EXPOSITION OF T. T. CHENG'S EDGEWORTH APPROXIMATION OF THE POISSON DISTRIBUTION, IMPROVED

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by R. M. Dudley

For $0 < \lambda < +\infty$ and $k = 0, 1, ...,$ let $P(k, \lambda) := \sum_{j=0}^{k} e^{-\lambda}\lambda^j/j!$ and let $x := x(k, \lambda) := (k + \frac{1}{2} - \lambda)/\sqrt{\lambda}$. For $u \in \mathbb{R}$ let $\phi(u) := (2\pi)^{-1/2}\exp(-u^2/2)$ and $\Phi(x) := \int_{-\infty}^{x} \phi(u)\,du$. The following is essentially due to T. T. Cheng (1949, Theorem I):

Theorem 1 (T. T. Cheng)**.** *We have for any* $0 < \lambda < \infty$ *and* $k = 0, 1, 2, ...$ *that*

$$P(k, \lambda) \;=\; \Phi(x) + \frac{1 - x^2}{6\sqrt{\lambda}}\phi(x) + R \tag{1}$$

where the remainder $R = R(k, \lambda)$ *satisfies for all* $\lambda > 0$

$$|R| \;\le\; \delta_\lambda \;:=\; c_1\lambda^{-1} + c_2\lambda^{-3/2} + c_3\lambda^{-2} \tag{2}$$

with $c_1 = 0.0752$, $c_2 = 0.01$ *and* $c_3 = 0.122$.

Remarks. Cheng (1949) gave the statement with $c_1 = 0.076$, $c_2 = 0.043$ and $c_3 = 0.13$. His proof as it stands gives the slightly smaller values of c_1 and c_3 stated in Theorem 1. An improvement in the proof here will give the smaller c_2. Johnson, Kotz and Kemp (1992, §4.5, (4.40)) give further terms in the "formal" Edgeworth expansion, implying that as $\lambda \to +\infty$,

$$R(k, \lambda) \;=\; \phi(x)\left(\frac{-x^5 + 7x^3 - 3x}{72\lambda}\right) + O(\lambda^{-3/2}).$$

Now, $|x^5 - 7x^3 + 3x|\phi(x)$ is maximized at $x \doteq \pm 1.5352$. It follows that necessarily in Theorem 1, $c_1 \ge 0.02079$. So it's possible that the given $c_1 = 0.0752$ could be substantially improved.

Proof. The proof will be given in detail, following Cheng's proof but filling in further steps.

The Poisson characteristic function is, for all $t \in \mathbb{R}$,

$$e^{-\lambda}\sum_{j=0}^{\infty}(\lambda e^{it})^j/j! \;=\; \exp(\lambda(e^{it} - 1)).$$

The series is absolutely and uniformly convergent, so for $r = 0, 1, 2, ...$

$$\int_{-\pi}^{\pi} e^{-irt}e^{\lambda(e^{it}-1)}dt \;=\; 2\pi e^{-\lambda}\lambda^r/r!. \tag{3}$$

Next, for t not a multiple of 2π,

$$\sum_{r=0}^{k} e^{-irt} \;=\; (e^{-i(k+1)t} - 1)/(e^{-it} - 1) \;=\; (e^{it/2} - e^{-i(k+\frac{1}{2})t})/(2i\sin(t/2)).$$

Also,

$$\int_{-\pi}^{0} \exp(\lambda(e^{it}-1))e^{-irt}dt \;=\; \int_{0}^{\pi} \exp(\lambda(e^{-it}-1))e^{irt}dt.$$

Thus by (3)

$$\begin{aligned} P(k,\lambda) \quad &= \frac{1}{2\pi}\int_0^\pi \exp(\lambda(e^{it}-1))\frac{e^{it/2}-e^{-i(k+\frac{1}{2})t}}{2i\sin(t/2)} \\ &\quad + \exp(\lambda(e^{-it}-1))\frac{e^{i(k+\frac{1}{2})t}-e^{-it/2}}{2i\sin(t/2)}dt. \end{aligned}$$

Let

$$z \;:=\; z(t,k,\lambda) \;:=\; e^{i\lambda\sin t}(e^{it/2}-e^{-i(k+1/2)t})/(2i\sin(t/2)).$$

Then

$$\begin{aligned} 2\pi P(k,\lambda) \quad &= \int_0^\pi e^{\lambda(\cos t-1)}2\Re(z)dt \\ &= I_2 + \int_0^\pi e^{\lambda(\cos t-1)}\sin\left(\left(k+\frac{1}{2}\right)t-\lambda\sin t\right)/(\sin(t/2))\,dt \end{aligned}$$

where

$$I_2 := \int_0^\pi e^{\lambda(\cos t-1)}\left[\cos(\lambda\sin t)+\frac{\cos(t/2)}{\sin(t/2)}\sin(\lambda\sin t)\right]dt.$$

Claim. I_2 does not depend on λ.

