CONTINUED FRACTIONS

Lecture notes, R. M. Dudley, Math Lecture Series, January 15, 2014

1. Basic definitions and facts

A continued fraction is given by two sequences of numbers $\{b_n\}_{n\geq 0}$ and $\{a_n\}_{n\geq 1}$. One traditional way to write a continued fraction is:

$$Q \;=\; b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}} \tag{1}$$

Recall that an infinite sum $\sum_{i=1}^{\infty} a_i$ means the limit as $n \to \infty$ (if it exists) of the finite sum $\sum_{i=1}^{n} a_i$, and an infinite product $\Pi_{i=1}^{\infty} a_i$ means the limit, if it exists, of the finite products $a_1 a_2 \cdots a_n$ (sometimes with the proviso that none of the factors a_i is 0). Similarly, an infinite continued fraction will be the limit, if it exists, of the sequence of numbers

$$Q_0 = b_0, \quad Q_1 = b_0 + a_1/b_1, \quad Q_2 = b_0 + a_1/(b_1 + (a_2/b_2)), \ldots .$$

To be more precise, let $T_j(z) := T_j(z; a_j, b_j) := a_j/(b_j + z)$ for any number z and $j = 1, 2, \ldots$ (here ":=" means "equals by definition"). Then the *nth convergent* of the continued fraction is given by

$$Q_n \;=\; b_0 + T_1(T_2(\cdots(T_n(0))\cdots)) \tag{2}$$

if the expression is defined. Here $0/0$ is undefined but we define $a/0 := \infty$ for $a \neq 0$ and $b/(c + \infty) := 0$ for any finite b, c. To multiply the continued fraction by a constant c, one can multiply both b_0 and a_1 by c. Clearly, $Q_n = Q_n(b_0, ..., b_n; a_1, ..., a_n)$ is a rational function (quotient of polynomials) of its $2n + 1$ arguments. The continued fraction $Q = Q(\{b_k\}_{k\geq 0}, \{a_k\}_{k\geq 1})$ will be called *convergent* to a finite value Q if for n large enough, Q_n is defined and finite and $\lim_{n\to\infty} Q_n = Q$. For example, if $b_0 = a_1 = b_1 = 0$ and $a_2 = b_2 = 1$ then Q_1 is not defined but Q_2 is well defined and equals 0. A convergent continued fraction is said to *terminate* at the nth term for the smallest positive integer n such that $a_n = 0$ and Q_k is defined for all $k > n$. Then $Q_k = Q_{n-1}$ for all $k > n$.

Since expressions like (1) can take up a lot of space, we will follow several other authors in writing the continued fraction (1) as

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\;\cdots .$$

The following formula for a sequence $\{X_n\}_{n\geq -1}$ is called the *Wallis-Euler recurrence formula*:

$$X_n = b_n X_{n-1} + a_n X_{n-2}, \quad n = 1, 2, \ldots . \tag{3}$$

Theorem 1 (Wallis–Euler)**.** *For the continued fraction (1) and $n = 0, 1, 2, \ldots$, we have $Q_n = A_n/B_n$ where each of the two sequences $\{A_n\}$ and $\{B_n\}$ satisfies (3) for $A_{-1} := 1$, $B_{-1} := 0$, $A_0 := b_0$, and $B_0 := 1$. Here $Q_n = A_n/B_n$ means that either both sides are defined and equal, or neither side is defined.*

Proof. Define A_n recursively for $n \geq 1$ by (3) with A_n in place of X_n, and likewise define B_n. Clearly $Q_n = A_n/B_n$ for $n = 0$ or 1. To prove this for $n \geq 2$ by induction, for general sequences and not just fixed sequences $\{a_j\}, \{b_j\}$, suppose it holds for a given n. By (2), Q_{n+1} equals Q_n with $T_n(0) = a_n/b_n$ replaced by $T_n(T_{n+1}(0))$, in other words with b_n replaced by $b_n + (a_{n+1}/b_{n+1})$ (which may be infinite or undefined if $b_{n+1} = 0$). Then by (3) for n and the induction hypothesis, if $b_{n+1} \neq 0$,

$$\begin{aligned} Q_{n+1} &= \frac{\left(b_n + \frac{a_{n+1}}{b_{n+1}}\right) A_{n-1} + a_n A_{n-2}}{\left(b_n + \frac{a_{n+1}}{b_{n+1}}\right) B_{n-1} + a_n B_{n-2}} \\ &= \left(A_n + \frac{a_{n+1}}{b_{n+1}} A_{n-1}\right) / \left(B_n + \frac{a_{n+1}}{b_{n+1}} B_{n-1}\right) \\ &= (b_{n+1}A_n + a_{n+1}A_{n-1})/(b_{n+1}B_n + a_{n+1}B_{n-1}) \;=\; A_{n+1}/B_{n+1} \end{aligned}$$

by (3) for $n+1$, finishing the proof if $b_{n+1} \neq 0$. Or if $b_{n+1} = 0$ and $a_{n+1} \neq 0$ then $T_{n+1}(0) = \infty$ and $T_n(T_{n+1}(0)) = 0$ so $Q_{n+1} = Q_{n-1}$ and

$$\frac{A_{n+1}}{B_{n+1}} = \frac{a_{n+1}A_{n-1}}{a_{n+1}B_{n-1}} = \frac{A_{n-1}}{B_{n-1}} = Q_{n-1} = Q_{n+1}$$

as stated. Lastly if $a_{n+1} = b_{n+1} = 0$ then $A_{n+1}/B_{n+1} = 0/0$ undefined and $T_{n+1}(0)$ is undefined so Q_{n+1} is also undefined, finishing the proof. Q.E.D.

