18.466 Problem set 3, due Monday, March 11, 2013

1. Let X have a binomial (n, p) distribution. For $\psi = p$, let the parameter space be $\Psi = (0, 1)$, $0 < p < 1$. Recall that as mentioned in a lecture Feb. 22(?), or note, that we get an exponential family with $\theta_1(p) = \log(p/(1-p))$ and $T_1(X) = X$. Recall, or note, also, that the MLE of p is X/n.
(a) Find p as a function of $\theta = \theta_1(p)$.
(b) Find the MLE of θ, when it exists, by exponential family methods, and show that this agrees with the usual MLE of p.
(c) Find the probability, as a function of p, that the MLE of θ exists in the natural parameter space of the family.

2. Background: an item in a cargo shipment may or may not contain a radioactive element. If it does, assume that the number X of emitted radioactive particles detected by a detector in a given time at a given distance from the detector has a Poisson distribution with some parameter $\lambda > 0$. If $X = 0$, it may be either that the item does not contain the radioactive element, or that it does, but that the Poisson(λ) variable X happens to equal 0, as it will with probability $e^{-\lambda}$.

The problem: find an unbiased estimate of $e^{-\lambda}$ given one observation Y of a variable Poisson with parameter λ, conditional on $Y > 0$. Is it unique? What properties does it have? *Hint*: Write an equation for a function $a(Y)$, expressible as a sequence of numbers, to be an unbiased estimator of $e^{-\lambda}$. When two power series converge to the same function, what can one say about their coefficients?

3. In the same situation, for a given Y, show that a unique maximum likelihood estimate of $e^{-\lambda}$ exists, but you need not find it in closed form. Say what you can about whether this estimate is reasonable. *Hints*: The case $Y = 1$ is special, so treat that case and $Y \geq 2$ separately. By definition of MLE for functions of a parameter, if $\hat{\lambda}$ is the MLE of λ, then $\exp(-\hat{\lambda})$ is the MLE of $e^{-\lambda}$, where in this case the function $\lambda \mapsto e^{\lambda}$ is one-to-one, so this is just a reparameterization of the family. What are expected monotonicity relations between Y, an estimate of λ, and an estimate of $e^{-\lambda}$, which may have been violated in problem 1. Do they hold for the MLE solution?

4. Let $X_1, ..., X_n$ be i.i.d. $N(0, \sigma^2)$ where σ^2 is unknown, $0 < \sigma < +\infty$.
(a) Find a sufficient statistic for σ^2.
(b) What is an MLE for σ^2?
(c) What is an unbiased estimate?
(d) Do the results of (b) and (c) agree?
(e) The usual method of moments estimation doesn't apply to estimating σ^2 in this case, why?
(f) To minimize mean-square error in estimating σ^2 when μ is unknown, one uses $V(X) = (1/(n+1)) \sum_{j=1}^{n} (X_j - \overline{X})^2$. Compare its mean-square error to those of the estimates found in (a) or (b).
(g) For μ unknown, minimizing mean-square error required a different factor than for other purposes (unbiasedness, MLE). For $\mu = 0$ known, is another factor preferable to the one(s) found in (b) and (c)?

5. Consider the family of mixtures of two normal distributions, having densities of the form

$$f(x,\theta) \;=\; \frac{\lambda}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)+\frac{1-\lambda}{\sqrt{2\pi}\sigma_2}\exp\left(-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right)$$

where $\theta = (\lambda, \mu_1, \sigma_1, \mu_2, \sigma_2)$ is a 5-dimensional parameter with μ_1 and μ_2 any real numbers, $0 < \sigma_j < \infty$ for $j = 1, 2$, and $0 < \lambda \leq 1/2$. Suppose given n observations $X_1, ..., X_n$, not all equal, assumed to be i.i.d. from such a distribution. If a value θ' of a parameter is such that as θ approaches θ' (possibly under some restrictions), the likelihood approaches $+\infty$, then we may consider θ' as *a* maximum likelihood estimate (MLE) of θ, or *the* MLE if it's unique.

(a) For the given family of normal mixture densities, do there exist such θ'? Are they unique? *Hint*: the exponential of a nonpositive number is at most 1, so the likelihood can only approach $+\infty$ if at least one of the σ_j approaches 0. But if say σ_1 approaches 0, then $\exp(-(X_j - \mu_1)^2/(2\sigma_1^2))$ will approach 0 very fast if μ_1 is fixed and unequal to X_j. In the likelihood function the X_j are fixed and the parameters are free to vary, so for what value(s) of μ_1 would we get large likelihood as $\sigma_1 \downarrow 0$?

(b) Is maximum likelihood a good method of estimating the parameters in this case?