = (b_1, \ldots, b_j)$ be two sequences of nonnegative integers of the same length j. We say that α is less than β in *reverse lexicographic order* (or *reverse lex order* for short), denoted $\alpha \overset{R}{<} \beta$, if for some $0 \le i \le j-1$ we have

$$a_j = b_j,\ a_{j-1} = b_{j-1}, \ldots, a_{j-i+1} = b_{j-i+1},\ \text{and } a_{j-i} < b_{j-i}.$$

Equivalantly, if we regard the nonnegative integers as the letters of an alphabet in their usual order, then the reverse sequences $(a_j, \ldots, a_1)$ and $(b_j, \ldots, b_1)$ are in dictionary (lexicographic) order. If S and T are two j-element subsets of $\mathbb{N}$, then we say that $S \overset{R}{<} T$ if $S' \overset{R}{<} T'$, where A' denotes the sequence of elements of the set A written in increasing order. If we abbreviate a set like $\{2,4,7\}$ as 247, then the one-element subsets of $\mathbb{N}$ in reverse lex order are

$$0 \overset{R}{<} 1 \overset{R}{<} 2 \overset{R}{<} 3 \overset{R}{<} 4 \overset{R}{<} 5 \overset{R}{<} 6 \overset{R}{<} \cdots.$$

The two-element subsets are

$$01 \overset{R}{<} 02 \overset{R}{<} 12 \overset{R}{<} 03 \overset{R}{<} 13 \overset{R}{<} 23 \overset{R}{<} 04 \overset{R}{<} \cdots.$$

The three-element subsets are

$$012 \overset{R}{<} 013 \overset{R}{<} 023 \overset{R}{<} 123 \overset{R}{<} 014 \overset{R}{<} 024 \overset{R}{<} 124 \overset{R}{<} 034 \overset{R}{<} 134 \overset{R}{<} 234 \overset{R}{<} 015 \overset{R}{<} \cdots.$$

The next result explains the connection between the j-binomial expansion and reverse lex order on j-element subsets of $\mathbb{N}$.

13.7 Theorem. Let $S_0, S_1, \ldots$ be the sequence of j-element subsets of $\mathbb{N}$ in reverse lex order. Suppose that $S_n = \{a_1, \ldots, a_j\}$

with $a_1 < \cdots < a_j$. Then

$$n = \binom{a_j}{j} + \binom{a_{j-1}}{j-1} + \cdots + \binom{a_1}{1},$$

and this formula gives the j-binomial expansion of n.

Before beginning the proof, here is an example. What is the 1985th term (calling the first term S_0 the 0th term) S_{1985} of the rlex order on four-element subsets of $\mathbb{N}$? We have

$$1985 = \binom{16}{4} + \binom{11}{3}.$$

Hence $S_{1985} = \{16, 11, 1, 0\}$.

Proof. There are numerous ways to present the proof. Here we use induction on n. The assertion is clear for $n = 0$ since $S_0 = \{0, 1, \ldots, j-1\}$ and $0 = \binom{j-1}{j} + \binom{j-2}{j-1} + \cdots + \binom{0}{1}$.

Assume the assertion for n. Let $S_n = \{a_1, \ldots, a_j\}$ with $a_1 < \cdots < a_j$. Suppose that the sequence $a_1, \ldots, a_j$ begins $b, b+1, \ldots, b+c, d, \ldots$, where $d > b+c+1$. Clearly b, c, d are uniquely defined. Then the elements of S_{n+1} begin $0, 1, \ldots, c-1, b+c+1, d, \ldots$ and are otherwise identical to those of S_n. But

$$\binom{b}{1} + \binom{b+1}{2} + \cdots + \binom{b+c}{c+1} + 1 = \binom{0}{1} + \binom{1}{2} + \cdots + \binom{c-1}{c} + \binom{b+c+1}{c+1}$$

by equation (2). Thus if $S_{n+1} = \{b_1, \ldots, b_j\}$ with $b_1 < \cdots < b_j$ then

$$n+1 = \binom{a_j}{j} + \cdots + \binom{a_1}{1} + 1 = \binom{b_j}{j} + \cdots + \binom{b_1}{1},$$

and the proof follows by induction. $\square$

Now suppose that $\boldsymbol{f} = (f_0, \ldots, f_{d-1}) \in \mathbb{P}^d$. Define a collection $\Gamma_{\boldsymbol{f}}$ of subsets of $\mathbb{N}$ to consist of the empty set $\emptyset$ together with the first f_i of the $(i+1)$-element subsets of $\mathbb{N}$ in rlex order. For example, if $\boldsymbol{f} = (6, 8, 5, 2)$ then (writing as usual $\{1, 2, 3\} = 123$, etc.)

$$\Gamma_{\boldsymbol{f}} = \{\emptyset, 0, 1, 2, 3, 4, 5, 01, 02, 12, 03, 13, 23, 04, 14,$$
$$012, 013, 023, 123, 014, 0123, 0124\}.$$

Note that for this example, $\Gamma_{\boldsymbol{f}}$ is not a simplicial complex.

13.8 Theorem. The set $\Gamma_{\boldsymbol{f}}$ is a simplicial complex if and only if $f_{i+1} \leq f_i^{(i+1)}$ for $0 \leq i \leq d-2$.

Proof. Let us use the notation $[0, m] = \{0, 1, \ldots, m\}$ and for any set S,

$$\binom{S}{k} = \{T \subseteq S \,:\, \#T = k\}.$$

Let $f_i = \binom{n_{i+1}}{i+1} + \binom{n_i}{i} + \cdots + \binom{n_1}{1}$ be the $(i+1)$-binomial expansion of f_i. By the definition of rlex order, we see that the set X of the first f_i $(i+1)$-elements of $\mathbb{N}$ in rlex order is given by

$$X = \binom{[0, n_{i+1} - 1]}{i+1} \bigcup \left(\{n_{i+1}\} \cup \binom{[0, n_i - 1]}{i} \right)$$
$$\bigcup \left(\{n_{i+1}, n_i\} \cup \binom{[0, n_{i-1} - 1]}{i-1} \right) \bigcup \cdots .$$

The set of $(i+2)$-elements subsets F of $\mathbb{N}$ all of whose $(i+1)$-element subsets belong to X is given by

$$X = \binom{[0, n_{i+1} - 1]}{i+2} \bigcup \left(\{n_{i+1}\} \cup \binom{[0, n_i - 1]}{i+1} \right)$$

$$\bigcup\left(\{n_{i+1}, n_i\} \cup \binom{[0, n_{i-1}-1]}{i}\right) \bigcup \cdots .$$

These are just the first $f_i^{(i+1)}$ $(i+2)$-element subsets of $\mathbb{N}$ in rlex order, and the proof follows. $\square$

Theorem 13.8 establishes the "if" direction of the Kruskal-Katona theorem (Theorem 13.6), i.e., condition (3) is sufficient for the existence of a simplicial complex with f-vector $(f_0, f_1, \ldots, f_{d-1})$. We have in fact constructed a "canonical" simplicial complex with this f-vector. Such a simplicial complex is called *compressed*.

The difficult part of the Kruskal-Katona theorem is the "only if" direction. We need to show that every simplicial complex Δ has the same f-vector as some compressed simplicial complex $\Gamma_{\boldsymbol{f}}$. This is proved by transforming Δ to $\Gamma_{\boldsymbol{f}}$ by a sequence of steps preserving the simplicial complex property and preserving the f-vector. It is not necessary to understand this argument (or in fact even the statement of the Kruskal-Katona theorem) in order to understand the main result of this chapter (Theorem 13.25) and its proof, so we will omit it here. See the "Notes for Chapter 13" below for a reference to a readable proof.

13.9 Example. Is $\boldsymbol{f} = (5, 7, 5)$ an f-vector? Of course we could simply check whether the Kruskal-Katona conditions (3) hold. Alternatively, we can construct $\Gamma_{\boldsymbol{f}}$ and check whether it is a simplicial complex. In fact,

$$\Gamma_{\boldsymbol{f}} = \{\emptyset, 0, 1, 2, 3, 4, 01, 02, 12, 03, 13, 23, 04, 012, 013, 023, 123, 014\}.$$

This is *not* a simplicial complex since 14 is a subset of $014 \in \Gamma_{\boldsymbol{f}}$, but $14 \not\in \Gamma_{\boldsymbol{f}}$. Hence $(5, 7, 5)$ is not an f-vector. In fact, we have $7 = \binom{4}{2} + \binom{1}{1}$ and $7^{(2)} = \binom{4}{3} + \binom{1}{2} = 4 < 5$.

We next want to characterize the f-vectors of a certain class of simplicial complexes, called *shellable* simplicial complexes. The result will be very similar to the Kruskal-Katona theorem, but the proof is vastly different. It will use tools from commutative algebra. First we will define shellable simplicial complexes and state the characterization of their f-vectors. We will then develop the algebraic tools necessary for the proof. Finally we will discuss a connection with an analogue of the Kruskal-Katona theorem.

We say that a simplicial complex is *pure* if every facet (maximal face) has the same dimension. For instance, the simplicial complex of Figure 1 is not pure; it has facets of dimensions zero, one and two. A *subcomplex* Δ' of a simplicial complex Δ is a subset of Δ that is itself a simplicial complex. (We don't require that Δ' has the same vertex set as Δ.)