From (3) and Theorem 3, it's clear that A_n and B_n are polynomials with integer coefficients in the $2n+1$ variables $b_0, a_1, b_1, ..., a_n, b_n$. Actually B_n doesn't depend on b_0 or a_1. For $j = 0, 1, ..., n$ let

$$Q_{n,j} \;:=\; Q_{n,j}(b_0, b_1, ..., b_n; a_1, ..., a_n) \;:=\; Q_{n-j}(0, b_{j+1}, ..., b_n; a_{j+1}, ..., a_n).$$

Then for $j = 0, ..., n$,

$$Q_n = Q_n(b_0, b_1, ..., b_n; a_1, ..., a_n) = Q_j(b_0, ..., b_{j-1}, b_j + Q_{n,j}; a_1, ..., a_j). \tag{4}$$

Theorem 2. *Suppose that for given $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 0}$, and a given nonnegative integer j, the vectors (A_{j-1}, A_j) and (B_{j-1}, B_j) are linearly independent in the plane. Then the set of all sequences $\{X_i\}_{i\geq j-1}$ satisfying (3) for $n \geq j+1$ is two-dimensional and has a basis given by $\{A_i\}_{i\geq j-1}$ and $\{B_i\}_{i\geq j-1}$. The linear independence is true if and only if the determinant $D_j := A_{j-1}B_j - B_{j-1}A_j \neq 0$, and we have*

$$D_0 = 1 \quad \text{and} \quad D_j = (-1)^j a_1 a_2 \cdots a_j, \quad j \geq 1. \tag{5}$$

Proof. $D_0 = 1$ follows from the definitions, and

$$D_1 = b_0 B_1 - A_1 = b_0 b_1 - b_1 b_0 - a_1 = -a_1$$

as stated. To prove (5) for larger j by induction, suppose it holds for a given j. Then by (3)

$$\begin{aligned} D_{j+1} &= A_j B_{j+1} - B_j A_{j+1} \\ &= A_j(b_{j+1}B_j + a_{j+1}B_{j-1}) - B_j(b_{j+1}A_j + a_{j+1}A_{j-1}) \;=\; -a_{j+1}D_j \end{aligned}$$

and (5) follows.

The set of all $\{X_i\}_{i\geq j-1}$ satisfying (3) for the given a_i and b_i is a vector space since the equations (3) are linear in the X_i. Once X_{j-1} and X_j are given, the X_i for $i > j$ are uniquely and linearly determined by (3) applied for $n =$

$j+1, j+2, \ldots$. So the set of solutions is indeed two-dimensional and if the two vectors $(A_{j-1}, A_j), (B_{j-1}, B_j)$ are linearly independent, then clearly so are the sequences $\{A_i\}_{i \geq j-1}$, $\{B_i\}_{i \geq j-1}$. The equivalence of linear independence of vectors and the given determinant not vanishing is well known from linear algebra, and the conclusion follows. Q.E.D.

Now, some inequalities for continued fractions will be developed. A continued fraction (1) will be called *fully positive* if $a_n \geq 0$ and $b_n > 0$ for all $n \geq 1$.

Theorem 3 (Euler). *If a continued fraction (1) is fully positive, and if Q converges, then*

$$Q_0 \leq Q_2 \leq Q_4 \leq \cdots \leq Q \leq \cdots \leq Q_5 \leq Q_3 \leq Q_1.$$

If also $a_n > 0$ for all n then each "$\leq$" can be replaced by the strict inequality "$<$." If Q doesn't converge, the inequalities remain true if "$\leq Q \leq$" is omitted.

Proof. By induction on n, from (2), Q_n is a nondecreasing function of b_n (increasing if all $a_j > 0$) for n even, and a nonincreasing (resp. decreasing) function of it for n odd. Thus by (4), if j is even, then $Q_j \leq Q_n$ for all $n \geq j$, while if j is odd, $Q_j \geq Q_n$ for all $n \geq j$, and the Theorem follows. Q.E.D.

Theorem 3 implies that for a fully positive convergent continued fraction Q, if two successive convergents Q_n and Q_{n+1} are close together, then since Q is between them we have good lower and upper bounds for it. If A is an approximation to Q, the *relative error* of the approximation is defined as $|(A/Q) - 1|$. So given $\varepsilon > 0$, to compute Q with a relative error $< \varepsilon$ we can take n large enough so that $(Q_{2n+1}/Q_{2n}) - 1 < \varepsilon$ and let $A = Q_{2n+1}$.

A similar thing happens for continued fractions with terms a_j alternating in sign, as follows.