To prove the claim let

$$\begin{aligned} G(\lambda,t) \quad &:= e^{\lambda(\cos t-1)}\sin\left(\frac{t}{2}+\lambda\sin t\right)/\sin(t/2) \\ &= e^{-2\lambda\sin^2(t/2)}\sin\left(\frac{t}{2}+2\lambda\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\right)/\sin(t/2). \end{aligned}$$

Then

$$\partial G(\lambda,t)/\partial\lambda \;=\; 2e^{-\lambda(\cos t-1)}\cos(t+\lambda\sin t).$$

Subclaim. For $n=1,2,...,$

$$\frac{\partial^n G(\lambda,t)}{\partial\lambda^n} \;=\; 2e^{\lambda(\cos t-1)}\sum_{j=1}^{n}(-1)^{n-j}\binom{n}{j}\cos(jt+\lambda\sin t).$$

Proof of subclaim. For $n = 1$ the statement holds. If it holds for a given n, then

$$\begin{aligned}
\frac{\partial^{n+1} G(\lambda,t)}{\partial\lambda^{n+1}} &= 2e^{\lambda(\cos t-1)}\sum_{j=1}^{n}(-1)^{n-j}\binom{n}{j}\Big[(\cos t-1)\cos(jt+\lambda\sin t) \\
&\qquad -(\sin t)\sin(jt+\lambda\sin t)\Big] \\
&= 2e^{\lambda(\cos t-1)}\sum_{j=1}^{n}(-1)^{n-j}\binom{n}{j}\Big[\cos((j+1)t+\lambda\sin t)-\cos(jt+\lambda\sin t)\Big] \\
&= 2e^{\lambda(\cos t-1)}\sum_{m=1}^{n+1}(\cos(mt+\lambda\sin t)(-1)^{n+1-m}\left[\binom{n}{m}+\binom{n}{m-1}\right]
\end{aligned}$$

which gives the subclaim by the Pascal's triangle identity.

Now to continue with the proof of the Claim, G is an entire analytic function of the variables λ and t. (The possible singularities when t is a multiple of 2π are removable.) Thus, I_2 is an entire analytic function of λ and we can differentiate under the integral sign. We have by the subclaim for all $n \ge 1$

$$\left.\frac{\partial^n G(\lambda,t)}{\partial\lambda^n}\right|_{\lambda=0} = \sum_{j=1}^{n}(-1)^{n-j}\binom{n}{j}\cos(jt),$$

and $\int_0^\pi \cos(jt)dt = 0$ for all $j = 1, 2, ...$ Thus by analyticity $dI_2/d\lambda \equiv 0$ and the claim follows. □

So $I_2 \equiv I_2(0) = \int_0^\pi 1\,dt = \pi$. It follows that

$$P(k,\lambda) - \frac{1}{2} = I := I(\lambda,k) \tag{4}$$

where

$$I := \frac{1}{2\pi}\int_0^\pi e^{\lambda(\cos t-1)}\sin\left(\left(k+\frac{1}{2}\right)t-\lambda\sin t\right)/(\sin(t/2))\,dt. \tag{5}$$

Let

$$I_1 := \frac{1}{2\pi}\int_0^\infty \exp(-\lambda t^2/2)\frac{2}{t}\sin[\sqrt{\lambda}xt+\lambda(t-\sin t)]dt. \tag{6}$$

By definition of x, $\sqrt{\lambda}x + \lambda = k + (1/2)$. Let

$$\begin{aligned}
J_0 &:= \tfrac{1}{2\pi}\textstyle\int_\pi^\infty \exp(-\lambda t^2/2)(2/t)dt, \\
J_1 &:= \tfrac{1}{2\pi}\textstyle\int_0^\pi \frac{\exp(-2\lambda\sin^2(t/2))}{\sin(t/2)} - \frac{2\exp(-\lambda t^2/2)}{t}\,dt,
\end{aligned} \tag{7}$$

where $J_1 > 0$ because $0 < \sin u < u$ for $0 < u \le \pi/2$. Then for the same reason and because $|\sin u| \le 1$ for all u,

$$|I - I_1| \le J_0 + J_1. \tag{8}$$

Now $J_1 = J_2 + J_3$ where

$$J_2 := \frac{1}{2\pi}\int_0^\pi \exp(-2\lambda\sin^2(t/2))\cot(t/2) - \exp(-\lambda t^2/2)(2/t)\,dt$$

and

$$J_3 \ := \ \frac{1}{2\pi}\int_0^{\pi} \exp(-2\lambda\sin^2(t/2))\frac{1-\cos(t/2)}{\sin(t/2)}\,dt.$$