13.10 Definition. A $(d-1)$-dimensional simplicial complex Δ is *shellable* if Δ is pure and there exists an ordering $F_1, F_2, \ldots, F_t$ of its facets (so $t = f_{d-1}$) such that the following property holds. For $0 \le j \le t$ let $\Delta_j = \langle F_1, \ldots, F_j \rangle$, the subcomplex of Δ generated by $F_1, \ldots, F_j$. In particular, $\Delta_0 = \emptyset$. Now let $1 \le j \le t$. Then we require that the set of faces of F_j (that is, the set of all subsets of F_j) has a *unique* minimal element G_j not belonging to Δ_{j-1}. Call the sequence $F_1, \ldots, F_t$ a *shelling order* or just *shelling* of Δ, and call G_j the *restriction* of F_j (with respect to the shelling $F_1, \ldots, F_t$).

NOTE. Let Δ be a pure $(d-1)$-dimensional simplicial complex. It is easy to see (Exercise 2) that a facet ordering $F_1, \dots F_t$ is a shelling if and only if for all $2 \le i \le t$, the subcomplex $\langle F_1, \dots, F_{i-1}\rangle \cap \langle F_i \rangle$ (i.e., the set of faces of F_i that already belong to $\langle F_1, \dots, F_{i-1}\rangle$) is a pure simplicial complex of dimension $d-2$. More informally, F_i attaches along some nonempty union of its facets.

It takes some time looking at examples to develop a feeling for shellings. First note that since $\Delta_0 = \emptyset$, the empty set is the unique minimal element of F_1 not in Δ_0. Thus we always have $G_1 = \emptyset$.

13.11 Example.

(a) Consider the one-dimensional simplicial complex Δ of Figure 3(a). The ordering 1,2,3 of the facets is a shelling order, with $G_1 = \emptyset$, $G_2 = \{c\}$ (abbreviated as c), $G_3 = d$. For instance, when we attach facet 2 to facet 1, we create the two new faces c and bc. The unique minimal element (with respect to inclusion) of the two sets c and bc is $G_2 = c$. Another shelling order is 2,1,3, with $G_1 = \emptyset$, $G_2 = a$, $G_3 = d$. In fact, there are exactly four shelling orders: 123, 213, 231, 312. For instance, 132 is *not* a shelling order. When we adjoin 3 to 1 we create the new faces c, d, cd, and now we have *two* minimal elements c and d.

(b) One shelling order of the simplicial complex of Figure 3(b) is 1,2,3,4, with $G_1 = \emptyset$, $G_2 = c$, $G_3 = d$, $G_4 = ad$.

(c) As essentially explained in (a) above, the simplicial complex Δ_c of Figure 3(c) is not shellable.

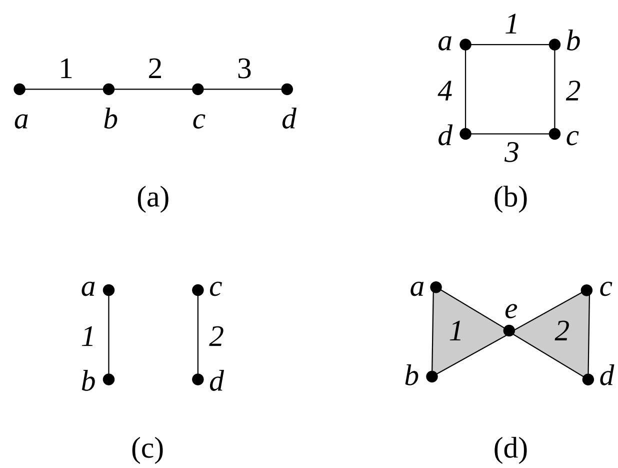

Figure 3: Some simplicial complexes

(d) The simplicial complex Δ_d of Figure 3(d) is also not shellable. Otherwise by symmetry we can assume 1,2 is a shelling order. But when we adjoin facet 2 to 1, we introduce the new faces c, d, cd, de, cde. There are *two* minimal new faces: c and d.

There is in fact a close connection between Δ_c and Δ_d. Given any simplicial complex Δ with vertex set V, define the *cone* over Δ, denoted $C(\Delta)$, to be the simplicial complex with vertex set $V \cup \{v\}$, where v is a new vertex not in V, and with faces

$$C(\Delta) = \Delta \cup \{\{v\} \cup F \,:\, F \in \Delta\}.$$

Then $\Delta_d = C(\Delta_c)$. Morevover, it is not hard to see (Exercise 7) that a simplicial complex Δ is shellable if and only if $C(\Delta)$ is shellable.

13.12 Example. For a somewhat more complicated example of a shellable simplicial complex, let Δ be the simplicial

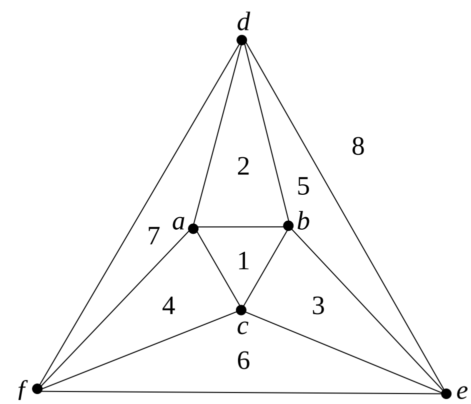

Figure 4: A shelling order of the octahedron

complex realized by the boundary of an octahedron. Figure 4 shows a shelling of Δ. This figure is a projection of the octahedron into the plane. All eight triangular regions, including the unbounded outside region with vertices, d, e, f, represent faces of Δ. The sets G_i of minimal new faces are as follows:

$$\begin{array}{llll} G_1 = \emptyset, & G_2 = d, & G_3 = e, & G_4 = f \\ G_5 = de, & G_6 = ef, & G_7 = df, & G_8 = def. \end{array}$$

13.13 Example. Figure 5 shows a more subtle example of a nonshellable simplicial complex Δ. It has nine triangular facets. There is no "local" obstruction to shellabilty. That is, we cannot look at just a small part of Δ and conclude that it is nonshellable. We will explain why Δ is nonshellable in Corollary 13.16 (see the paragraph after its proof). In general, however, there is no simple way to tell whether a simplicial complex is shellable.

We now want to discuss a connection between f-vectors and shellability. Suppose that $F_1, \ldots, F_t$ is shelling of a $(d-1)$-

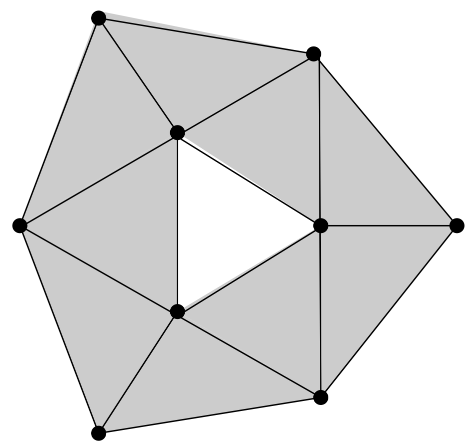

Figure 5: A nonshellable simplicial complex

dimensional simplicial complex Δ. Let G_j and Δ_{j-1} have the meaning in Definition 13.10. Suppose that $\#G_j = m$. Thus some m-element subset S of F_j is the unique minimal face of F_j not belonging to Δ_{j-1}. This set S is contained in $\binom{d-m}{i}$ $(m+i)$-element subsets T of F_j, since $\#F_j = d$. Therefore, knowing the number of elements of G_j tells us exactly how many new faces of each dimension we have adjoined to Δ at the jth shelling step.

There is an elegant and very useful way of organizing the above information. Given the f-vector $(f_0, f_1, \ldots, f_{d-1})$ of a $(d-1)$-dimensional simplicial complex Δ, define numbers $h_0, h_1, \ldots, h_d$ by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}, \tag{4}$$

where as usual $f_{-1} = 1$ unless $\Delta = \emptyset$. We call the vector

$$h(\Delta) = (h_0, h_1, \ldots, h_d)$$

the *h-vector* of Δ. It is clear from equation (4) that the f-vector and h-vector contain equivalent information—$f(\Delta)$ determines $h(\Delta)$ and *vice versa.*

13.14 Example.

(a) The f-vector of the simplicial complex of Figure 3(a) is $(3, 2)$. We compute that

$$(x-1)^2 + 3(x-1) + 2 = x^2 + x.$$

Hence $h(\Delta) = (1, 1, 0)$.

(b) For Figure 3(c) we have $f(\Delta) = (4, 2)$ and

$$(x-1)^2 + 4(x-1) + 2 = x^2 + 2x - 1.$$

Hence $h(\Delta) = (1, 2, -1)$.

(c) For Figure 2 (the boundary of an octahedron) we have $f(\Delta) = (6, 12, 8)$ and

$$(x-1)^3 + 6(x-1)^2 + 12(x-1) + 8 = x^3 + 3x^2 + 3x + 1.$$

Hence $h(\Delta) = (1, 3, 3, 1)$.

