SEMI-LINEAR DIFFRACTION OF CONORMAL WAVES

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Abstract. We study the conormal regularity of bounded solutions to semi-linear strictly hyperbolic equations on domains with diffractive boundaries:

$$Pu = f(x,u) \text{ in } X,\ \ u\restriction_{\partial X} = 0,\ \ u \in L^\infty_{\text{loc}}(X).$$

If $X_- \subset X$ and X is the domain of influence of X_- we consider solutions such that $\operatorname{singsupp}(u) \cap X_- \cap \partial X = \emptyset$ and further suppose that $u\restriction_{X_-}$ is conormal with respect to a smooth characteristic hypersurface, the incoming front.

For the linear equation, $f \equiv 0$, the singular support of u is contained in the incoming front and the reflected front obtained using the rules of geometrical optics; these two characteristic surfaces are tangent at the glancing set, the locus of points at which the incoming bicharacteristics are tangent to the boundary. We prove that in the semi-linear case the only new singularites which may occur appear on the characteristic half-cone over the glancing set. The actual conormal regularity result presented in the paper is considerably more precise.

Our assumptions are best illustrated by taking for P the constant coefficient wave equation with X the product of $\mathbb{R}_t$ and the exterior of a strictly convex obstacle. Then $X_- = X \cap \{t < -T\}$ and for the initial data one can take locally an anti-derivative of the plane wave $\delta(t - \langle x, \omega\rangle)$ with T appropriately large. The geometry of this problem in two space dimensions is shown in Figures 1 and 2.

Résumé. Nous étudions la régularité conormale de solutions bornées d'équations semi-linéaires strictement hyperboliques dans des domaines à bord diffractif:

$$Pu = f(x,u) \text{ dans } X,\ \ u\restriction_{\partial X} = 0,\ \ u \in L^\infty_{\text{loc}}(X).$$

Si $X_- \subset X$ et X est le domaine d'influence de X_-, nous considérons des solutions u telles que $\operatorname{singsupp}(u) \cap X_- \cap \partial X = \emptyset$; de plus nous supposons que $u\restriction_{X_-}$ est conormale par rapport à une hypersurface caractéristique lisse, le front entrant.

Dans le cas de l'équation linéaire $f \equiv 0$, le support singulier de u est contenu dans la réunion du front entrant et du front réfléchi obtenu par les lois de l'optique géométrique. Ces deux surfaces caractéristiques sont tangentes à l'ensemble des rayons rasants, le lieu des points où les bicaractéristiques entrantes sont tangentes au bord. Dans le cas semi-linéaire, nous démontrons que si de nouvelles singularités apparaissent alors elles apparaissent sur le demi-cône caractéristique au-dessus de l'ensemble des rayons rasants. En fait, le théorème de régularité conormale établi dans cet article est beaucoup plus précis.

Pour illustrer notre propos, nous choisirons pour P l'opérateur des ondes à coefficients constants et pour X le produit de $\mathbb{R}_t$ et de l'extérieur d'un obstacle strictement convexe. Alors $X_- = X \cap \{t < -T\}$. Comme donnée initiale, on pourra prendre une primitive locale de l'onde plane $\delta(t - \langle x, \omega\rangle)$ avec T suffisamment grand. La géométrie de ce problème est figurée sur les schémas 1 et 2.

Contents

1. Introduction and statement of results

The purpose of this paper is to describe the conormal regularity for a class of mixed problems for the semi-linear hyperbolic equations.

The study of $\mathcal{C}^\infty$ regularity of solutions to non-linear wave equations has had two main directions: finding estimates on the strength of the anomalous singularities, i.e. those not present in the linear interaction, and obtaining geometric restrictions on the location of singularities. Our work is of the latter type. The strength of singularities for non-linear mixed problems has already been investigated with considerable success in [45, 10, 21, 48]. The estimates on the location of singularities are much finer, so stronger assumptions are needed on the incoming waves or the initial data. The most striking example of this was provided by [2] where it is shown that wave-front set restrictions alone still allow the self-spreading of singularities, making the singular support propagate essentially in the same way as the support of the solution. Thus, in full generality, the location of singularities cannot be related to the original geometry except in a trivial way. A technically more challenging construction of a similar example for gliding mixed problems was then given in [47].

The appropriate class of distributions to consider for the incoming waves or the initial data are the *conormal distributions*, as was first noted in [6]. The conormal distributions appear naturally in the linear theory and are a subclass of the *Lagrangian distributions* motivated by geometrical optics. The interaction of conormal waves for interior problems has been investigated in [40, 32, 7, 9, 3, 42, 34] and the formation of non-linear caustics in [18, 19, 11, 27, 43, 44]. For mixed problems, with only transversal reflections allowed, it was shown in [4, 5] that no anomalous singularities appear. One should also mention that examples of 'new' non-linear singularities were provided at an early stage in [39]: namely, the interaction of three plane waves carrying conormal singularities produces a conic surface of new singularities propagating from the triple interaction point. However, in more complicated settings such as the propagation of the swallowtail or diffraction, where the 'new' cones are expected, no examples have yet been constructed. For the interior problems the methods developed in [20] provide a systematic approach to such constructions. Energy estimates used in the work on the lifespan of solutions to semi-linear hyperbolic equations [15, 22] are also, in essence, of conormal type.