Definition. A continued fraction (1) will be called *alternating* if the following all hold:
(i) $b_0 \geq 0$ and $b_j \geq 1$ for all $j \geq 1$.
(ii) Let $K := i + 1$ for the least i such that $a_i = 0$, or $K := +\infty$ if there is no such i. Then for all positive integers $j < K$, $a_j = (-1)^{j+1} c_j$ where $c_j \geq 0$ and if j is even, $c_j < 1$.

A monotonicity argument like the proof of Theorem 3 also can be applied to alternating continued fractions. This was noted at least in special cases by A. A. Markov around 1920. The following formulation is not proved here, see Dudley (1987). As shown there, some functions (e.g. hypergeometric functions) can be evaluated more efficiently, in some ranges, via fully positive or alternating continued fractions than by summing their Taylor series.

Theorem 4. *For any alternating continued fraction Q, if Q converges, then*

$$Q_1 \leq Q_4 \leq Q_5 \leq Q_8 \leq \cdots \leq Q \leq \cdots \leq Q_7 \leq Q_6 \leq Q_3 \leq Q_2.$$

For a convergent alternating continued fraction Q, and any $n \geq 1$, Q is between Q_n and Q_{n+2}, so if Q_n and Q_{n+2} are close, then we have good upper and lower bounds for Q. To compute an alternating continued fraction Q to within a relative error $< \varepsilon$, one can find k large enough so that $(Q_{4k+2}/Q_{4k+1}) - 1 < \varepsilon$ and approximate Q by Q_{4k+2}. Alternating continued fractions won't be mentioned further in this talk.

2. Convergence conditions

A continued fraction (1) and a series $\sum_{j\geq 0} c_j$ are called *equivalent* if for each $n = 0, 1, 2, ...,$ $Q_n = \sum_{j=0}^{n} c_j$. In particular, all Q_n must be defined. Clearly, for any continued fraction (1) with all Q_n defined, there is a unique equivalent series, with $c_0 = b_0$ and $c_n = Q_n - Q_{n-1}$ for all $n \geq 1$. Thus by Theorems 1 and 3,

$$c_n \ = \ -D_n/(B_{n-1}B_n) \ = \ (-1)^{n+1}a_1a_2\cdots a_n/(B_{n-1}B_n).$$

Since convergence of a series doesn't depend on its first term, it follows that:

Theorem 5. *A continued fraction (1) with all Q_n defined is convergent if and only if, for D_j as in (5), the following series converges:*

$$\sum_{j=1}^{\infty} \frac{D_j}{B_{j-1}B_j}. \tag{6}$$

A continued fraction (1) will be called *unary* if $a_n = 1$ for all $n \geq 1$. For such a continued fraction, if b_n are small, approaching 0, putting in one more term makes a big difference: b_n is small, but $b_n + (1/b_{n+1})$ is large, and so on. So for the continued fraction to converge, b_n should not be too small.

Theorem 6. *A unary continued fraction (1) with $\sum_n |b_n| < \infty$ does not converge.*

Proof. For a unary continued fraction we have $|B_n| \leq \Pi_{j=1}^{n}(1+|b_j|)$ for all $n \geq 1$, as follows from $B_{-1} = 0$, $B_0 = 1$, Theorem 1, which gives $|B_n| \leq |b_n||B_{n-1}| + |B_{n-2}|$, and by induction on n. Now $\sum_j |b_j| < \infty$ implies $\Pi_{j=1}^{\infty}(1+|b_j|) < \infty$ since $1+x \leq e^x$ for $x \geq 0$. Thus $|B_n|$ remain bounded, and since $D_j = \pm 1$, the terms of the series (6) don't approach 0, so it diverges, Q.E.D.

Theorem 7 (Seidel and Stern)**.** *A unary, fully positive continued fraction (1) is convergent if and only if $\sum_{n=1}^{\infty} b_n = +\infty$.*

Proof. "Only if" follows from Theorem 6. To prove "if," suppose $\sum_{n=1}^{\infty} b_n = +\infty$. By Theorems 2 and 5, we need to show that the series (6), $\sum_{j=1}^{\infty}(-1)^j/(B_{j-1}B_j)$ in this case, converges. By Theorem 1, $B_j = b_jB_{j-1} + B_{j-2} > B_{j-2}$. Thus the terms of (6) are alternating in sign and decreasing in absolute value, so it is enough to show that their absolute values approach 0. It will be shown by induction on n that for $n \geq 1$,

$$B_{2n} \geq 1 + b_1(b_2 + b_4 + \cdots + b_{2n}), \quad B_{2n+1} > b_1 + b_3 + \cdots + b_{2n+1}. \tag{7}$$

We have $B_1 = b_1$, $B_2 = b_2b_1 + 1$, and $B_3 = b_1b_2b_3 + b_3 + b_1 > b_1 + b_3$, so (7) holds for $n = 1$. Assuming (7) for some $n \geq 1$, we get from Theorem 1

$$\begin{aligned} B_{2n+2} &= b_{2n+2}B_{2n+1} + B_{2n} \ > \ b_{2n+2}b_1 + B_{2n}, \\ B_{2n+3} &= b_{2n+3}B_{2n+2} + B_{2n+1} \ > \ b_{2n+3} + B_{2n+1}, \end{aligned}$$

which gives the induction step. Since $\sum_n b_n = +\infty$ and $b_1 > 0$, either $B_{2n} \to +\infty$ or $B_{2n+1} \to +\infty$, so the terms of (6) approach 0 and it converges. Q.E.D.