(d) For a more general example, let Δ be generated by a single d-element face F, i.e., Δ consists of all subsets of F. A geometric realization of Δ is a $(d-1)$-dimensional simplex. Now

$$f(\Delta) = \left(\binom{d}{1}, \binom{d}{2}, \binom{d}{3}, \ldots, \binom{d}{d} \right),$$

and

$$\sum_{i=0}^{d} \binom{d}{i} (x-1)^{d-i} = x^d,$$

by the binomial theorem. Hence $h(\Delta) = (1, 0, 0, \ldots, 0)$.

There are some elementary properties of the h-vector worth noting:

- By taking coefficients of x^d on both sides of equation (4), we see that $h_0 = 1$ unless $\Delta = \emptyset$.
- Taking coefficients of x^{d-1} shows that $h_1 = f_0 - d$.
- If we set $x = 1$ in equation (4) then we obtain[1]
$$f_{d-1} = h_0 + h_1 + \cdots + h_d. \tag{5}$$
The left-hand side f_{d-1} is the number of $(d-1)$-faces of Δ. It would be nice if h_i were the number of such faces with some property (depending on i). Example 13.14(b) shows that we can have $h_i < 0$, so in general h_i does not have such a nice combinatorial interpretation. However, for *shellable* simplicial complexes h_i has a simple interpretation given by Theorem 13.15 below.
- (for readers with some knowledge of topology) Putting $x = 0$ on both sides of equation (4) shows that
$$h_d = (-1)^{d-1}(-f_{-1} + f_0 - f_1 + f_2 - \cdots + (-1)^{d-1} f_{d-1}). \tag{6}$$
If X is any topological space that possesses a finite triangulation, then for any triangulation Γ, say with f-vector $(f_0, \ldots, f_{d-1})$, the alternating sum $-f_{-1} + f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1}$ is independent of the triangulation and is known as the *reduced Euler characteristic* of X, denoted $\tilde{\chi}(X)$. We also write $\tilde{\chi}(\Gamma) = \tilde{\chi}(X)$ for any triangulation Γ of X. We

[1]Since we have $(x-1)^0 = 1$ in the term indexed by $i = d$ on the left-hand side of equation (4), we need to interpret $0^0 = 1$ when we set $x = 1$. Although 0^0 is an indeterminate form in calculus, in combinatorics it usually makes sense to set $0^0 = 1$.

say that $\tilde{\chi}(\Gamma)$ is a *topological invariant* of Γ since it depends only on the geometric realization $|\Gamma|$ as a topological space. Equation (6) therefore shows that

$$h_d = (-1)^{d-1}\tilde{\chi}(\Gamma). \tag{7}$$

Recall also that the ordinary *Euler characteristic* $\chi(X)$ is given by $f_0 - f_1 + f_2 - \cdots + (-1)^{d-1} f_{d-1}$ for any triangulation Γ as above. Thus if $\Gamma \neq \emptyset$ then

$$\tilde{\chi}(X) = \chi(X) - 1$$

since $f_{-1} = 1$. Hence the difference between the reduced and ordinary Euler characteristics depends on whether or not we regard $\emptyset$ as a face.

We now come to the relationship between shellings and h-vectors.

13.15 Theorem. Let $F_1, \ldots, F_t$ be a shelling of the simplicial complex Δ, with restrictions $G_1, \ldots, G_t$. Then

$$\sum_{i=0}^{d} h_i x^i = \sum_{j=1}^{t} x^{\# G_j}.$$

In other words, h_i is the number of restrictions with i elements (independent of the choice of shelling).

Proof. We noted after Example 13.13 that when we adjoin a facet F_j to a shelling with restriction G_j satisfying $\# G_j = m$, then we adjoin $\binom{d-m}{i}$ new faces with $m+i$ elements. Hence the contribution to the polynomial $\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}$ from adjoining F_j is (using the symmetry $\binom{d-m}{i} = \binom{d-m}{d-m-i}$ and the binomial theorem) is given by

$$\begin{aligned} \sum_{i=0}^{d-m} \binom{d-m}{i} (x-1)^{d-(m+i)} &= \sum_{i=0}^{d-m} \binom{d-m}{i} (x-1)^{i} \\ &= x^{d-m}, \end{aligned}$$

and the proof follows. $\square$

13.16 Corollary. A necessary condition for a (pure) $(d-1)$-dimensional simplicial complex Δ to be shellable is that $h_i(\Delta) \geq 0$ for all $0 \leq i \leq d$. Moreover, if Δ triangulates a topological space X, then a necessary condition for shellability is $(-1)^{d-1}\tilde{\chi}(X) \geq 0$.

Proof. Assume that Δ is shellable. By Theorem 13.15 we have $h_i(\Delta) \geq 0$ for all $0 \leq i \leq d$. The second assertion then follows from equation (7). $\square$

Corollary 13.16 explains why the simplicial complex Δ of Figure 5 is not shellable. We have

$$h_3(\Delta) = (-1)^2(-1+9-18+9) = -1.$$

The geometric realization of Δ is a cylinder (or more accurately, homeomorphic to a cylinder). Since $h_3(\Delta) = -1$, it follows that *any* triangulation Γ of a cylinder X satisfies $h_3(\Gamma) = -1 = \tilde{\chi}(X)$. Similarly, the two-dimensional torus T satisfies $(-1)^2\tilde{\chi}(T) = -1$, so no triangulation of T can be shellable.

The condition of Corollary 13.16 is necessary but not sufficient for shellability. For instance, the disjoint union of two cycles (a one-dimensional simplicial complex) satisfies $h_d = 1$ but isn't shellable. (See Exercise 24.) For some more subtle examples, see Exercises 9 and 10.

13.2 The face ring

Our goal is a complete characterization of the f-vector of a shellable simplicial complex, analogous to the characterization of the f-vector of all simplicial complexes given by the Kruskal-Katona theorem (Theorem 13.6). The main tool will be a certain commutative ring associated with a simplicial complex Δ on the vertex set $V = \{x_1, \ldots, x_n\}$. To keep the presentation as simple as possible, we will develop the necessary ring theory to prove the main result of this chapter (Theorem 13.25), but no more. Most of our definitions, results, and proofs can be extended to a far greater context. We make a brief remark on one of these generalizations in Remark 13.26.

Let K be a field. Any infinite field will do for our purposes. Think of the elements of the vertex set V as indeterminates. Let $K[x_1, \ldots, x_n]$ or $K[V]$ denote the polynomial ring in the indeterminates $x_1, \ldots, x_n$. For any subset S of $\{x_1, \ldots, x_n\}$, write

$$x_S = \prod_{x_i \in S} x_i. \tag{8}$$

Let I_Δ denote the ideal of $K[V]$ generated by all monomials x_S such that $S \notin \Delta$. We call such a set S a *nonface* of Δ. If S is a nonface and $T \supset S$ then clearly T is a nonface. Hence I_Δ is generated by the *minimal nonfaces* of Δ, that is, those nonfaces for which no proper subset is a nonface. A minimal nonface is also called a *missing face*.

13.17 Example. For the simplicial complexes of Figure 3 we have the following minimal generators of I_Δ, i.e., the monomials corresponding to missing faces: (a) ac, ad, bd, (b) ac, bd, (c) ac, ad, bc, bd, (d) ac, ad, bc, bd. Note that (c) and (d) have the same missing faces. This is because (d) is a cone over (c). The cone vertex e is attached to every face F of (c) (i.e., $\{e\} \cup F$ is a face of (d)), so e belongs to no missing face.

For Figure 1, the missing faces all have two elements except for $\{3, 5, 6\}$. For the octahedron of Figure 2, the missing faces are (writing as usual $11'$ for $\{1, 1'\}$, etc.) $11'$, $22'$, and $33'$.

The quotient ring $K[\Delta] \coloneqq K[V]/I_\Delta$ is called the *face ring* (also called the *Stanley-Reisner ring*) of Δ. It is the fundamental algebraic object of this chapter.

If face rings are to be useful in characterizing f-vectors, we need to connect the two together. For this aim, define the *support* $\mathrm{supp}(u)$ of a monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ by

$$\mathrm{supp}(u) = \{x_i : a_i > 0\}.$$

Note that a K-basis for the ideal I_Δ consists of all monomials u satisfying $\mathrm{supp}(u) \notin \Delta$ [why?]. Hence a K-basis for $K[\Delta]$ consists of all monomials u satisfying $\mathrm{supp}(u) \in \Delta$, including (unless $\Delta = \emptyset$) the monomial 1, whose support is $\emptyset$. (More precisely, we mean the images of these monomials in $K[\Delta]$ under the quotient map $K[V] \to K[\Delta]$, but in such situations we identify elements of $K[V]$ with their images in $K[\Delta]$.) For $i \geq 0$ define $K[\Delta]_i$ to be the span of all monomials u of degree i satisfying $\operatorname{supp}(u) \in \Delta$. Then

$$K[\Delta] = K[\Delta]_0 \oplus K[\Delta]_1 \oplus \cdots \ \text{(vector space direct sum)}.$$

We define the *Hilbert series* of $K[\Delta]$ to be the power series

$$L(K[\Delta], \lambda) = \sum_{i \geq 0} (\dim_K K[\Delta]_i)\, \lambda^i,$$

where λ is an indeterminate. Thus $L(K[\Delta], \lambda)$ is some kind of measurement of the "size" of $K[\Delta]$.