If $\Sigma \subset X$ is a C^∞ hypersurface in a C^∞ manifold X, let $\mathcal{V}(X,\Sigma)$ be the Lie algebra of C^∞ vector fields in X tangent to Σ. The space of distributions of finite L^2-based conormal regularity with respect to Σ is then defined by the stability of regularity under the applications of the elements of $\mathcal{V}(X,\Sigma)$:

$$I_k L^2_{\text{loc}}(X,\Sigma) = \{u \in L^2_{\text{loc}}(X) \,:\, V_1 \cdots V_l u \in L^2_{\text{loc}}(X) \text{ for } \; l \le k \text{ and } \; V_i \in \mathcal{V}(X,\Sigma)\}$$

This modifies the definition of the Sobolev space $H_{(k)}$ by placing some geometric restrictions on the differentiations. Nevertheless, as observed in [32], bounded conormal functions have very good multiplicative properties in view of Gagliardo-Nirenberg type inequalities.

Let us now consider a mixed hyperbolic problem with a diffractive boundary (see Sect.2 for a review of definitions). Our object of study is the semi-linear equation:

$$Pu = f(x,u) \text{ in } \; X, \; u|_{\partial X} = 0, \; u|_{X_-} = u_0 \tag{1.1}$$

where f is a C^∞ function of its arguments, P is a strictly hyperbolic operator, X is a C^∞ manifold with the boundary ∂X, $X_- = \{x \in X : \phi(x) < -T\}$ with $\phi \in C^\infty(X)$ a time function for P and the time T fixed.

The initial data is assumed to be conormal to the *incident front* F. The reflection rule of geometrical optics produces the *reflected front* R. With the motivation coming again from the geometrical optics we define the *shadow boundary* on ∂X as

$$\Gamma = \partial X \cap \text{cl}[R \cap F \setminus \partial X]$$

The front obtained from the nonlinear interaction is the *forward half-cone*, S_+, of $P-$bicharacteristics starting on Γ. Let us also denote by D_+ and B_+ the two components of the set of glancing characteristics on S_+. A more detailed discussion of the fronts is presented in Sect.2. Fig.1 shows three different time slices and Fig.2 is a space-time picture. Note that R and F are hypersurfaces with singular boundaries.

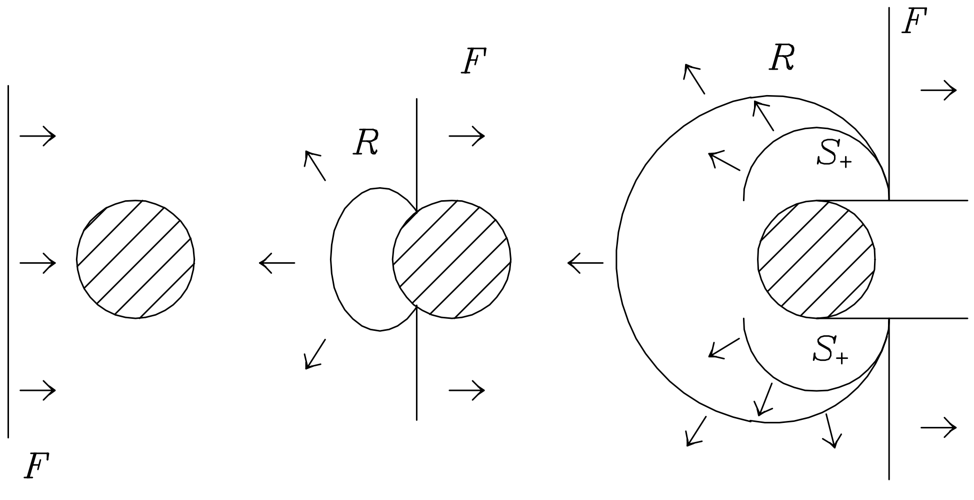

FIGURE 1. The fronts projected to the space variables at fixed times

The crudest form of our result is

Theorem 1.1. *Let* $u \in L^\infty(X)$ *be a bounded solution of* (1.1) *with*

$$u_0 \in I_\infty L^2_{loc}(X_-; F)$$

Then

$$WF_b(u) \subset {}^bN^*R \cup {}^bN^*F \cup {}^bN^*S_+ \cup {}^bN^*B_+ \cup {}^bN^*D_+ \cup {}^bT^*_\Gamma X \setminus 0$$

We refer the reader to [25] and [14], Sect.18.3 for the definition of the b-wave front set, WF_b, which reduces to the ordinary WF away from the boundary ∂X. We use the natural map $\jmath : T^*X \setminus 0 \to {}^bT^*X \setminus 0$ (see Sect.4 and the references given above) to define ${}^bN^*\Sigma = \jmath(N^*\Sigma)$.