3. Rational approximation of real numbers

A continued fraction (1) will be called *canonical* if it is unary, $b_0 \in \mathbb{Z} := \{0, \pm 1, \pm 2, ...\}$, and b_n is a positive integer for all $n \geq 1$. By Theorem 7, every canonical continued fraction is convergent.

Theorem 8. *A number q is rational, $q \in \mathbb{Q}$, if and only if there is a canonical continued fraction Q such that $Q_n = q$ for some n. Each rational number q has exactly two such representations. If it has one with $n = 0$, or $n \geq 1$ and $b_n \geq 2$, then the other has n replaced by $n+1$, b_n by $b_n - 1$ and $b_{n+1} := 1$. Or if $q = Q_n$ with $n \geq 1$ and $b_n = 1$, the other representation has n replaced by $n-1$ and b_{n-1} by $b_{n-1} + 1$.*

Proof. In a canonical (or any unary) continued fraction, Q_n only depends on $b_0, b_1, \ldots, b_n$, so once we have defined them, the values of b_k for $k > n$ won't change Q_n, e.g. we can set $b_k = 1$ for $k > n$. If $q \in \mathbb{Z}$ the theorem holds, with either $n = 0$ and $b_0 = q$, or $n = 1$, $b_0 = q - 1$, and $b_1 = 1$. If $q \notin \mathbb{Z}$, let b_0 be the largest integer $< q$. So we can reduce to the case $b_0 = 0$ and $0 < q < 1$. Let $q = k_1/m_1$ where $0 < k_1 < m_1$ are integers and k_1/m_1 is in lowest terms. There is a unique positive integer b_1 such that $1/(b_1 + 1) < k_1/m_1 \leq 1/b_1$. Thus $b_1 \leq m_1/k_1 < b_1 + 1$. If $b_1 = m_1/k_1$ (so $k_1 = 1$) let $n = 1$ and we get $Q_1 = q$ as desired. Otherwise iterate the process and take the unique positive integer b_2 such that

$$1/(b_2 + 1) \;<\; k_2/m_2 \;:=\; (m_1/k_1) - b_1 \leq 1/b_2,$$

where k_2/m_2 is in lowest terms, thus $m_2 = k_1 < m_1$. We get a decreasing sequence m_j of positive integers, so after finitely many steps, the process ends and gives $q = Q_n$ for some $n \leq m_1$.

It's clear that given one representation $q = Q_n$, q has another representation as described. For any positive integers k and m_j for $j = 1, ..., k$,

$$0 < Q_k(0, m_1, ..., m_k; 1, 1, ..., 1) \leq 1,$$

with equality if and only if $k = 1 = m_1$. This fact, applied to fractions $Q_{n,j}$ as in (4), implies that there are exactly two representations of q of the given form. Q.E.D.

For irrational numbers we have:

Theorem 9. *There is a one-to-one correspondence between irrational real numbers x and canonical continued fractions $Q = x$.*

Proof. Let x_0 be irrational. Let b_0 be the largest integer $< x_0$. Then $b_0 < x_0 < b_0 + 1$. For $n \geq 1$ a sequence of positive integers b_n and irrational numbers $x_n > 1$ will be defined recursively. Let $x_1 := 1/(x_0 - b_0) > 1$. Then x_1 is irrational. Given $x_n > 1$ irrational there is a unique positive integer b_n such that $b_n < x_n < b_n + 1$. Let $x_{n+1} := 1/(x_n - b_n)$, which is irrational and > 1. Then

$$x_0 \;=\; b_0 + \frac{1}{x_1} \;=\; b_0 + \frac{1}{b_1+}\;\frac{1}{x_2} \;=\; \ldots \;=\; b_0 + \frac{1}{b_1+}\;\frac{1}{b_2+}\cdots\frac{1}{x_n}.$$

By Theorem 7, the continued fraction

$$Q \;=\; b_0 + \frac{1}{b_1+}\;\frac{1}{b_2+}\cdots$$

converges. By Theorem 3 and its proof, $Q_{2n} < x_0 < Q_{2n+1}$ for all n. So $Q = x_0$.

The positive integers b_j are functions of x_0. Let x_0 equal a canonical continued fraction $\beta_0 + \frac{1}{\beta_1+}\;\frac{1}{\beta_2+}\cdots$. Then it's easily seen that $0 < x_0 - \beta_0 < 1$ so $\beta_0 = b_0$. Considering $1/(x_0 - \beta_0)$ we likewise get $\beta_1 = b_1$, and iterating we get $\beta_n = b_n$ for all n. So the irrational numbers are in 1–1 correspondence with the canonical continued fractions, Q.E.D.