13.18 Theorem. If $\dim \Delta = d-1$ and $h(\Delta) = (h_0, h_1, \ldots, h_d)$, then

$$L(K[\Delta], \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_d\lambda^d}{(1-\lambda)^d}. \tag{9}$$

Proof. We have seen that a K-basis for $K[\Delta]$ consists of monomials whose support is a face F of Δ. Let $\mathcal{M}_F$ be the set of monomials with support F. Then

$$\begin{aligned} \sum_{u \in \mathcal{M}_F} \lambda^{\deg(u)} &= \prod_{x_i \in F} \left(\sum_{a_i \geq 1} \lambda^{a_i} \right) \\ &= \frac{\lambda^{\#F}}{(1-\lambda)^{\#F}}. \qquad (10) \end{aligned}$$

In particular, when $F = \emptyset$ the two sides of equation (10) are equal to 1. Summing over all $F \in \Delta$ gives

$$\begin{aligned} L(K[\Delta], \lambda) &= \sum_{F \in \Delta} \frac{\lambda^{\#F}}{(1-\lambda)^{\#F}} \\ &= \sum_{i=0}^{d} f_{i-1} \frac{\lambda^i}{(1-\lambda)^i} \\ &= \frac{\sum_{i=0}^{d} f_{i-1} \lambda^i (1-\lambda)^{d-i}}{(1-\lambda)^d}. \end{aligned}$$

Now

$$\begin{aligned} \sum_{i=0}^{d} f_{i-1} \lambda^i (1-\lambda)^{d-i} &= \lambda^d \sum_{i=0}^{d} f_{i-1} \left(\frac{1}{\lambda} - 1 \right)^{d-i} \\ &= \lambda^d \sum_{i=0}^{d} h_i \lambda^{-(d-i)} \quad \text{(by (4))} \\ &= \sum_{i=0}^{d} h_i \lambda^i, \end{aligned}$$

and the proof follows. $\square$

The integer $d = 1 + \dim \Delta$ is called the *Krull dimension* of $K[\Delta]$, denoted $\dim K[\Delta]$. Do no confuse the vector space dimension $\dim_K$ with the Krull dimension dim! By equations (5) and (9) $\dim K[\Delta]$ is the order to which 1 is a pole of $L(K[\Delta], \lambda)$, i.e., the least integer k for which $(1-\lambda)^k L(K[\Delta], \lambda)$ does not have a singularity at $\lambda = 1$. It is known (but not needed here) that $\dim K[\Delta]$ is also the most number of elements of $K[\Delta]$ that are algebraically independent over K, and is also the length ℓ of the longest chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_\ell$ of prime ideals of $K[\Delta]$.

There is a special situation in which the h_i's have a direct algebraic interpretation. In any commutative ring R, recall that an element u is called a *non-zero-divisor* (or NZD) if whenever $y \in R$ and $uy = 0$, then $y = 0$. Now let $\theta \in K[\Delta]_1$ be an NZD. (Note that $\theta \in K[\Delta]_1$ means that θ is a linear combination $\sum_{i=1}^n \alpha_i x_i$ ($\alpha_i \in K$) of the vertices $x_1, \ldots, x_n$ of Δ.) Since θ is an NZD, we have that for $i \geq 0$ the map $K[\Delta]_i \to K[\Delta]_{i+1}$ defined by $y \mapsto \theta y$ is injective (one-to-one). Hence

$$\dim_K \theta K[\Delta]_i = \dim_K K[\Delta]_i. \tag{11}$$

Let (θ) denote the ideal of $K[\Delta]$ generated by θ. Since θ is homogeneous, the quotient ring $K[\Delta]/(\theta)$ has the vector space grading

$$K[\Delta]/(\theta) = (K[\Delta]/(\theta))_0 \oplus (K[\Delta]/(\theta))_1 \oplus \cdots, \tag{12}$$

where $(K[\Delta]/(\theta))_i$ is the image of $K[\Delta]_i$ under the quotient homomorphism $K[\Delta] \to K[\Delta]/(\theta)$.

If $A(\lambda) = \sum_{i\geq 0} a_i \lambda^i$ and $B(\lambda) = \sum_{i\geq 0} b_i \lambda^i$ are two power series with real coefficients, write $A(\lambda) \leq B(\lambda)$ to mean $a_i \leq b_i$ for all i.

13.19 Lemma. Let $\theta \in K[\Delta]_1$. Then

$$L(K[\Delta], \lambda) \leq \frac{L(K[\Delta]/(\theta), \lambda)}{1 - \lambda}, \tag{13}$$

with equality if and only if θ in an NZD.

Proof. If θ is an NZD then by equation (11) we have

$$H(K[\Delta]/(\theta), i+1) = H(K[\Delta], i+1) - H(K[\Delta], i).$$

Multiplying both sides by λ^{i+1} and summing on $i \geq -1$ gives

$$L(K[\Delta]/(\theta), \lambda) = L(K[\Delta], \lambda) - \lambda L(K[\Delta], \lambda),$$

so

$$L(K[\Delta], \lambda) = \frac{L(K[\Delta]/(\theta), \lambda)}{1-\lambda}.$$

If θ is not an NZD, then we always have

$$\dim_K \theta K[\Delta]_i \leq \dim_K K[\Delta]_i,$$

and for at least one i strict inequality holds. This is easily seen to imply that strict inequality holds in equation (13). $\square$

By iteration of Lemma 13.19 we have the following result.

13.20 Theorem. Let $\theta_1, \ldots, \theta_j \in K[\Delta]_1$.Then

$$L(K[\Delta], \lambda) \leq \frac{L(K[\Delta]/(\theta_1, \ldots, \theta_j), \lambda)}{(1-\lambda)^j},$$

with equality if and only θ_i is an NZD in the ring $K[\Delta]/(\theta_1, \ldots, \theta_{i-1})$ for $1 \leq i \leq j-1$.

If $\theta_1, \ldots, \theta_j \in K[\Delta]_1$ has the property that θ_i is an NZD in the ring $K[\Delta]/(\theta_1, \ldots, \theta_{i-1})$ for $1 \leq i \leq j-1$, then we say that $\theta_1, \ldots, \theta_j$ is a *regular sequence*. The number of elements of the largest regular sequence in $K[\Delta]_1$ is called the *depth* of $K[\Delta]$, denoted depth $K[\Delta]$. Let us remark that it can be shown that all maximal regular sequences have the same number of elements, though we do not need this fact here.

It is easy to see that a regular sequence $\theta_1, \ldots, \theta_j \in K[\Delta]_1$ is algebraically independent over K (Exercise 19). In other words, there does not exist a polynomial $0 \neq P(t_1, \ldots, t_k) \in K[t_1, \ldots, t_k]$ for which $P(\theta_1, \ldots, \theta_k) = 0$ in $K[\Delta]$. Let us point out that if the sequence $\theta_1, \ldots, \theta_j \in K[\Delta]$ is algebraically independent and moreover each θ_i is an NZD in $K[\Delta]$, then these conditions are *not* sufficient for $\theta_1, \ldots, \theta_j$ to be a regular sequence. For instance, if Δ has vertices a, b, c and the single edge ab, then $a-c$ and $b-c$ are algebraically independent NZDs. However, in the ring $K[\Delta]/(a-c)$ we have $c \neq 0$ but $(b-c)c = 0$. In fact, we have $\operatorname{depth} K[\Delta] = 1$, e.g., by Exercise 22. For another example, let Δ have vertices a, b, c and edges ab, bc, and assume that $\operatorname{char}(K) \neq 2$. Now $a + b$ and $a - b$ are algebraically independent NZDs but not a regular sequence since in $K[\Delta]/(a+b)$ we have $c \neq 0$ and $c(a - b) = 0$. Unlike the previous example, this time we have $\operatorname{depth} K[\Delta] = 2$. A regular sequence of length two is given by, for instance, $a - c, b$.

Suppose that $\theta_1, \ldots, \theta_d \in K[\Delta]_1$ is a regular sequence, where as usual $d = \dim K[\Delta] = \dim \Delta + 1$. Let $R = K[\Delta]/(\theta_1, \ldots, \theta_d)$. Thus R inherits a grading $R = R_0 \oplus R_1 \oplus \cdots$ from $K[\Delta]$. By Theorem 13.18 and the definition of regular sequence we have

$$L(R, \lambda) = h_0 + h_1\lambda + \cdots + h_d\lambda^d, \tag{14}$$

a polynomial in λ. Hence R is a finite-dimensional vector space, and $\dim_K R = \sum h_i = f_{d-1}$. Clearly R_1 cannot contain an NZD ψ, since e.g. if u is a nonzero element of R of maximal degree (which must exist since $\dim_K R < \infty$) then $\psi u = 0$. Hence $\operatorname{depth} K[\Delta] = d = \dim K[\Delta]$, the maximum possible. This motivates the following key definition.

13.21 Definition. Assume that K is an infinite field. We say that the simplicial complex Δ is *Cohen-Macaulay* (with respect to the field K) and that the ring $K[\Delta]$ is a *Cohen-Macaulay ring* if $\dim K[\Delta] = \operatorname{depth} K[\Delta]$.

NOTE. Note that the above definition assumes that K is infinite. There is a more algebraic definition of Cohen-Macaulay that coincides with our definition when K is infinite but not always when K is finite. For our purposes it doesn't hurt to assume that K is infinite.

It follows from equation (14) that a Cohen-Macaulay simplicial complex Δ satisfies $h_i(\Delta) \geq 0$. However, two basic problems remain, as follows.