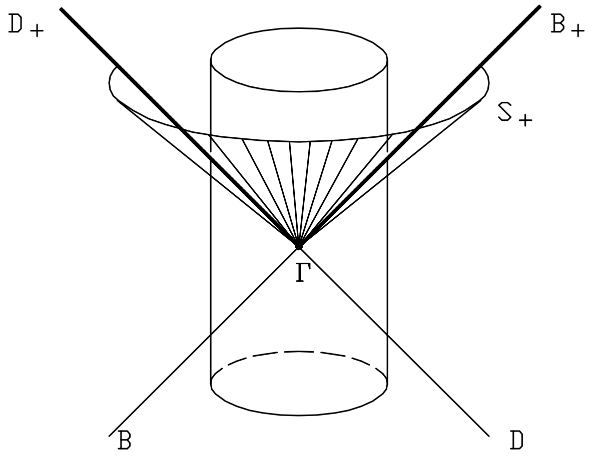

FIGURE 2. The forward half-cone and the glancing boundaries B and D.

Theorem 1.1 immediately gives the singular support statement:

Corollary 1.1. *Under the assumptions of Theorem 1.1*

$$\text{sing supp } u \subset F \cup R \cup S_+$$

Since the data u_0 is conormal, one would like to describe precisely the conormal regularity of the solution u. In fact the proof is based on the construction of an appropriate space with good multiplicative and propagative properties – see Sect.3. Since the precise definition of this 'strong', but not quite conormal, space is rather involved we shall content ourselves with a weaker statement here, referring the reader to Definition 3.2 and Theorem 8.2 for the full result.

Theorem 1.2. *Let $u \in L^\infty_{loc}(X)$ be a bounded solution of* (1.1) *with*

$$u_0 \in I_k L^2_{loc}(X_-; F)$$

If Ω is an open subset of X such that

$$\Omega \cap (D_+ \cup B_+) = \emptyset$$

then

$$u|_\Omega \ \in \ I_k L^2_{loc}(\Omega, F) + I_k L^2_{loc}(\Omega, R) + I_k L^2_{loc}(\Omega, S_+)$$

Already in the transversal case this is slightly stronger than the result in [4] as conormal singularities with respect to the boundary are excluded.

Our conclusions are concerned purely with the L^2-based regularity. The present existence theory [45] requires higher Sobolev regularity for u_0 to guarantee local existence of bounded solutions, so one needs to assume $u_0 \in I_k L^2_{\text{loc}}(X_-; F) \cap H_{(s)}(X_-)$ for $s > n/2$. However, the conormal results described above should lead to an improvement in the style of [41]. It should be noted that our present method does not treat the fully semi-linear equation $Pu = f(x, u, \nabla u)$, essentially because the iteration procedure in k proceeds in steps of $1/2$.

Acknowledgements

The last two authors would like to thank J.-M. Delort, G. Lebeau and J. Sjöstrand for a number of stimulating discussions, their interest and encouragement. The research of the three authors was partially supported by the National Science Foundation and the last two are Alfred P. Sloan Research Fellows. The third author is also grateful to University of Toronto for the hospitality in the summer of 1992.

2. Diffractive geometry

First we describe the interaction of a characteristic hypersurface for a second-order hyperbolic operator with a bicharacteristically concave (diffractive) boundary. In particular the reflected front is shown to have a cusp singularity when continued across the boundary.

Let X be a manifold with boundary equipped with a pseudo-Riemannian metric of hyperbolic signature, $+,-,-,-\dots$. The metric symbol $p \in S^2(T^*X)$ is therefore a polynomial of degree two on each fibre and it can be reduced, in linear coordinates in each fibre, to

$$\tau^2 - \xi_1^2 - \cdots - \xi_n^2, \quad \dim X = n+1.$$

The boundary of X is said to be time-like if p is negative-definite on $N^*\partial X$; this is always assumed below. It will be convenient to assume that X is time oriented; this amounts to the continuous selection of one of the solid cones, $p > 0$, in the fibres. A function $t \in C^\infty(X)$ is a time-function if $p(dt) > 0$.