It's easily shown that $\cot u = \frac{1}{u} - \frac{u}{3} + O(u^3)$ as $u \to 0$, from which $|J_2| < \infty$. In J_2 we can thus replace $\int_0^\pi$ by $\lim_{\delta\downarrow 0}\int_\delta^\pi$. We have setting $v := 2\sin(t/2)$ that

$$\int_\delta^\pi \exp(-2\lambda\sin^2(t/2))\cot(t/2)dt \ = \ 2\int_{2\sin(\delta/2)}^2 \exp(-\lambda v^2/2)dv/v.$$

Now

$$0 < 2\int_{2\sin(\delta/2)}^{\delta} \exp(-\lambda v^2/2)dv/v \ < \ 2\int_{2\sin(\delta/2)}^{\delta} dv/v \le 2[\ln(\delta) - \ln(2\sin(\delta/2))] \to 0$$

as $\delta \to 0$. Thus $J_2 = -\int_2^\pi \exp(-\lambda v^2/2)dv/v < 0$, so

(9) $$J_1 \ < \ J_3.$$

We have

(10) $$J_0 \ < \ \frac{1}{32\pi}\int_\pi^\infty t^3\exp(-2\lambda t^2/\pi^2)dt$$

because $\exp(-\lambda t^2/2)(2/t) < \frac{1}{16}t^3\exp(-2\lambda t^2/\pi^2)$ for all $t > \pi$. Next,

(11)
$$J_3 \ < \ \frac{1}{2\pi}\int_0^\pi \frac{\exp(-2\lambda\sin^2(t/2))}{\sin(t/2)}\left\{\frac{\sin^2(t/2)\cos(t/2)}{2} + \left(1-\cos\left(\frac{t}{2}\right)\right)\sin^2\left(\frac{t}{2}\right)\right\}dt$$

beause $1-\cos u < (\sin^2 u)\left(1-\frac{\cos u}{2}\right)$ for $0 < u < \pi/2$, as follows from $0 < v(1-v)^2$, $0 < v = \cos u < 1$. It will be shown that

(12) $$J_3 \ < \ \frac{1}{8\pi}\int_0^2 v\exp(-\lambda v^2/2)dv + \frac{1}{32\pi}\int_0^\pi t^3\exp(-2\lambda t^2/\pi^2)dt.$$

For this, the first summand on the right in (11), via the subsitution $v = 2\sin(t/2)$, gives the first term on the right in (12). For the other terms it will be shown for $u := t/2$ that

$$(1-\cos u)(\sin u)\exp(-2\lambda(\sin^2 u - 4u^2/\pi^2) \ < \ u^3/2$$

for $0 < u < \pi/2$. We have $\sin u > 2u/\pi$ on this range by concavity, $0 < \sin u < u$, and $1 - \cos u < u^2/2$ by derivatives, so indeed (12) holds.

By (8), (10), (9), and (12) we get

(13)
$$\begin{aligned} |I - I_1| &< \tfrac{1}{32\pi}\int_0^\infty t^3\exp(-2\lambda t^2/\pi^2)dt + \tfrac{1}{8\pi}\int_0^2 v\exp(-\lambda v^2/2)dv \\ &= \tfrac{1}{64\pi}\int_0^\infty u\exp(-2\lambda u/\pi^2)du + \tfrac{1}{8\pi\lambda}\int_0^2 -d\exp(-\lambda v^2/2)dv \\ &= \tfrac{1}{64\pi}\left(\tfrac{\pi^2}{2\lambda}\right)^2 + \tfrac{1}{8\pi\lambda}(1-e^{-2\lambda}) \ < \ \tfrac{\pi^3}{256\lambda^2} + \tfrac{1}{8\pi\lambda} \ < \ \tfrac{1}{8\pi\lambda} + \tfrac{0.122}{\lambda^2}. \end{aligned}$$

For I_1 itself we have by (6)

$$\begin{aligned} I_1 \ = \ & \tfrac{1}{\pi}\int_0^\infty t^{-1}\exp(-\lambda t^2/2)\sin(\sqrt{\lambda}xt)\cos(\lambda(t-\sin t))dt \\ & +\tfrac{1}{\pi}\int_0^\infty t^{-1}\exp(-\lambda t^2/2)\cos(\sqrt{\lambda}xt)\sin(\lambda(t-\sin t))dt \end{aligned}$$

$$(14)\qquad \begin{aligned} &= J_4 + \tfrac{1}{\pi}\int_0^\infty t^{-1}\exp(-\lambda t^2/2)\sin(\sqrt{\lambda}xt)dt \\ &+ J_5 + \tfrac{1}{\pi}\int_0^\infty t^{-1}\exp(-\lambda t^2/2)\cos(\sqrt{\lambda}xt)\lambda t^3\,dt/6 \end{aligned}$$

where

$$J_4 := \frac{1}{\pi}\int_0^\infty t^{-1}\exp(-\lambda t^2/2)\sin(\sqrt{\lambda}xt)[\cos(\lambda(t-\sin t))-1]dt,$$

$$J_5 := \frac{1}{\pi}\int_0^\infty t^{-1}\exp(-\lambda t^2/2)\cos(\sqrt{\lambda}xt)[\sin(\lambda(t-\sin t))-\lambda t^3/6]dt.$$