- What simplicial complexes are Cohen-Macaulay?
- What more can be said about the h-vector (or f-vector) of a Cohen-Macaulay simplicial complex?

The answer to the first question is beyond the scope of this chapter, but for the benefit of readers with some knowledge of algebraic topology we discuss the answer in Remark 13.26. What we will prove is that shellable simplicial complexes are indeed Cohen-Macaulay. Regarding the second question, we will obtain a complete characterization of the h-vector of a Cohen-Macaulay simplicial complex, which will also characterize h-vectors of shellable simplicial complexes. This characterization is a multiset analogue of the Kruskal-Katona theorem (Theorem 13.6).

Let us first consider the second question. A *multicomplex* Γ on a set V is a multiset analogue of a simplicial complex whose vertex set is contained in V. More precisely, Γ is a collection of multisets (sets with repeated elements, as discussed on page 1 of *Algebraic Combinatorics*), such that every element of Γ is contained in V, and if $M \in \Gamma$ and $N \subseteq M$, then $N \in \Gamma$. We will assume from now on that the underlying set V is *finite*. For example (writing 112 for $\{1, 1, 2\}$, etc.), $\Gamma = \{\emptyset, 1, 2, 3, 11, 12, 112, 1112\}$ is *not* a multicomplex, since $1112 \in \Gamma$ and $111 \subseteq 1112$, but $111 \notin \Gamma$.

If Γ is a multicomplex with e_i elements of size i, then we call the sequence $e(\Gamma) = (e_0, e_1, \dots)$ the *e-vector* of Γ. Any integer vector $(e_0, e_1, \dots)$ which is the e-vector of some multicomplex is called an *e-vector*. Our e-vectors are also called *M-vectors* after F. S. Macaulay and *O-sequences*, where O stands for "order ideal of monomials," defined below.

Note on terminology. It might seem more natural to let f_i be the number of elements of Γ of size $i + 1$, and define $(f_0, f_1, \dots)$ to be the f-vector of Γ in complete analogy with simplicial complexes. Historically, the indexing of f-vectors is explained by f_i being the number of faces of *dimension* (rather than cardinality) i. For multicomplexes, we have no need for the concept of the dimension of a face F (and if we did, the "best" definition would be that $\dim F$ is one less than the number of *distinct* elements of F). Counting elements of multicomplexes by their cardinality is more natural for almost all purposes. In the literature our e_i is often replaced with h_i, and our e-vector is called an h-vector. This is because e-vectors of multicomplexes do sometimes coincide with h-vectors of simplicial complexes (e.g., Theorem 13.25). Moreover, e-vectors of multicomplexes coincide with the sequence of Hilbert function values of standard graded K-algebras (not defined here, though $K[\Delta]$ and its quotients considered here are special cases), so one can think that h stands for "Hilbert." To avoid any possible confusion we will use the new notation e_i and terminology e-vector.

We now discuss a "multiplicative equivalent" of multicomplexes. If u and v are monomials in the variables $x_1, \ldots, x_n$, we say that u *divides* v (written $u \mid v$) if there is a monomial w for which $uw = v$. Equivalently, if $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$, then $u \mid v$ if and only if $a_i \leq b_i$ for all i. An *order ideal of monomials* in the variables $x_1, \ldots, x_n$ is a collection $\mathcal{O}$ of monomials in these variables such that if $v \in \mathcal{O}$ and $u \mid v$, then $u \in \mathcal{O}$. Equivalently, associate with the monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ the multiset $M_u = \{1^{a_1}, \ldots, n^{a_n}\}$ (i.e., i has multiplicity a_i). Then a set $\mathcal{M}$ of monomials is an order ideal of monomials if and only if the collection $\{M_u \ : \ u \in \mathcal{M}\}$ is a multicomplex.

While we need the next result only for quotients of face rings by a regular sequence, it involves no extra work to prove it in much greater generality. For this purpose, we define a *homogeneous ideal* of the polynomial ring $K[x_1, \ldots, x_n]$ to be an ideal I generated by homogeneous polynomials.

13.22 Theorem. Let I be a homogeneous ideal of $K[x_1, \ldots, x_n]$, and let $P = K[x_1, \ldots, x_n]/I$. Then P has a K-basis that is an order ideal of monomials in the variables $x_1, \ldots, x_n$.

Proof. We define *reverse lex order* on monomials of degree m as follows: define

$$x_1^{a_1} \cdots x_n^{a_n} \overset{R}{<} x_1^{b_1} \cdots x_n^{b_n},$$

where $\sum a_i = \sum b_i$, if

$$(1^{a_1}, \ldots, n^{a_n}) \overset{R}{<} (1^{b_1}, \ldots, n^{b_n}),$$

where i^j denotes a sequence i's of length j. For instance, the reverse lex order on monomials of degree three in the variables x_1, x_2, x_3 is

$$x_1^3 \overset{R}{<} x_1^2 x_2 \overset{R}{<} x_1 x_2^2 \overset{R}{<} x_2^3 \overset{R}{<} x_1^2 x_3 \overset{R}{<} x_1 x_2 x_3 \overset{R}{<} x_2^2 x_3 \overset{R}{<} x_1 x_3^2 \overset{R}{<} x_2 x_3^2 \overset{R}{<} x_3^3.$$

For each $m \geq 0$ let P_m denote the span (over the field K) of the homogeneous polynomials in P of degree m. Because I is generated by homogeneous polynomials we have the vector space direct sum

$$P = P_0 \oplus P_1 \oplus P_2 \oplus \cdots,$$

a direct generalization of equation (12).

For each degree m, let B_m be the least K-basis for P_m in reverse lex order. In other words, first choose the least monomial u_1 of degree m in rlex order that is nonzero in P. Then choose the least monomial u_2 of degree m in rlex order such that $\{u_1, u_2\}$ are linearly independent, etc. We eventually obtain a K-basis for P_m by this process since every linearly independent subset of a vector space can be extended to a basis.

We claim that the set $B_0 \cup B_1 \cup B_2 \cup \cdots$ is a basis for P which is an order ideal of monomials. Suppose not. Let $v \in B_j$, $u \,|\, v$, but $u \notin B_i$ where $i = \deg u$. Then u is a linear combination of monomials $w \overset{R}{<} u$, say

$$u = \sum_{w \overset{R}{<} u} \alpha_w w.$$

Multiply both sides by v/u. It is easy to see that if $w \overset{R}{<} u$ then $wv/u \overset{R}{<} v$. Thus we have expressed v as a linear combination of monomials $wv/u \overset{R}{<} v$, contradicting $v \in B_j$. $\square$

13.23 Corollary. Let Δ be a Cohen-Macaulay simplicial complex. Then the h-vector of Δ is an e-vector.

Proof. Let $\theta_1, \ldots, \theta_d$ be a regular sequence in $K[\Delta]_1$, and let

$$R = R_0 \oplus R_1 \oplus \cdots = K[\Delta]/(\theta_1, \ldots, \theta_d).$$

According to the definition of a Cohen-Macaulay simplicial complex and equation (14) we have $h_i(\Delta) = \dim_K R_i$. Thus if D_i is any K-basis for R_i, then $\#D_i = h_i$. By Theorem 13.22 there is a K-basis B_i for each i such that $B_0 \cup B_1 \cup \cdots \cup B_d$ is an order ideal Γ of monomials. Thus if Γ has e-vector $(e_0, e_1, \ldots)$ then $e_i = \#B_i = h_i$, and the proof follows. $\square$

The next step is to find some simplicial complexes to which we can apply Corollary 13.23. The next result is the primary algebraic result of this chapter. Recall the notation $x_S = \prod_{x_i \in S} x_i$ of equation (8).

13.24 Theorem. If Δ is a shellable simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$ then the face ring $K[\Delta]$ is

Cohen-Macaulay for any infinite field K. Moreover, if $F_1, \ldots, F_t$ is a shelling of Δ with restrictions $G_1, \ldots, G_t$, then $x_{G_1}, \ldots, x_{G_t}$ is a K-basis for $R = K[\Delta]/(\theta_1, \ldots, \theta_d)$ for any regular sequence $\theta_1, \ldots, \theta_d \in K[\Delta]_1$.

Proof. Let $B = \{x_{G_1}, \ldots, x_{G_t}\}$. Let $\theta_1, \ldots, \theta_d \in K[\Delta]_1$ satisfy the following property.

(P) The restriction of $\theta_1, \ldots, \theta_d$ to any facet (or face) F spans the K-vector space KF with basis F. In other words, if we define

$$\psi_i = \theta_i|_{x_j=0 \text{ if } x_j \notin F},$$

then $\psi_1, \ldots, \psi_d$ span KF.

Note that K being infinite guarantees that there is enough "room" to find such $\theta_1, \ldots, \theta_d$. This is the reason why we require K to be infinite.

Now let $R = K[\Delta]/(\theta_1, \ldots, \theta_d)$. By Theorems 13.15 and 13.20 (in the case $j = d$) it follows that if B spans R (as a vector space over K) then $\theta_1, \ldots, \theta_d$ is regular, and B is a K-basis for R. Thus we need to show that B spans R.

The proof is by induction on t.