So, here is an iteration to find a canonical continued fraction for any real number x_0: for each $j = 0, 1, 2, \ldots$, let b_j be the largest integer $\leq x_j$. If $x_j - b_j = 0$, then $x_0 = Q_j$, which eventually happens if and only if x_0 is rational, as in Theorem 8. Otherwise, $0 < x_j - b_j < 1$. Let $x_{j+1} = 1/(x_j - b_j)$. So $x_j = b_j + (1/x_{j+1})$. Then for $i \geq 1$, b_i, if defined, will be a positive integer, $b_i \geq 1$.

Example. Let $x_0 = \pi = 3.14159265358979... \doteq 3.14159265359$. Then $b_0 = 3$, $x_1 = 1/(\pi - 3) \doteq 7.0625133059$, $b_1 = 7$, $x_2 = 1/(x_1 - b_1) \doteq 1/(0.0625133059) \doteq 15.9965944$, $b_2 = 15$, $x_3 \doteq 1/0.9965944 \doteq 1.003417$, $b_3 = 1$, $x_4 \doteq 1/0.003417 \doteq 292.6$, and $b_4 = 292$. For a calculator working to some fixed number of digits of accuracy, some digits are lost at each stage, and eventually b_k would become incorrect.

The canonical continued fraction for π, for which we just found the first few terms, is

$$\pi = 3 + \frac{1}{7+} + \frac{1}{15+} + \frac{1}{1+} + \frac{1}{292+} + \frac{1}{1+} + \frac{1}{1+} + \frac{1}{1+} + \frac{1}{2+} + \frac{1}{1+} + \frac{1}{3+} + \frac{1}{14+} + \frac{1}{1+} + \cdots,$$

with no discernible pattern. This continued fraction was found (as far as given) by J. Wallis in his book *Tractatus de Algebra* in 1685 and is stated in Perron, p. 242.

As rational approximations to π the continued fraction gives $Q_0 = 3$, $Q_1 = 22/7$ (over 2200 years ago Archimedes proved $223/71 < \pi < 22/7$), $Q_2 = 333/106$, $Q_3 = 355/113$, an approximation found in China over 1500 years ago. After that the denominators get much larger: $Q_4 = 103993/33102$.

Note that for canonical continued fractions Q, the quantities A_n and B_n as defined in Theorem 1 are all integers and $B_n > 0$ for $n \geq 0$ by (7).

Theorem 10. *For any canonical continued fraction Q and any $n \geq 1$, the fraction $Q_n = A_n/B_n$ is in lowest terms, i.e. A_n and B_n are relatively prime.*

Proof. By (5), $A_{n-1}B_n - A_n B_{n-1} = (-1)^n a_1 \cdots a_n = (-1)^n$. A common factor of A_n and B_n would be a factor of $(-1)^n$, which is impossible, Q.E.D.

Any real number x can be approximated by a rational number r/s for any positive integer s and some $r \in \mathbb{Z}$ with an error $|x - (r/s)| \leq 1/(2s)$. It turns out that for suitable s, it's possible to make the error less than $1/s^2$, and that continued fractions will give us such rational approximations, as the next fact shows. Conversely, if the error is less than $1/(2s^2)$, r/s must equal some Q_n from the canonical continued fraction Q (Corollary 1).

Theorem 11 (Lagrange)**.** *Let x_0 be an irrational real number with canonical continued fraction (1) from Theorem 9*

$$x_0 \ = \ Q \ = \ Q(\{b_k\}_{k\geq 0}) \ := \ Q(\{b_k\}_{k\geq 0}, \{a_k\}_{k\geq 1})$$

where $a_k = 1$ for all k. Then for B_k as usual (Theorem 1), for all $n \geq 0$,

$$|x_0 - Q_n| \ < \ 1/(B_n B_{n+1}) \leq 1/B_n^2.$$

Proof. Let $x^{(n)} \ := \ Q(\{b_{k+n}\}_{k\geq 0})$ for $n = 0, 1, \ldots$, and for any $\beta_0, \ldots, \beta_n$,

$$Q_n(\{\beta_k\}_{k=0}^n) \ := \ Q_n(\beta_0, \ldots, \beta_n; a_1, \ldots, a_n)$$

where $a_1 = a_2 = \cdots = a_n = 1$. Then $x_0 = Q_n(\{\beta_k\}_{k=0}^n)$ where $\beta_k = b_k$ for $k = 0, 1, \ldots, n-1$ and $\beta_n = x^{(n)}$. It follows from Theorem 1 that for any $n = 1, 2, \ldots$,

$$x_0 \ = \ [x^{(n)} A_{n-1} + A_{n-2}]/[x^{(n)} B_{n-1} + B_{n-2}]. \tag{8}$$

Therefore

$$x_0 - Q_{n-1} \;=\; \frac{A_{n-2}B_{n-1} - A_{n-1}B_{n-2}}{B_{n-1}(x^{(n)}B_{n-1} + B_{n-2})}.$$

We have $B_k \geq 0$ for all k, $B_k > 0$ for $k \geq 0$, and $x^{(n)} > b_n$. By Theorem 1 and (5) we then have $|x_0 - Q_{n-1}| < 1/(B_{n-1}B_n)$. Replacing n by $n+1$ gives the first conclusion. Then noting that for a canonical continued fraction, $B_0 \leq B_1 < B_2 < \cdots$, we get the second bound. Q.E.D.