The assumption that ∂X is time-like means that it carries an induced pseudo-Riemannian metric of hyperbolic signature. If g is the dual quadratic form to p, on TX, then $g_\partial = g\restriction_{T\partial X}$ fixes the induced structure. Let p_∂ denote the metric symbol on $T^*\partial X$. Set

$$\begin{aligned} \mathcal{H} &= \{p_\partial > 0\} = \mathcal{H}_+ \cup \mathcal{H}_- \\ \mathcal{G} &= \{p_\partial = 0\} \qquad\qquad \text{in } T^*\partial X\backslash 0 \\ \mathcal{E} &= \{p_\partial < 0\}, \end{aligned}$$

respectively the hyperbolic, glancing and elliptic regions of $T^*\partial X\backslash 0$. The time-orientation of X induces a time-orientation of ∂X, giving the decomposition of the hyperbolic region.

The restriction to the boundary of the characteristic variety

$$\Sigma = \{p = 0\} \subset T^*X\backslash 0$$

projects onto $\mathcal{H} \cup \mathcal{G} = \bar{\mathcal{H}}$:

$$\Sigma_\partial = \Sigma \cap T^*_{\partial X}X \xrightarrow{\iota^*} \mathcal{H} \cup \mathcal{G}. \tag{2.1}$$

Here $\iota^* : T^*_{\partial X}X \longrightarrow T^*\partial X$ is the pull-back map induced by the inclusion $\partial X \overset{\iota}{\hookrightarrow} X$.

The map (2.1) is a fold, $2-1$ over $\mathcal{H}$ and $1-1$ over $\mathcal{G}$. To see this, note that $\iota^* : T^*_{\partial X}X \longrightarrow T^*\partial X$ is the quotient map with respect to $N^*\partial X$. The restriction of p to each fibre of ι^* is of this form

$$p = \iota^* p_\partial - n^2 \tag{2.2}$$

where $n : T^*_{\partial X}X \longrightarrow \mathbb{R}$ is the symbol of the *inward-pointing vector field*, V, satisfying $g(V,\cdot) = 0$ on $T\partial X$, $g(V,V) = -1$. Thus $p_\partial \geq 0$ on the range of ι^* on Σ_∂ and the involution exchanging the points in Σ_∂ with the same image satisfies

$$I_\partial : \Sigma_\partial \longleftrightarrow \Sigma_\partial,\ \ I^*_\partial n = -n,\ \ \iota^* \cdot I_\partial = \iota^*. \tag{2.3}$$

Let $\tilde{\mathcal{H}}_\pm, \tilde{\mathcal{G}}$ be the preimages of $\mathcal{H}_\pm$ and $\mathcal{G}$ under ι^*, so in particular $\iota^* : \tilde{\mathcal{G}} \longleftrightarrow \mathcal{G}$ is an isomorphism.

The projection ι^* can be expressed symplectically. Let $x \in C^\infty(X)$ be a defining function for ∂X and let $q \in C^\infty(T^*X)$ be its lift to T^*X. Then the Hamilton vector field H_q satisfies $H_q q = 0$, i.e. H_q is tangent to the leaves of $\iota^* : T^*_{\partial X}X \longrightarrow T^*\partial X$. Since the leaves are tangent to Σ exactly at $\tilde{\mathcal{G}}$,

$$\tilde{\mathcal{G}} = \{p = 0,\ q = 0,\ \{p,q\} = 0\}$$

where $\{p,q\} = H_p q = -H_q p$ is the Poisson bracket. The simple tangency of H_q to Σ_∂, corresponding to the fact that (2.1) is a fold, is expressed by

$$\{q,\{q,p\}\} < 0.$$

This holds throughout $T^*_{\partial X}X$, since ∂X is time-like. Applying H_q to both sides of (2.2) and noting that $\{q,n\}$ is the lift of a function from the base, so $\{q,\{q,n\}\} \equiv 0$

$$\begin{aligned}\{q,p\} &= -2n\{q,n\}\\ \{q,\{q,p\}\} &= -2\{q,n\}^2.\end{aligned}$$

Thus (2.2) can be written in terms of Poisson brackets

$$p = \iota^* p_\partial + \frac{\{q,p\}^2}{2\{q,\{q,p\}\}}. \tag{2.4}$$

The denominator $\{q,\{q,p\}\}$ is also the lift of a function from the base. Thus, if the involution is extended to $T^*_{\partial X}X$ so that the second two conditions in (2.3) hold, then

$$I^*_\partial\{q,p\} = -\{q,p\}. \tag{2.5}$$

The points of $\mathcal{G}$ are further distinguished by the behaviour of the second Poisson bracket:

$$\begin{aligned}\tilde{\mathcal{G}}_d &= \{m \in \tilde{\mathcal{G}}; H_p^2 q(m) > 0\}\\ \tilde{\mathcal{G}}_h &= \{m \in \tilde{\mathcal{G}}; H_p^2 q(m) = 0\}\\ \tilde{\mathcal{G}}_g &= \{m \in \tilde{\mathcal{G}}; H_p^2 q(m) < 0\}\end{aligned} \tag{2.6}$$

These, and similarly their images in $\mathcal{G}$ under ι^*, are respectively the sets of diffractive, higher-order and gliding points. The boundary of X is said to be diffractive (or bicharacteristically concave) if $\mathcal{G} = \mathcal{G}_\lceil$; this is always assumed below.