First assume that $t = 1$. Then Δ is just a simplex and $K[\Delta] = K[x_1, \ldots, x_d]$. Moreover, $x_1, \ldots, x_d$ is a regular sequence and $K[\Delta]/(x_1, \ldots, x_d) = K$. The Hilbert series of the field K is just 1. Finally, if F is the unique facet of Δ then F (regarded as a one-term sequence) is a shelling of Δ with $G_1 = \emptyset$. Since $x_\emptyset = 1$ is a basis for K, the theorem is true for $t = 1$.

Now assume the theorem for $t - 1$, and let $F_1, \ldots, F_t$ be a shelling of Δ.

Claim. $x_i x_{G_t} = 0$ in R for all $1 \leq i \leq n$.

Case 1. Suppose that $x_i \notin F_t$. By definition of shelling the new faces F obtained by adjoining F_t to the shelling are given by $G_t \subseteq F \subseteq F_t$. Thus $\{x_i\} \cup G_t$ cannot be a new face, so $\{x_i\} \cup G_t \notin \Delta$. Hence $x_i x_{G_t} = 0$ in $K[\Delta]$, so also in R.

Case 2. Suppose that $x_i \in F_t$. Set

$$K[F_t] = K[\Delta]/(x_j \,:\, x_j \notin F_t) = K[x_j \,:\, x_j \in F_t],$$

a polynomial ring in the vertices of F_t. By property (P), the restrictions $\psi_1, \ldots, \psi_d$ of $\theta_1, \ldots, \theta_d$ to F span the space KF. Hence there exists a linear combination of $\theta_1, \ldots, \theta_d$ of the form

$$\eta = x_i + \sum_{x_j \notin F_t} \alpha_j x_j, \quad \alpha_j \in K.$$

Then in the ring R we have

$$\begin{aligned} x_i x_{G_t} &= (x_i - \eta) x_{G_t} \quad (\text{since } \eta = 0 \text{ in } R) \\ &= -\left(\sum_{x_j \notin F_t} \alpha_j x_j \right) x_{G_t} \\ &= 0 \quad (\text{by Case 1}). \end{aligned}$$

This completes the proof of the claim.

Now let $R' = R/(x_{G_t})$ and $\Delta_{t-1} = \langle F_1, \dots, F_{t-1} \rangle$. By definition of G_t we have

$$K[\Delta_{t-1}] = K[\Delta]/(x_{G_t}).$$

Condition (P) still holds for $K[\Delta_{t-1}]$ (since the facets of Δ_{t-1} are also facets of Δ). Moreover,

$$R' = K[\Delta_{t-1}]/(\theta_1, \dots, \theta_d).$$

By the induction hypothesis, $x_{G_1}, \dots, x_{G_{t-1}}$ span R'. By the claim, the ideal (x_{G_t}) of R is a vector space of dimension at most one. Hence $x_{G_1}, \dots, x_{G_t}$ span R, and the proof follows for any (regular) sequence $\theta_1, \dots, \theta_d$ satisfying (P).

It remains to show that *every* regular sequence $\theta_1, \dots, \theta_d \in K[\Delta]_1$ satisfies (P). This result is an easy exercise; a somewhat more general result is given by Exercise 21. $\square$

Note. Note the structure of the previous proof. We pick $\theta_1, \dots, \theta_d \in K[\Delta]_1$ satisfying Property (P). Let $F_1, \dots, F_t$ be a shelling of Δ, and set $R = K[\Delta]/(\theta_1, \dots, \theta_d)$. As we successively quotient R by the monomials $x_{G_1}, \dots, x_{G_t}$, the vector space dimension drops by at most one, and we end up with the ring 0. Hence $\dim_K R \le t = f_{d-1}(\Delta)$. On the other hand, by Theorems 13.18 and 13.20 we have $\dim_K R \ge \sum h_i = t$. Hence $x_{G_1}, \dots, x_{G_t}$ must be a K-basis for R.

We are finally ready for the main theorem of this chapter.

13.25 Theorem. Let $\boldsymbol{h} = (h_0, h_1, \dots, h_d)$ be a sequence of integers. The following three conditions are equivalent.

(a) There exists a $(d-1)$-dimensional Cohen-Macaulay simplicial complex (over any infinite field) Δ with $h(\Delta) = \boldsymbol{h}$.

(b) There exists a $(d-1)$-dimensional shellable simplicial complex Δ with $h(\Delta) = \boldsymbol{h}$.

(c) The sequence $\boldsymbol{h}$ is an e-vector.

Proof. (b)⇒(a) Immediate from Theorem 13.24.

(a)⇒(c) This is Corollary 13.23.

(c)⇒(b) Given the e-vector $\boldsymbol{h}$, we need to construct a shellable simplicial complex Δ whose h-vector is $\boldsymbol{h}$. We will identify a (finite) multiset M of positive integers with the increasing sequence of its elements. Given $0 \leq i \leq d$, let $\alpha_1, \alpha_2, \ldots, \alpha_{h_i}$ be the first h_i terms of the rlex order on i-element multisets of positive integers. For instance, when $i = 3$ and $h_3 = 8$, we have

$$(\alpha_1, \ldots, \alpha_8) = (111, 112, 122, 222, 113, 123, 223, 133). \tag{15}$$

If $\alpha_j = a_1 a_2 \cdots a_i$, define

$$\beta_j = 1, 2, 3, \ldots, d-i, a_1+d-i+1, a_2+d-i+2, \ldots, a_i+d. \tag{16}$$

For the example of equation (15) and $d = 5$ we have

$$(\beta_1, \ldots, \beta_8) = (12456, 12457, 12467, 12567, 12458, 12468, 12568, 12478).$$

Now let σ be the concatenation $\beta_1, \beta_2, \ldots, \beta_d$ of the sequences $\beta_1, \ldots, \beta_d$. For instance, if $\boldsymbol{h} = (1, 4, 2, 1)$ (so $d = 3$) then we get (where we separate the different β_j's with semicolons, and we write in boldface the terms $a_1+d-i+1, a_2+d-i+2, \ldots, a_i+d$ of each β_j)

$$\sigma = (123; 12\mathbf{4}, 12\mathbf{5}, 12\mathbf{6}, 12\mathbf{7}; 1\mathbf{34}, 1\mathbf{35}; \mathbf{234}). \tag{17}$$

We leave as an exercise (Exercise 25) to show that σ is a shelling of a $(d-1)$-dimensional simplicial complex Δ on the vertex set $\{1, 2, \ldots, h_1+d\}$. Moreover, if we write $\sigma = (F_1, \ldots, F_m)$ (where $m = \sum h_i = f_{d-1}(\Delta)$) and if F_k is given by the sequence on the right-hand side of equation (16), then the restriction G_k is given by $G_k = \{a_1 + d - i + 1, a_2 + d - i + 2, \ldots, a_i + d\}$. This being the case, exactly h_i restrictions G_k have i elements, so indeed $h(\Delta) = \boldsymbol{h}$. $\square$

As an example, the sequence σ of equation (17) is a shelling $(F_1, \ldots, F_8)$ of a simplicial complex Δ with vertices $1, \ldots, 7$. The restrictions $G_1, \ldots, G_8$ are given by $\emptyset, 4, 5, 6, 7, 34, 35, 234$ (the elements in boldface).

13.26 Remark. We mentioned earlier that the complete characterization of Cohen-Macaulay simplicial complexes is beyond the scope of this chapter. For readers familiar with some algebraic topology (only the rudiments of simplicial homology are needed), we will state without proof the theorem of Gerald Reisner that provides this characterization. For this purpose, if $F \in \Delta$ then define the *link* of F, denoted $\mathrm{lk}_\Delta(F)$, by

$$\mathrm{lk}_\Delta(F) = \{G \in \Delta \,:\, F \cap G = \emptyset,\ F \cup G \in \Delta\}.$$

It is clear that $\mathrm{lk}_\Delta(F)$ is a subcomplex of Δ. In particular, $\mathrm{lk}_\Delta(\emptyset) = \Delta$. For any simplicial complex Γ we write $\widetilde{H}_i(\Delta; K)$ for the ith reduced homology group of Δ over the field K.

13.27 Theorem. Let K be an infinite field. The following two conditions on a simplicial complex Δ are equivalent.

- Δ is Cohen-Macaulay with respect to K.

- For every $F \in \Delta$ (including $F = \emptyset$), we have $\widetilde{H}_i(\mathrm{lk}_\Delta(F); K) = 0$ for all $i \neq \dim \mathrm{lk}_\Delta(F)$.

It can be shown from Reisner's theorem that for fixed K (or actually, for fixed characteristic of K), Cohen-Macaulayness is a topological property, i.e., it depends only on the geometric realization of Δ (as a topological space). For instance, all triangulations of spheres and balls (of any dimension) are Cohen-Macaulay over any (infinite) field. A triangulation of the real projective plane is Cohen-Macaulay with respect to K if and only if $\mathrm{char}(K) \neq 2$.

Theorem 13.25 gives an elegant characterization of h-vectors (and hence f-vectors) of shellable simplicial complexes, but one ingredient is still missing—a "nice" characterization of e-vectors. Since an e-vector is a multiset analogue of an f-vector of a simplicial complex, it is not unreasonable to expect a characterization of e-vectors similar to the Kruskal-Katona theorem (Theorem 13.6) for ordinary f-vectors. We conclude this chapter by discussing such a characterization.