In the example of the canonical continued fraction of π given before Theorem 10 we got $Q_3 = 355/113 \doteq 3.1415929...$ where the given digits equal those of π except for the last one. From Theorem 11, we see that

$$\frac{355}{113} - \pi \;=\; \frac{\theta}{113^2}$$

for some θ with $|\theta| < 1$. In fact in this case $\theta \doteq 0.0034$, which is small, as is connected with the unusually large number 292 occurring in the canonical continued fraction expansion. So 355/113 is a remarkably good rational approximation of π in relation to its denominator, which is not very large. According to some websites, this approximation was first discovered in China by Zu or Zhu Chongzhi (spelled Tsu Ch'ung Chi on another site), who lived from about 430 to 500 A.D., and a son. It was not improved until about 1,000 years later.

Let $|x_0 - (r/s)| = \theta/s^2$ where $r, s \in \mathbb{Z}$, $s > 0$, and r/s is in lowest terms. Then for r/s to equal some Q_n for the canonical continued fraction of x_0, we have just seen that $\theta < 1$ is necessary, and it will be shown that $\theta < 1/2$ is sufficient. The precise necessary and sufficient condition is as follows.

Theorem 12. *Let x_0 be a real irrational number. Let $r, s \in \mathbb{Z}$ with $s > 0$ where r/s is in lowest terms. Represent $r/s = Q_n(\{\gamma_k\}_{k=0}^n)$ by Theorem 8 for integers γ_k with $\gamma_k > 0$ for $k \geq 1$ and n such that $(-1)^n(x_0 - (r/s)) > 0$ (by Theorem 8, we can choose n even or odd). Define*

$$\theta \;:=\; s^2(-1)^n\left[x_0 - \frac{r}{s}\right] \;>\; 0.$$

Let $\mathcal{A}_k$ and $\mathcal{B}_k$ for $k = -1, 0, ..., n$ be defined as A_k and B_k for $Q_n(\{\gamma_k\}_{k=0}^n)$ by Theorem 1. Then for the canonical continued fraction Q of x_0, $r/s = Q_m$ for some m if and only if

$$\theta < \mathcal{B}_n/(\mathcal{B}_n + \mathcal{B}_{n-1}), \tag{9}$$

and then $n = m$.

Proof. The case $m = 0$ can occur if and only if $s = 1$. Then $r/s = Q_0$ if and only if $r = b_0$. Since $\mathcal{B}_0 = 1$ and $\mathcal{B}_{-1} = 0$ by definition, $\mathcal{B}_0/(\mathcal{B}_0 + \mathcal{B}_{-1}) = 1$ and the equivalence holds in this case. So we can assume $m \geq 1$ and $s \geq 2$. Then also $n \geq 1$ ($n = 0$ is only possible for $s = 1$).

To prove "if," let $w \;:=\; (\mathcal{B}_n - \theta\mathcal{B}_{n-1})/(\theta\mathcal{B}_n)$. Since $0 < \theta < 1$ and $\mathcal{B}_n \geq \mathcal{B}_{n-1} > 0$, we have $w > 0$. Solving for θ gives

$$\theta = \mathcal{B}_n/(w\mathcal{B}_n + \mathcal{B}_{n-1}). \tag{10}$$

By Theorem 10, $r = \mathcal{A}_n$ and $s = \mathcal{B}_n$. Thus by definition of θ,

$$x_0 \;=\; \frac{r}{s} + \frac{(-1)^n\theta}{s^2} \;=\; \frac{r}{s} + \frac{(-1)^n}{\mathcal{B}_n^2}\cdot\frac{\mathcal{B}_n}{(w\mathcal{B}_n + \mathcal{B}_{n-1})} \;=\; \frac{\mathcal{A}_n(w\mathcal{B}_n + \mathcal{B}_{n-1}) + (-1)^n}{\mathcal{B}_n(w\mathcal{B}_n + \mathcal{B}_{n-1})}.$$

Now $\mathcal{A}_n\mathcal{B}_{n-1} + (-1)^n = \mathcal{A}_{n-1}\mathcal{B}_n$ by (5), so

$$x_0 \;=\; (\mathcal{A}_n w + \mathcal{A}_{n-1})/(\mathcal{B}_n w + \mathcal{B}_{n-1}). \tag{11}$$

Since θ is irrational, so is w by (10). We have $w > 1$ if and only if (9) holds, by (10). If $w > 1$, then by Theorem 9, we have the canonical continued fraction $w = Q(\{\zeta_k\}_{k\geq 0})$ for some positive integers ζ_k ($\zeta_0 \geq 1$ since $w > 1$). Let $\gamma_k := \zeta_{k-n-1}$ for $k \geq n+1$. Then $Q(\{\gamma_k\}_{k\geq 0})$ is a canonical continued fraction, so it converges to some ξ, with $\xi^{(n+1)} = w$ from the definitions. Then by (8) applied to ξ and to $n+1$ in place of n, $\xi = (\mathcal{A}_n w + \mathcal{A}_{n-1})/(\mathcal{B}_n w + \mathcal{B}_{n-1}) = x_0$ by (3.4). Thus $\mathcal{A}_k = A_k$ and $\mathcal{B}_k = B_k$ for $k = -1, 0, 1, ..., n$, and $r/s = Q_n$. So "if" holds, with $n = m$ as stated. By Theorem 3, since $a_j \equiv 1 > 0$, m is uniquely determined.