Consider the differential of the involution I_∂ on Σ_∂, at $\mathcal{G}$. In particular note that the type of a glancing point, in the sense of (2.6), is reflected in:

$$(I_\partial)_* H_p - H_p = 2\frac{\{p,\{p,q\}\}}{\{q,\{q,p\}\}} H_q \tag{2.7}$$

In fact, since $\iota^* \cdot I_\partial = I_\partial$, the projection of $(I_\partial)_* H_p$ under ι^* is equal to the projection of H_p. The null space of the projection is spanned by H_q, so it is only necessary to compute the coefficient on the right. Applying both sides to $\{q,p\}$ and using (2.5) gives (2.7).

We will be concerned with the local geometry near a base point $x_0 \in \partial X$, so we are free to shrink X as necessary. In this sense the assumption that the boundary is diffractive is really that $\mathcal{G} \cap \mathcal{T}^*_{\S,}\partial\mathcal{X} \subset \mathcal{G}_{\lceil}$. In case $X = \mathbb{R} \times Y$ carries a product metric, $g = dt^2 - h$, with h a Riemannian metric on Y, the boundary is diffractive if and only if ∂Y is strictly geodesically concave. In case $Y = \mathbb{R}^n \backslash K$ where K is an open, smoothly bounded region and h is the Euclidean metric this is equivalent to the strict convexity of K (cf. [26]).

It is convenient to consider an extension, $\widetilde{X}$, of X to a manifold without boundary. A corresponding extension of this pseudo-Riemannian structure will be denoted $\tilde{p}$. The defining function $x \in C^\infty(X)$ extends to $\tilde{x} \in C^\infty(\widetilde{X})$ and if $\widetilde{X}$ is chosen small enough, $\partial X = \{\tilde{x} = 0\}$ is an embedded hypersurface. The freedom to shrink X will be used to choose $\widetilde{X}$ to be bicharacteristically convex.

In $\widetilde{X}$ we consider a closed characteristic hypersurface for $\tilde{p}$, passing through this point x_0. Thus $\widetilde{F} \subset \widetilde{X}$ satisfies

$$\widetilde{F} = \{\tilde{f} = 0\},\ \tilde{f} \in C^\infty(\widetilde{X}),\ d\tilde{f} \neq 0 \text{ on } \widetilde{F},\ \tilde{p}|_{N^*\widetilde{F}} = 0. \tag{2.8}$$

Since ∂X is time-like, $N^*\partial X$ and $N^*\widetilde{F}$ are linearly independent and hence

$$\widetilde{F}_\partial = \widetilde{F} \cup \partial X \hookrightarrow \partial X$$

is an embedded hypersurface. Since $N^*\widetilde{F}_\partial = \iota^*(N^*_{\partial X}\widetilde{F})$, with $N^*_{\partial X}\widetilde{F} \subset \Sigma_\partial$, we have $N^*\widetilde{F}_\partial \subset \bar{\mathcal{H}} = \mathcal{H} \cup \mathcal{G}$. For us the most interesting points are the diffractive points for $\widetilde{F}$:

Lemma 2.1. *If $\widetilde{F} \subset \widetilde{X}$ is an embedded characteristic hypersurface then*

$$\Gamma = \pi\left(N^*\widetilde{F}_\partial \cap \mathcal{G}_\lceil\right) \subset \widetilde{F}_\partial \tag{2.9}$$

is an embedded hypersurface.

Proof. Consider the function $\tilde{\gamma} = \{p, q\}$ restricted to $N^*\widetilde{F}$. By assumption $N^*\widetilde{F} \subset \Sigma$ so H_p is tangent to it. At any point of $\tilde{\mathcal{G}}_d \cap N^*_{\partial X}\widetilde{F}$, $\{p, q\} = 0$ and $\{p, \{p, q\}\} > 0$, by definition. Thus $H_p\{p, q\} \neq 0$ and so $d\tilde{\gamma} \neq 0$ on $N^*\widetilde{F}$. Since $H_p q = \{p, q\} = 0$ on $\tilde{\mathcal{G}}_d$, q and $\{p, q\}$ have independent differentials on $N^*\widetilde{F}$ at $\tilde{\mathcal{G}}_d \cap N^*_{\partial X}\widetilde{F}$. It follows that

$$N^*_{\partial X}\widetilde{F} \cap \mathcal{G}_\lceil \subset \mathcal{N}^*_{\partial \mathcal{X}}\widetilde{\mathcal{F}}$$

is an embedded hypersurface. Since it is homogeneous it projects to the embedded hypersurface Γ. □

The characteristic hypersurface $\widetilde{F}$ is to be thought of as the extension through the boundary of X of the incident front. It is important to separate which parts of $\widetilde{F}$ are intrinsic and which depend on the choice of extension—the latter being necessarily irrelevant to the final form of the results.