Given positive integers n and j, let

$$n = \binom{n_j}{j} + \binom{n_{j-1}}{j-1} + \cdots + \binom{n_1}{1}$$

be the j-binomial expansion of n (equation (1)). Now define

$$n^{\langle j \rangle} = \binom{n_j + 1}{j+1} + \binom{n_{j-1} + 1}{j} + \cdots + \binom{n_1 + 1}{2}.$$

Thus instead of adding 1 to the bottom of each binomial coefficient as we did when we defined $n^{(j)}$, now we add 1 to the bottom *and* top. We now have the following exact analogue of the Kruskal-Katona theorem.

13.28 Theorem. A vector $(e_0, e_1, \ldots, e_d) \in \mathbb{P}^{d+1}$ is an e-vector if and only if $e_0 = 1$ and

$$e_{i+1} \leq e_i^{\langle i \rangle}, \quad 0 \leq i \leq d-1. \tag{18}$$

The proof is analogous to that of the Kruskal-Katona theorem. Namely, we identify a finite multiset M on $\mathbb{N}$ with the increasing sequence of its elements. For instance, the multiset $\{0, 0, 2, 3, 3, 3\}$ becomes the sequence 002333, and the sequence of 3-element multisets on $\mathbb{N}$ in rlex order begins

000 001 011 111 002 012 112 022 122 222 $003 \cdots$.

It can easily be checked that if $a_1 a_2 \cdots a_j$ is the nth term (beginning with term 0) in the rlex ordering of j-element multisets on $\mathbb{N}$, then $a_1, a_2+1, a_3+2, \ldots, a_j+j-1$ is the nth term in the rlex ordering of j-element *subsets* of $\mathbb{N}$. Hence Theorem 13.7 applies equally well to multisets on $\mathbb{N}$.

We next have the following multiset analogue of Theorem 13.8. The proof is completely analogous to that of Theorem 13.8.

13.29 Theorem. Given $\boldsymbol{e} = (e_0, e_1, \ldots, e_d) \in \mathbb{P}^d$ with $e_0 = 1$, let $\Omega_{\boldsymbol{e}}$ consist of the union over all $i \geq 1$, together with $\emptyset$, of the first e_i of the i-element multisets on $\mathbb{N}$ in rlex order. The set $\Omega_{\boldsymbol{e}}$ is a multicomplex if and only if $e_{i+1} \leq e_i^{\langle i \rangle}$ for $1 \leq i \leq d-1$.

Theorem 13.29 proves the "if" direction of Theorem 13.28.

The proof of the "only if" direction is similar to that of the Kruskal-Katona theorem. A multicomplex as in Theorem 13.29 is called *compressed*. Given any multicomplex Γ, we transform it by a sequence of simple operations into a compressed multicomplex, at all steps preserving the e-vector. We omit the details, which are somewhat more complicated than in the simplicial complex case.

13.30 Example. Is $(e_0, e_1, e_2, e_3) = (1, 4, 5, 7)$ an e-vector? The first e_i multisets on $\mathbb{N}$ in rlex order for $1 \leq i \leq 3$ are given by

$$\begin{array}{c} 0\ \ 1\ \ 2\ \ 3 \\ 00\ \ 01\ \ 11\ \ 02\ \ 12 \\ 000\ \ 001\ \ 011\ \ 111\ \ 002\ \ 012\ \ 112 \end{array}.$$

These multisets (together with $\emptyset$) form a multicomplex, so $(1, 4, 5, 7)$ is an e-vector. On the other hand, $(1, 4, 5, 8)$ is *not* an e-vector. We need to add the multiset 022, but 22 does not appear. We can also check this by writing $5 = \binom{3}{2} + \binom{2}{1}$. Then $5^{\langle 2 \rangle} = \binom{4}{3} + \binom{3}{2} = 7$, so if in an e-vector we have $e_2 = 5$ then $e_3 \leq 7$.

Notes for Chapter 13.

The embedding theorem of Menger discussed in Example 13.3 appears (in much greater generality) in Menger [13]. The statement that the simplicial complex whose facets are all $(d+1)$-element subsets of a $(2d+3)$-element set cannot be realized in $\mathbb{R}^{2d}$ is due to A. Flores and E. R. van Kampen. For a modern treatment see Matoušek [12].

The Kruskal-Katona theorem (Theorem 13.6) was first stated by M.-P. Schützenberger in a rather obscure journal [15]. The first published proofs were by J. Kruskal [9] and later independently by G. O. H. Katona [7]. A nice survey of this area is given by Greene and Kleitman, [6], including a good presentation of a proof of the Kruskal-Katona theorem due to Clements and Lindström [3].

The first indication of a connection between commutative algebra and combinatorial properties of simplicial complexes appears in a paper of Melvin Hochster [5]. The face ring of a simplicial complex first appeared in the Ph.D. thesis of Gerald Reisner (published version in [14]), which was supervised by Hochster, and independently in two papers of Stanley [17][18]. For an exposition of the connections between combinatorics and commutative algebra see Stanley [19].

The concept of shelling goes back to nineteenth century geometers, but perhaps the first substantial result on shellings is due to Bruggesser and Mani [1]. The characterization of h-vectors of shellable simplicial complexes (Theorem 13.25) is a special case of a result of Stanley [17]. Our proof here is based on that of B. Kind and P. Kleinschmidt [8].

The characterization of e-vectors (Theorem 13.28) is due to F. S. Macaulay [11], who gave a very complicated proof as part of his characterization of Hilbert series of graded algebras. It is interesting that Macaulay's theorem preceded the Kruskal-Katona theorem, though the latter is somewhat easier to prove. Simpler proofs of Macaulay's theorem were later given by by Sperner [16], Whipple [20], and Clements and Lindström [3], among others.

Cohen-Macaulay rings are named after I. Cohen [4] and F. S. Macaulay [10], who were interested in them primarily because of their connection with "unmixedness" theorems. For a modern treatment see the text of W. Bruns and J. Herzog [2].

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Exercises for Chapter 13

As in the book *Algebraic Combinatorics*, the notation (*) indicates that a hint is provided. Here the hints are at the end of this set of exercises.

1. A simplicial complex Δ has 2528 three-dimensional faces. Show that it has at most 6454 four-dimensional faces, and that this result is best possible. ("Best possible" means that there exists some Δ with $f_3 = 2528$ and $f_4 = 6454$).

2. Prove the assertion of the Note following Definition 13.10. That is, let Δ be a pure $(d-1)$-dimensional simplicial complex. Then a facet ordering $F_1, \dots F_t$ is a shelling if and only if for all $2 \leq i \leq t$, the subcomplex $\langle F_1, \dots, F_{i_1} \rangle \cap \langle F_i \rangle$ is a pure simplicial complex of dimension $d-2$.

3. (a) Find the number of shellings of a path of length n, i.e, the simplicial complex with $n+1$ vertices and n edges forming a path.

   (b) Find the number of shellings of a cycle of length n.

4. Find explicitly every simplicial complex Δ with the property that *every* ordering of its facets is a shelling.

5. Suppose that $F_1, F_2, \dots, F_t$ is shelling of a simplicial complex Δ. Is it always the case that the reverse order $F_t, F_{t-1}, \dots, F_1$ is also a shelling of Δ?

6. Show that if a simplicial complex Δ is shellable and $F \in \Delta$, then the link $\mathrm{lk}_\Delta(F)$ (as defined in Remark 13.26) is also shellable.

7. Prove the assertion of Example 13.11(d) that a simplicial complex Δ is shellable if and only if the cone $C(\Delta)$ is shellable.

8. A *matroid complex* is a simplicial complex Δ on the vertex set V such that for any $W \subseteq V$, the *restriction* Δ_W of Δ to W, i.e.,
$$\Delta_W = \{F \in \Delta \,:\, F \subseteq W\},$$
is pure.

   (a) Let $V = \{x_1, \ldots, x_n\}$ be a set of distinct nonzero vectors in some vector space over a field. Define
   $$\Delta = \{F \subseteq V \,:\, F \text{ is linearly independent}\}.$$
   Clearly Δ is a simplicial complex on V. Show that Δ is a matroid complex.

   (b) Show that a matroid complex is shellable.

9. (*) (for those who know a little topology) Let X be the topological space obtained by identifying the three sides (edges) of a solid triangle as shown in Figure 6. (The edges are identified in the direction of the arrows.) This space is called the *topological dunce hat*. Show that if Δ is a simplicial complex whose geometric realization is homeomorphic to Δ then Δ is not shellable.

10. (a) Show that every triangulation Δ of a two-dimensional ball X (i.e., the geometric realization of Δ is homeomorphic to X) is shellable.

    (b) (very difficult) Find a triangulation of a three-dimensional ball that is not shellable.

(c) (even more difficult) Find a triangulation of a three-dimensional sphere that is not shellable.

11. (*) A *partial shelling* of a pure $(d-1)$-dimensional complex is a sequence $F_1, \ldots, F_r$ of some *subset* of the facets such that this sequence is a shelling order for the simplicial complex $\langle F_1, \ldots, F_r \rangle$ which they generate. Clearly if $F_1, \ldots, F_t$ is a shelling of Δ, then $F_1, \ldots, F_r$ is a partial shelling for all $1 \leq r \leq t$. Give an example of a shellable simplicial complex that has a partial shelling that cannot be extended to a shelling.

