A differential Chevalley theorem

Eric Rosen

MIT

AMS Sectional Meeting
Wesleyan University
October 12, 2008

Tarski's version of Chevalley's theorem

Theorem (Tarski)

The theory of algebraically closed fields of a fixed characteristic has quantifier elimination.

Let $K \models ACF$. A set $X \subseteq K^n$ is an affine algebraic variety if it is the zero set of a finite set of polynomials. These are the closed sets in the Zariski topology on K^n. A constructible set is a boolean combination of closed sets.

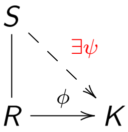
Corollary

In $K \models ACF_0$, the definable sets are the constructible sets in the Zariski topology. In particular, given a definable map $f : X \to Y$ between varieties, $f(X)$ is constructible.

Chevalley's Theorem
about extending ring homomorphisms

Theorem

Let S be a ring with no zero divisors, finitely generated over a subring R. For any nonzero $b \in S$, there is a nonzero $a \in R$, such that for any homomorphism $\phi : R \to K$, K an algebraically closed field, with $\phi(a) \neq 0$, there is a homomorphism $\psi : S \to K$ lifting ϕ, with $\psi(b) \neq 0$.



(with $\phi(a) \neq 0$ and $\psi(b) \neq 0$)

Chevalley's Theorem

about extending ring homomorphisms

Theorem

Let S be a ring with no zero divisors, finitely generated over a subring R. For any nonzero $b \in S$, there is a nonzero $a \in R$, such that for any homomorphism $\phi : R \to K$, K an algebraically closed field, with $\phi(a) \neq 0$, there is a homomorphism $\psi : S \to K$ lifting ϕ, with $\psi(b) \neq 0$.

Given a ring R, there is a Zariski topology on the affine scheme $\mathrm{Spec}(R)$, whose points are the prime ideals.

Corollary

Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then $f(X)$ is constructible.

Differential algebra

In this talk, a differential ring will be a $\mathbb{Q}$-algebra equipped with n commuting derivations, $n \geq 1$.

Example

$$(\mathbb{C}[x_1, \ldots, x_n], \partial/\partial x_1, \ldots, \partial/\partial x_n)$$

It is natural to ask what theorems from algebraic geometry have analogs in differential algebraic geometry.

Examples

- Differential Nullstellensatz
- Kolchin's theory of differential algebraic groups

Quantifier elimination for *DCF*

Theorem (A. Robinson, Blum, McGrail, Yaffe)

The theory DCF_n of differentially closed fields with n commuting derivations has quantifier elimination.

The following is a convenient restatement.

Corollary (Seidenberg elimination theorem)

Let $K \models DCF_n$, and let $f_i(\overline{x}, y)$, $i \leq m$, $g(\overline{x}, y)$ be differential polynomials in $\mathbb{Z}\{\overline{x}, y\}$. Then the set

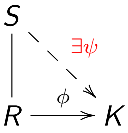
$$\{\overline{a} \in K : \exists y (\bigwedge f_i(\overline{a}, y) = 0 \wedge g(\overline{a}, y) \neq 0)\}$$

can be defined by a boolean combination of differential polynomials equations in $\mathbb{Z}\{\overline{x}\}$.

Differential Chevalley theorem

Theorem

Let S be a differential algebra with no zero divisors, R a differential subalgebra of S over which S is differentially finitely generated, and K a differentially closed field. Then for any nonzero $b \in S$, there is a nonzero $a \in R$ such that any homomorphism $\phi : R \to K$ with $\phi(a) \neq 0$ extends to a homomorphism $\psi : S \to K$ with $\psi(b) \neq 0$.



(with $\phi(a) \neq 0$ and $\psi(b) \neq 0$)

- (This generalizes a theorem of V. Kac, who proved the case with a single derivation.)

Differential Chevalley theorem

Proof Idea

1. The interesting case is $S = R\{t\}$, t differentially algebraic over R, and $b = g(t)$.
2. Embed S in a differentially closed field L.
3. Let $\mathcal{I} \subseteq R\{x\}$ be the ideal of differential polynomials $f(x)$ such that $f(t) = 0$. Let $A \subseteq \mathcal{I}$ be a finite set of 'generators'.
4. Let $\overline{c}$ be the set of all parameters in $A \cup \{g(x)\}$. So $\overline{c}$ *encodes the description of* b *in* R.
5. Using Seidenberg, can define $a \in R$ from the tuple $\overline{c}$.

Differential Chevalley theorem

Proof Idea

1. The interesting case is $S = R\{t\}$, t differentially algebraic over R, and $b = g(t)$.
2. Embed S in a differentially closed field L.
3. Let $\mathcal{I} \subseteq R\{x\}$ be the ideal of differential polynomials $f(x)$ such that $f(t) = 0$. Let $A \subseteq \mathcal{I}$ be a finite set of 'generators'.
4. Let $\overline{c}$ be the set of all parameters in $A \cup \{g(x)\}$. So $\overline{c}$ *encodes the description of b in R*.
5. Using Seidenberg, can define $a \in R$ from the tuple $\overline{c}$.

Complexity of commuting derivations

For $n > 1$, the structure of prime ideals in $K\{x\}$ is significantly more complicated than for $n = 1$.

A geometric differential Chevalley theorem

As a corollary, we generalize a theorem of Buium, who proved the case of a single derivation.

Theorem

Let $f : X \to Y$ be a morphism of differential finite type between affine differential schemes. Suppose that Y is noetherian. Then $f(X)$ is a constructible subset of Y.

New proof of a theorem of Kolchin

An immediate corollary is Kolchin's lifting theorem, which generalized earlier results of Ritt, Seidenberg, and Rosenfeld.

Corollary (Kolchin)

Let S be a differential ring with no zero divisors and R a subring over which S is differentially finitely generated. Given a nonzero $b \in S$, there is an $a \in R$ such that for every differential prime ideal $\mathcal{P} \subseteq R$ with $a \notin \mathcal{P}$, there is a differential prime ideal $\mathcal{Q} \subseteq S$, not containing b, with $\mathcal{Q} \cap R = \mathcal{P}$.

$$\begin{array}{ccc} \exists\mathcal{Q} & \text{———} & S \\ \Big| \subseteq & & \Big| \subseteq \\ \mathcal{P} & \text{———} & R \end{array}$$

New characterization of differentially closed fields

Theorem

Let K be a differential field. The following properties are equivalent.

1. *K is differentially closed.*
2. *"The main theorem holds for K."*
3. *For any finitely generated K-algebra S with no zero divisors and nonzero $b \in S$, there is an algebra homomorphism $\psi : S \to K$ with $\psi(b) \neq 0$.*

$(1 \Rightarrow 2)$ is the main theorem.
$(2 \Rightarrow 3)$ follows by letting $R = K$.
$(3 \Rightarrow 1)$ is a straightforward argument.

Further results and questions

Results

1. *For infinite cardinal λ, version of the main theorem when S is integral over R and generated by $\leq \lambda$ elements over R and K is λ-saturated. (Uses new Differential lying over and going up theorem.)*
2. *Easy counterexample shows characteristic p analog of main theorem fails. Also for difference fields (ACFA).*

Open questions

1. *Is there a geometric differential Chevalley theorem in characteristic p? For difference fields?*
2. *Is there a model theoretic version of Chevalley's homomorphism extension theorem?*