By assumption $N^*\widetilde{F}$ is closed, so

$$\Lambda_F \stackrel{\text{def}}{=} N^*\widetilde{F}\backslash 0$$

is the union of the maximally extended bicharacteristic intervals, i.e. integral curves of H_p, through each of its points. Set

$$F = \{z \in \widetilde{F} \cap X;\ \text{the bicharacteristics through } N^*_z\widetilde{F} \text{ stay in } T^*X \text{ for } t \leq t(z)\}.$$

Here, t is a time function. The submanifold $\Gamma \subset F$ is the singular locus in F near which it is not even a manifold with corners. Indeed the boundary of F consists of two smooth manifolds with boundary (each of codimension two in X)

$$\partial F = F_\partial \cup B,\ F_\partial \cap B = \partial F_\partial = \partial B = B \cap \partial X = \Gamma. \tag{2.10}$$

Here F_∂ is half of $\widetilde{F}_\partial$ and B, the shadow boundary, is the projection into X of the forward half-bicharacteristic starting at points of $N^*_\Gamma \widetilde{F}$.

The main objective of this section is to consider the reflected front generated by $\widetilde{F}$ and ∂X. To do so we need to recall the notion of a hypersurface with cusp singularity. By definition a cusp hypersurface is one which is diffeomorphic to $C = \{x_2^3 = x_1^2\}$ in $\mathbb{R}^n$, $n \geq 2$.

A simple characterization can be obtained in terms of the closure of the conormal bundle to the regular part of the hypersurface. As is easily checked

$$\Lambda_C = \operatorname{cl} N^*\{x_2^3 = x_1^2; x_2 > 0\} \subset T^*\mathbb{R}^n \backslash 0$$

is a smooth, homogeneous Lagrangian. Now a point of the singular locus, $L = \{x_1 = x_2 = 0\}$,

$$\begin{gathered}\pi\colon \Lambda_C \longrightarrow \mathbb{R}^n \text{ has differential with} \\ \text{two-dimensional null space at } \Lambda_C \cap T^*_l\mathbb{R}^n, l \in L.\end{gathered} \tag{2.11}$$

Moreover, any vector field V on $T^*\mathbb{R}^n$ which is tangent to $T^*_l\mathbb{R}^n$, independent of the radial vector field and takes the value $v \in T_m\Lambda_C \cap T_m(T^*_l\mathbb{R}^n)$ at m is only simply tangent to Λ_C at m. Conversely (see Arnol'd [1]) if these two conditions hold for Λ_C near $m \in T^*_l\mathbb{R}^n \cap \Lambda_C$ then the projection of a neighbourhood of $m \in \Lambda_C$ is a cusp. We use this abstract characterization, with $\mathbb{R}^n$ replaced by $\widetilde{X}$ (as can obviously be done) to analyze the reflected front.

Set $\widetilde{\Lambda}^0_R = I_\partial(N^*_{\partial X}\widetilde{F})$ and let $\widetilde{\Lambda}_R$ be the H_p-flow-out in $T^*\widetilde{X}\backslash 0$ of $\widetilde{\Lambda}^0_R$. Thus $\widetilde{\Lambda}_R$ is just the union of the maximally extended H_p integral curves passing through points of $\widetilde{\Lambda}^0_R$.

Proposition 2.1. *If $\widetilde{F} \subset \widetilde{X}$ is a smooth characteristic hypersurface for which $x_0 \in \partial X$ is a diffractive point then, for $\widetilde{X}$ shrunk to a sufficiently small bicharacteristically convex neighbourhood of x_0, $\widetilde{\Lambda}_R \subset T^*\widetilde{X}\backslash 0$ is a smooth closed conic Lagrangian submanifold which is the closure of the conormal bundle to a hypersurface with cusp singularity, $\widetilde{R}$, through x_0.*

Proof. To see that $\widetilde{\Lambda}_R$ is smooth it suffices to observe that H_p is not tangent to the initial surface $\widetilde{\Lambda}^0_R$. By definition $\widetilde{\Lambda}^0_R$ is the image under I_∂ of $N^*_{\partial X}\widetilde{F}$ and $N^*_{x_0}\widetilde{F}\backslash 0 \subset \mathcal{G}_\Gamma$, by assumption. Thus H_p is tangent to $N^*_{\partial X}\widetilde{F}$ at $N^*_{x_0}\widetilde{F}\backslash 0$ and hence $(I_\partial)_*H_p$ is tangent to $\widetilde{\Lambda}^0_R$ at $N^*_{x_0}\widetilde{F}$. Now H_q cannot be tangent to $N^*_{\partial X}\widetilde{F}\backslash 0$ (since this would mean $\widetilde{F}$ was tangent to ∂X) and hence cannot be tangent to $\widetilde{\Lambda}^0_R$ at $N^*_{x_0}\widetilde{F}$. From (2.7) it follows that H_p is *not* tangent to $\widetilde{\Lambda}^0_R$, so $\widetilde{\Lambda}_R \subset T^*\widetilde{X}\backslash 0$ is smooth, closed and conic if $\widetilde{X}$ is chosen small enough. It is also invariant under reflection in the fibres.