12. Let $(f_0, f_1, \ldots, f_{d-1})$ be the f-vector of a $(d-1)$-dimensional simplicial complex Δ. We will illustrate a certain procedure with the example $(6, 12, 8)$ (the f-vector of an octahedron). Write down the numbers $f_0, f_1, \ldots, f_{d-1}$ on a diagonal, and put 1 to the left of f_0:

$$\begin{array}{cccc} 1 & 6 & & \\ & & 12 & \\ & & & 8 \end{array}$$

Think of the 1 as being preceded by a string of 0's. Turn the array into a difference table by writing below each pair

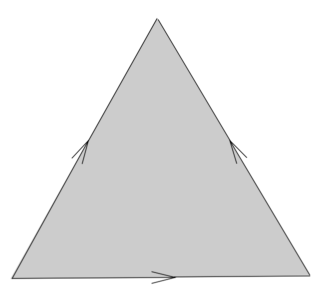

Figure 6: The topological dunce hat

of consecutive numbers their difference:

$$\begin{array}{ccccccc} & & 1 & & 6 & & \\ & 1 & & 5 & & 12 & \\ 1 & & 4 & & 7 & & 8 \end{array}$$

Now write down one further row of differences:

$$\begin{array}{cccccccc} & & & 1 & & 6 & & \\ & & 1 & & 5 & & 12 & \\ & 1 & & 4 & & 7 & & 8 \\ \hline 1 & & 3 & & 3 & & 1 & \end{array}$$

Show that this bottom row is the h-vector of Δ.

13. Find the f-vector and h-vector of the simplicial complex whose geometric realization is the boundary of an icosahedron.

14. Let Δ_d be the simplicial complex on the vertex set $V = \{x_1, \dots, x_d,\ y_1, \dots, y_d\}$ whose faces are those subsets of V that do not contain both x_i and y_i, for any $1 \le i \le d$. Compute the h-vector of Δ_d.

    Cultural note. Let δ_i be the ith unit coordinate vector in $\mathbb{R}^d$. A geometric realization of Δ_d consists of the boundary of the convex hull $\mathcal{C}_d$ of the vectors $\pm\delta_i$, $1 \le i \le d$. The polytope $\mathcal{C}$ is called the d-dimensional *cross-polytope* and is a d-dimensional generalization of an octahedron, the case $d = 3$.

15. Give an example of two simplicial complexes Δ_1 and Δ_2 such that the geometric realizations of Δ_1 and Δ_2 are homeomorphic, the h-vector of Δ_1 is nonnegative, and some $h_i(\Delta_2) < 0$. What is the smallest possible dimension of Δ_1 and Δ_2?

16. (difficult from first principles) (*) Let Δ be a triangulation of a $(d-1)$-dimensional sphere, and let $h(\Delta) = (h_0, h_1, \ldots, h_d)$. Show that $h_i = h_{d-i}$ for $0 \leq i \leq d$. This result is called the *Dehn-Sommerville equations* for spheres.

17. Let $\boldsymbol{h} = (h_0, h_1, \ldots, h_d)$ and $\boldsymbol{k} = (k_0, k_1, \ldots, k_d)$ be e-vectors. Define

$$\begin{aligned} \boldsymbol{h} \wedge \boldsymbol{k} &= (\min\{h_0, k_0\}, \min\{h_1, k_1\}, \ldots, \min\{h_d, k_d\}) \\ \boldsymbol{h} \vee \boldsymbol{k} &= (\max\{h_0, k_0\}, \max\{h_1, k_1\}, \ldots, \max\{h_d, k_d\}). \end{aligned}$$

Show that $\boldsymbol{h} \wedge \boldsymbol{k}$ and $\boldsymbol{h} \vee \boldsymbol{k}$ are e-vectors.

18. (*) Suppose that Γ and Δ are simplicial complexes whose face rings $K[\Gamma]$ and $K[\Delta]$ are isomorphic as K-algebras (or even as rings). Show that Γ and Δ are isomorphic.

19. Show that a regular sequence $\theta_1, \ldots, \theta_j \in K[\Delta]_1$ is algebraically independent over K.

20. (a) Let $\theta_1, \ldots, \theta_j \in K[\Delta]_1$ be a regular sequence. Show that any permutation of this sequence is also a regular sequence.

    (b) Show that each θ_i is an NZD in $K[\Delta]$.

21. Let Δ be any $(d-1)$-dimensional simplicial complex, and let $\theta_1, \ldots, \theta_d \in K[\Delta]_1$. Show that the quotient ring $R = K[\Delta]/(\theta_1, \ldots, \theta_d)$ is a finite-dimensional vector space over K if and only if $\theta_1, \ldots, \theta_d$ satisfy Property (P).

22. Show that the face ring $K[\Delta]$ of a simplicial complex Δ has depth one if and only if Δ is disconnected. Deduce that a disconnected simplicial complex of dimension at least one is not Cohen-Macaulay.

23. Let Γ and Δ be simplicial complexes on disjoint vertex sets V and W, respectively. Define the *join* $\Gamma * \Delta$ to be the simplicial complex on the vertex set $V \cup W$ with faces $F \cup G$, where $F \in \Gamma$ and $G \in \Delta$. (If Γ consists of a single point, then $\Gamma * \Delta$ is the *cone* over Δ. If Γ consists of two disjoint points, then $\Gamma * \Delta$ is the *suspension* of Δ.)

    (a) Compute the h-vector $h(\Gamma * \Delta)$ in terms of $h(\Gamma)$ and $h(\Delta)$.

    (b) Show that if Γ and Δ are Cohen-Macaulay, then so is $\Gamma * \Delta$.

    (c) Generalizing Exercise 7, show that if Γ and Δ are shellable, then so is $\Gamma * \Delta$.

24. Let Δ be a one-dimensional simplicial complex. Show that the following three conditions are equivalent: (a) Δ is Cohen-Macaulay, (b) Δ is shellable, and (c) Δ is connected.

25. Complete the proof of Theorem 13.25 by showing that the sequence σ is a shelling of Δ with the stated restrictions G_k.

26. (*) Let Δ be a four-dimensional shellable simplicial complex with $f_0 = 13$, $f_1 = 50$, and $f_2 = 129$. What is the most number of facets that Δ can have?

27. (*) Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex with h-vector $(h_0, h_1, \ldots, h_d)$. Let Δ' be a $(d-1)$-dimensional Cohen-Macaulay subcomplex of Δ with h-vector $(h'_0, h'_1, \ldots, h'_d)$. Show that $h'_i \leq h_i$ for all $0 \leq i \leq d$.

28. (*) Let Δ be a $(d-1)$-dimensional simplicial complex on the vertex set V. We say that Δ is *balanced* if we can write V as a disjoint union $V = V_1 \cup V_2 \cup \cdots \cup V_d$ such that for every $F \in \Delta$ and every $1 \le i \le d$ we have $\#(F \cap V_i) \le 1$. In particular, if Δ is pure then always $\#(F \cap V_i) = 1$. (Sometimes Δ is required to be pure in the definition of balanced.) Suppose that $(h_0, h_1, \ldots, h_d)$ is the h-vector of a Cohen-Macaulay balanced simplicial complex Δ. Show that $(h_1, h_2, \ldots, h_d)$ is the f-vector of a balanced simplicial complex. You may assume the following result: let Δ be a Cohen-Macaulay simplicial complex of dimension $d-1$. If $\theta_1, \ldots, \theta_d \in K[\Delta]_1$ satisfies (P) if and only if $\theta_1, \ldots, \theta_d$ is a regular sequence.

Hints for Some Exercises

13.9. Compute the reduced Euler characteristic of X and consider the last step of a possible shelling.

13.11. There are two examples with f-vector $(6, 14, 9)$ and none smaller.

13.16. The following property of triangulations Δ of spheres is sufficient for solving this exercise. For every $F \in \Delta$ we have

$$\tilde{\chi}(\mathrm{lk}_\Delta(F)) = (-1)^{\dim \mathrm{lk}_\Delta(F)}.$$

Here $\tilde{\chi}$ denotes reduced Euler characteristic, and lk denotes link, as defined in Remark 13.26.

13.18. One approach is to consider minimal ideals I of $K[\Gamma]$, say, for which $K[\Gamma]/I$ is a polynomial ring, i.e., isomorphic to $K[y_1, \ldots, y_r]$ for indeterminates $y_1, \ldots, y_r$.

13.26. *Answer:* 325.

13.27. Let I be the ideal of $K[\Delta]$ generated by all monomials x^F, where $F \notin \Delta'$. Clearly $K[\Delta']$ is isomorphic to $K[\Delta]/I$ (as a K-algebra). Let $\theta_1, \ldots, \theta_d \in K[\Delta]_1$ satisfy Property (P), and consider the natural map

$$f\colon K[\Delta]/(\theta_1, \ldots, \theta_d) \to (K[\Delta]/I)/(\theta_1, \ldots, \theta_d).$$

13.28. Consider $\theta_i \in K[\Delta]_1$ defined by $\theta_i = \sum_{x_j \in V_i} x_j$. Also consider the product $x_j\theta_i$ in the ring $K[\Delta]/(\theta_1, \ldots, \theta_d)$ for all $1 \leq i \leq d$ and $x_j \in V_i$.

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