[To confirm that the definitions of n and θ in Theorem 12 give $n = m$ as opposed to $n = m \pm 1$, if $r/s < x_0$ then n is even, and if $r/s = Q_m$ then m is also even by Theorem 3. Likewise if $Q_m = r/s > x_0$ then n and m are both odd.]

To prove "only if," still with $m \geq 1$, $n \geq 1$, and $s \geq 2$, let $w < 1$, i.e. (9) fails. Since $w > 0$ we then have $\gamma_n + (1/w) > \gamma_n + 1$. The canonical continued fraction expansion of $\gamma_n + 1/w$ is $Q(\{c_i\}_{i\geq 0})$ where $c_0 = \gamma_n + u$ with $u \geq 1$. I claim that $x_0 = Q_n(\{\gamma'_k\}_{k=0}^n)$ where $\gamma'_k = \gamma_k$ for $k = 0, 1, ..., n-1$ and $\gamma'_n = \gamma_n + (1/w)$. For this let $\mathcal{A}'_k$ and $\mathcal{B}'_k$ be defined like $\mathcal{A}_k$ and $\mathcal{B}_k$ respectively except with γ'_j in place of γ_j, and let $Q'_k = \mathcal{A}'_k/\mathcal{B}'_k$ for $k = -1, 0, \ldots, n$. Then $\mathcal{A}'_k = \mathcal{A}_k$ and $\mathcal{B}'_k = \mathcal{B}_k$ for $k = -1, 0, \ldots, n-1$, and we have by Theorem 1

$$Q'_n \;=\; \frac{w\mathcal{A}'_n}{w\mathcal{B}'_n} \;=\; \frac{(w\gamma_n + 1)\mathcal{A}_{n-1} + w\mathcal{A}_{n-2}}{(w\gamma_n + 1)\mathcal{B}_{n-1} + w\mathcal{B}_{n-2}} \;=\; \frac{w\mathcal{A}_n + \mathcal{A}_{n-1}}{w\mathcal{B}_n + \mathcal{B}_{n-1}} \;=\; x_0$$

by (3.4), proving the claim.

It follows that in the canonical continued fraction for x_0, $b_k = \gamma_k$ for $k = 0, 1, ..., n-1$ and $b_n = c_0 = \gamma_n + u$. Then $Q_{n-1} = A_{n-1}/B_{n-1} = \mathcal{A}_{n-1}/\mathcal{B}_{n-1}$ and $Q_n = (\mathcal{A}_n + u\mathcal{A}_{n-1})/(\mathcal{B}_n + u\mathcal{B}_{n-1})$, while $r/s = \mathcal{A}_n/\mathcal{B}_n$. Now

$$B_{n-1} \;=\; \mathcal{B}_{n-1} \leq \mathcal{B}_n < \mathcal{B}_n + u\mathcal{B}_{n-1} \;=\; B_n.$$

If $n \geq 2$, or $n = 1$ and $\gamma_1 > 1$, then $\mathcal{B}_{n-1} < \mathcal{B}_n$ and r/s cannot be any Q_k, since $s = \mathcal{B}_n$ is not equal to any B_k, using Theorem 10. If $n = \gamma_1 = 1$, then $s = 1$, a case treated at the beginning of the proof. Thus r/s is no Q_k of x_0 in any case. This completes the proof of Theorem 12, Q.E.D.

Corollary 1. *If x_0 is irrational and $|x_0 - (r/s)| < 1/(2s^2)$, where r and s are integers and $s > 0$, then $r/s = Q_n$ for some n where Q is the canonical continued fraction of x_0.*

Proof. Since $\mathcal{B}_n \geq \mathcal{B}_{n-1}$, $\theta < 1/2$ implies $\theta < \mathcal{B}_n/(\mathcal{B}_n + \mathcal{B}_{n-1})$ (9) for any n, Q.E.D.

4. Musical intervals

A vibrating string (of a piano, violin, etc.) has a basic frequency b and overtones $2b$, $3b, \ldots$. If basic frequencies b and b' have ratio $b/b' = m/n$ for small integers m and n they will have overtones in common and sound "consonant." The Pythagoreans over 2500 years ago noticed the consonance of "octaves" $b'/b = 2/1$ and "fifths" $b'/b = 3/2$. Combining these they got a scale, but to keep the scale finite, an approximation is needed somewhere because no power of 3/2 exactly equals a power of 2, for integer powers not both 0.