This discussion shows that both $(I_\partial)_*H_p$ and H_p are tangent to $\widetilde{\Lambda}_R$ at $N^*_{x_0}\widetilde{F}$, hence so is H_q. Since H_q is tangent to the fibres of $T^*\widetilde{X}$ and is non-radial at $N^*_{x_0}\widetilde{F}\backslash 0$ it follows that the differential of the projection $\pi : \widetilde{\Lambda}_R \longrightarrow \widetilde{X}$ has null space of dimension of at least two. In fact $\pi_* : T_m\widetilde{\Lambda}_R \longrightarrow T_{\pi(m)}\widetilde{X}$ has rank exactly $\dim \widetilde{X} - 2$, since

$\pi_*: T_m\widetilde{\Lambda}_R^0 \longrightarrow T_{\pi(m)}\partial X$ has rank $\dim \partial X - 1$ at $m \in N^*_{x_0}\widetilde{F}\backslash 0$. Finally note that H_q is only simply tangent to $\widetilde{\Lambda}_R$ at $N^*_{x_0}\widetilde{F}\backslash 0$ since it is only simply tangent to Σ. Thus $\widetilde{\Lambda}_R$ is the closure of the conormal bundle of a hypersurface with cusp singularity $\square$

Clearly the cusp locus $L \subset \widetilde{R}$ passes through Γ. It is important to check that

$$L\backslash\Gamma \subset \widetilde{X}\backslash X \text{ and } L \text{ is simply tangent to } \partial X \text{ at } \Gamma. \tag{2.12}$$

Since the tangent space to L is just the image of the tangent space to $\widetilde{\Lambda}_R$ under the projection, L is certainly tangent to ∂X at Γ. The reflected front $\widetilde{R} \cap X^\circ$ is smooth so the inclusion follows. To see the simple tangency we first choose coordinates so that $\widetilde{X} \subset \mathbb{R}^3_x \times \mathbb{R}^{n-3}_y$ and

$$\widetilde{R} = \{(x,y) : x_3^2 = x_2^3\} \cap \widetilde{X}, \quad \Gamma = \{(x,y) : y = 0\} \cap \widetilde{X}.$$

As in Proposition 2.1 of [34] it then follows that

$$p = a(4\xi_2^2 - 9x_2\xi_3^2 + \xi_1 p_1 + \langle \eta, \tilde{p} \rangle), \ a \neq 0. \tag{2.13}$$

The cusp locus is given by $L = \{x_2 = x_3 = 0\} \cap \widetilde{X}$ and if $\partial X = \{\rho = 0\}$, $X = \{\rho > 0\}$ then the tangency of L to ∂X at Γ implies

$$\partial_{x_1}\rho(0,y) = 0. \tag{2.14}$$

The points $m = (0, y; (0,0,1), 0) \in T^*\widetilde{X} \setminus 0$ are diffractive:

$$p(m) = \rho(m) = \{p, \rho\}(m) = 0, \quad \{p, \{p, \rho\}\}(m) > 0.$$

Hence (2.13) and (2.14) show that

$$\partial_{x_2}\rho(0,y) > 0, \tag{2.15}$$

and consequently

$$\rho(x,y) = x_2 - g(x_1, x_3, y), \quad \partial_{x_1} g(0,y) = g(0,y) = 0.$$

Since $L \setminus \Gamma \subset \{\rho < 0\}$ we also see that $g(x_1, 0, y) \geq 0$ and thus we can write

$$g(x_1, x_3, y) = g_1(x_1, y)x_1^2 + x_3 g_2(x_1, x_3, y). \tag{2.16}$$

The restriction of $\widetilde{R}$ to ∂X is given by

$$\mathrm{cl}(\partial X \cap \widetilde{R} \setminus \widetilde{F}) \cup (\partial X \cap \widetilde{F}) = \widetilde{R} \cap \partial X = \{(x,y) \in \partial X : g(x_1, x_3, y)^3 = x_3^2\},$$

and using (2.16) we see that

$$\begin{aligned}
\widetilde{F} \cap \partial X &= \{(x,y) : x_3 = G(x_1,x_3,y) g_1(x_1,y)^3 + F(x_1,x_3,y) g_1(x_1,y)^4, \\
&\quad\ x_2 = g(x_1,x_3,y)\}, \ G(0,y) \neq 0, \\
\mathrm{cl}(\widetilde{R} \setminus \widetilde{F}) \cap \partial &= \{(x,y) : x_3 = -G(x_1,x_3,y) g_1(x_1,y)^3 + F(x_1,x_3,y) g_1(x_1,y)^4 \\
&\quad\ x_2 = g(x_1,x_3,y)\}.
\end{aligned}$$