Since $|\sin u| \le 1$ for all real u and $|1-\cos v| \le v^2/2$ for all real v by derivatives,

$$|J_4| \le \frac{1}{\pi}\int_0^\infty t^{-1}\exp(-\lambda t^2/2)\lambda^2(t-\sin t)^2 dt/2,$$

and since $0 \le t - \sin t \le t^3/6$ for all $t \ge 0$ by derivatives, a substitution using the gamma function gives

$$(15)\qquad |J_4| \le \frac{\lambda^2}{72\pi}\int_0^\infty t^5\exp(-\lambda t^2/2)dt \;=\; \frac{1}{9\pi\lambda}.$$

For J_5 we have $\sin t < t$ for all $t > 0$, which applied twice gives

$$\sin(\lambda(t-\sin t)) - \lambda t^3/6 \le \lambda(t - \sin t - t^3/6) \le 0.$$

We have moreover $\sin t - t + t^3/6 \le t^5/120$ for all $t \ge 0$ by derivatives. It follows by another substitution and gamma function that

$$(16)\qquad \begin{aligned} |J_5| &< \frac{\lambda}{120\pi}\int_0^\infty t^4\exp(-\lambda t^2/2)dt \\ &= \frac{\lambda}{240\pi}\int_0^\infty u^{3/2}\exp(-\lambda u/2)du \\ &= \lambda^{-3/2}\sqrt{\frac{2}{\pi}}\cdot\frac{1}{80} \;<\; 0.01/\lambda^{3/2}. \end{aligned}$$

By (14) we have

$$(17)\qquad I_1 = J_4 + J_5 + U_1 + U_2$$

where

$$(18)\qquad U_1 := \frac{1}{\pi}\int_0^\infty \exp(-\lambda t^2/2)\sin(\sqrt{\lambda}xt)dt/t,$$

$$(19)\qquad U_2 := \frac{\lambda}{6\pi}\int_0^\infty \exp(-\lambda t^2/2)t^2\cos(\sqrt{\lambda}xt)dt.$$

Setting $u := \sqrt{\lambda}t$ gives

$$\begin{aligned} U_1 &= \frac{1}{\pi}\int_0^\infty \exp(-u^2/2)\sin(ux)du/u \\ &= \frac{1}{4\pi}\int_{-\infty}^\infty \exp(-u^2/2)[e^{iux}-e^{-iux}]du/(iu) \\ &= \frac{1}{4\pi}\int_{-\infty}^\infty \exp(-u^2/2)\int_{-x}^{x} e^{iut}\,dt\,du \\ &= \frac{1}{4\pi}\int_{-x}^{x}\int_{-\infty}^\infty \exp\left(-\frac{(u-it)^2}{2}-\frac{t^2}{2}\right)du\,dt \\ &= \frac{1}{4\pi}\sqrt{2\pi}\int_{-x}^{x} e^{-t^2/2}dt \;=\; \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}dt \;=\; \Phi(x)-\frac{1}{2}, \end{aligned}$$

and by an integration by parts we get

$$U_2 \;=\; \frac{1}{6\pi\sqrt{\lambda}}\int_0^\infty e^{-u^2/2}[\cos(ux)-xu\sin(ux)]du,$$

where

$$\int_0^\infty e^{-u^2/2}\cos(ux)du \;=\; \frac{1}{2}\int_{-\infty}^\infty \exp(-(u^2/2)-iux)du \;=\; \sqrt{\pi/2}\exp(-x^2/2).$$

Another integration by parts for the other term in U_2 gives

$$U_2 \;=\; \frac{1-x^2}{6\sqrt{2\pi\lambda}}\exp(-x^2/2).$$

We see that $U_1+U_2+\frac{1}{2}$ gives the first two terms of the expansion in the theorem. Using $I+\frac{1}{2}=P(k,\lambda)$ from (4), we get by (8), (13), (17), (15) and (16),

$$|R(k,\lambda)| \;<\; |I-I_1|+|J_4|+|J_5| \;<\; \left(\frac{1}{8}+\frac{1}{9}\right)\frac{1}{\pi\lambda}+\frac{0.01}{\lambda^{3/2}}+\frac{0.122}{\lambda^2},$$

which finishes the proof of the theorem. □

REFERENCES

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