To divide an octave into m notes so that the ratio of frequencies of successive notes is a constant, the constant must be $2^{1/m}$. To get a good approximation of a fifth, from each note to the kth note above it, we then need to have $2^{k/m}$ approximately $3/2$. In other words, we need a good rational approximation k/m to λ defined as $\lambda = \log_2(3/2) \doteq 0.58496$. The canonical continued fraction expansion of λ has denominators B_n forming an increasing sequence of possible m's: $1, 2, 5, 12, 41, \ldots$. Choosing among these, we can see that $m = 5$ or less would give a coarse scale of too few notes. Whereas, $m = 41$ gives too many notes: that many notes crowded into a single octave of a piano would give a fine approximation of fifths, but it wouldn't be worth it. So $m = 12$ is the adopted solution: in each octave there are 7 white keys and 5 black keys, counting only one of the two ends of the octave. Pianos, for about the past century, have "equal temperament" where the octave is divided into 12 half-tones, with ratio of successive frequencies $2^{1/12}$. This results in approximations et_q of the so-called "just" consonances $q = m/n$ as follows. Here d represents the absolute error $d = |et_q - q|$ and d/q is the corresponding relative error. $\mathrm{CF}(et_q)$ is the canonical continued fraction of et_q and the last column gives a convergent equal to q.

Interval	$q = \frac{m}{n}$	et_q	d	d/q	Q_k	$\mathrm{CF}(et_q)$
Octave	2	2	0	0	Q_0	2
Fifth	3/2	$2^{7/12}$	0.0017	0.0011	Q_1	$1 + \frac{1}{2+} \frac{1}{147+} \cdots$
Fourth	4/3	$2^{5/12}$	0.0015	0.0011	Q_2	$1 + \frac{1}{2+} \frac{1}{1+} \frac{1}{73+} \cdots$
Major third	5/4	$2^{1/3}$	0.0099	0.0079	Q_2	$1 + \frac{1}{3+} \frac{1}{1+} \frac{1}{5+} \cdots$
Minor third	6/5	$2^{1/4}$	0.0107	0.0090	Q_1	$1 + \frac{1}{5+} \frac{1}{3+} \cdots$

For the fifth and fourth, the approximations by equal temperament are close enough so that the human auditory system generally accepts them as consonances. For the thirds, the approximations are generally accepted by the broad musical community, but some specialists dislike them, e.g. Sethares (1998).

Only the first two of the following problems are assigned.

PROBLEMS

1. (a) Find the canonical continued fraction of the number e far enough to evaluate $q = Q_5 = m/n$ for positive integers $m = A_5$ and $n = B_5$.

(b) Evaluate $n^2|e - q|$ and check that it's less than 1, as it should be by Theorem 11. Also check if it's less than $1/2$, so that if we had been given q in advance, we would have known q must equal Q_k for some k (Corollary 1).

(c) Let $S_k = \sum_{j=0}^{k} \frac{1}{j!}$, a sequence of rational numbers (coming from the Taylor series of e^x around 0, evaluated at $x = 1$) which converges to e rather fast. Find $S_4 = \mu/\nu$ in lowest terms for positive integers μ and ν.

(d) Find $\nu^2|e - S_4|$. In terms of this, is the approximation of e by S_4 as good as those given by canonical continued fractions (as in part (b))?

2. In the table for musical intervals, consider the next interval beyond a fifth, which would have an approximation $et_q = 2^{8/12} = 2^{2/3}$.

(a) Find q for this case (a ratio of single-digit integers). *Hint*: note that $2^{2/3} = 2/2^{1/3}$ and look at the "octave" and "major third" lines of the table.

(b) Find the canonical continued fraction of $2^{2/3}$ to enough terms and with enough accuracy to do the next part.

(c) For what k does $Q_k = q$, as in the last column of the table?

(d) Let $q = m/n$ with $m = A_k$ and $n = B_k$ integers. Find $n^2|et_q - q|$ and verify it's less than 1.

UNASSIGNED PROBLEMS

You can do Problem 3 for extra credit.

3. In the canonical continued fraction expansion of $\sqrt{2}$,

(a) Show that $b_k = 2$ for all $k \geq 1$.

(b) Show that $A_{k+1} \equiv 5A_{k-1} + 2A_{k_2}$ and $B_{k+1} \equiv 5B_{k-1} + 2B_{k-2}$ for all $k \geq 0$.

4. For the same continued fraction:

(a) Find constants $K > 1$, C, D, E, F such that for all n, $A_n = CK^n + D(-K)^{-n}$ and $B_n = EK^n + F(-K)^{-n}$.

(b) Show that $B_n^2|(A_n/B_n) - \sqrt{2}|$ is never 0 and converges to a non-zero limit ζ as $n \to +\infty$. Find ζ.

(c) Prove that for any sequences m_k and n_k of positive integers and $q_k = m_k/n_k$, $n_k^2|q_k - \sqrt{2}|$ cannot approach 0 as $k \to +\infty$. So, the order of approximation of irrational numbers by rationals given by Theorem 11 can't be improved except by some constant factor.

5. Find the smallest possible value of $n^2|(m/n) - \sqrt{2}|$ for any positive integers m and n.

Notes. Authors mentioned, whose works don't appear in the following short list, are cited in one or more of the references given. Sections 2 and 3 were based mainly on Perron (1929).

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