The Lagrangian $N^*(\widetilde{F} \cap \partial X)$ is simply tangent to $\mathcal{G} \subset T^*\partial X \setminus 0$ and since $N^*(\mathrm{cl}(\widetilde{R} \setminus \widetilde{F}) \cap \partial X)$ is related to it by the billiard ball map (see (2.19)) below the two Lagrangians are simply tangent at $N^*(\widetilde{F} \cap \partial X) \cap \mathcal{G}$ (see the equivalent model case in Sect.7). Hence $\partial_{x_1} g_1(0,y) \neq 0$ and thus in view of (2.16), L is simply tangent to $\partial X = \{x_2 = g(x_1, x_3, y)\}$.

In the case of the wave equation in the exterior of a convex obstacle Proposition 2.1 was given in [49]. In that case the cusp locus L projected to the space variables is the envelope of the reflected rays, see Fig.3.

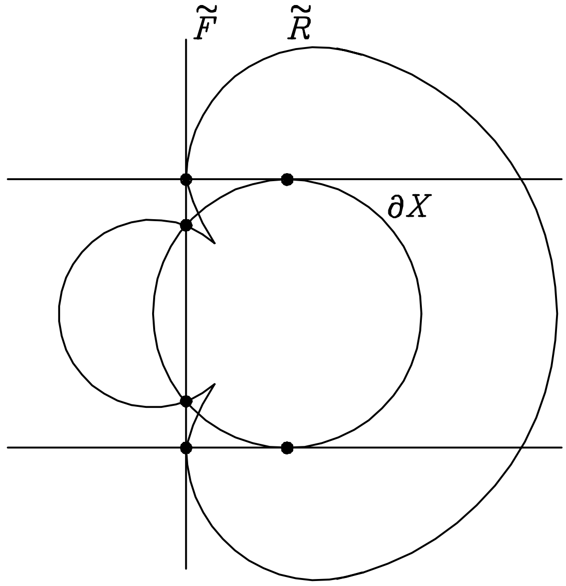

FIGURE 3. The extended reflected front projected to the space variables at a fixed time.

In the remainder of this section we review the geometry of the fronts. With the possible exception of Proposition 2.4 b) (see [34]) all the facts are essentially well known and are implicit in the proofs in Sect.3 and Sect.7 below.

The intersection properties of $\widetilde{F}$ and $\widetilde{R}$ are described in

Proposition 2.2. *Let B be the shadow boundary defined by (2.10). Then*
a) $\widetilde{F} \cap \widetilde{R} = (\widetilde{F} \cap \partial X) \cup B$
b) Λ_R *and* Λ_F *intersect cleanly and* $\Lambda_R \cap \Lambda_F = T^*_B X \setminus 0 \cap \Lambda_R = T^*_B X \setminus 0 \cap \Lambda_F$

It is important to remark that although the extension $\tilde{p}$ was used in the definition of Λ_R, the part of $\widetilde{R}$ corresponding to the true reflection is determined by p and F alone. It will be denoted by R and is defined as follows. Proposition 2.2 implies that $\widetilde{R} \setminus \widetilde{F}$ has four components, two of which are disjoint from L. We now take as R the closure of the one for which $R \cap \partial X = F \cap \partial X$. An alternative definition is provided by taking

$$R = \{z \in \widetilde{R} \cap X : \iota^*(B^-_z \cap T^*_{\partial X} X \setminus 0) \subset \iota^*(N^*_{\partial X} F), \text{ where } B^-_z \text{ is the bicharacteristic through } N^*_z \widetilde{R} \text{ with } t \le t(z)\}.$$

The bicharacteristic cone over the shadow boundary in ∂X, Γ is now defined in the standard way, as the union of the maximally extended bicharacteristic intervals over $N^*\Gamma \cap \Sigma$. We denote it by Λ_S and its projection by $\widetilde{S}$. We note however that $\widetilde{S}|_X$ depends on the extension $\tilde{p}$. Thus we need

Proposition 2.3. *The set* $D = \mathrm{cl}[\pi_*(\exp H_p(N^*\Gamma \cap \mathcal{G})) \setminus B]$ *is a smooth codimension two submanifold of X, tangent to ∂X at Γ and intersecting B cleanly,* $B \cap D = \Gamma$. *The components S_+ and S_- of* $\widetilde{S} \setminus (B \cup D)$ *such that* $S_\pm \cap (\widetilde{X} \setminus X) = \emptyset$ *are determined by Γ and p.*

We can separate the forward and retarded components, $S_\pm$, respectively, and similarly denote by $\widetilde{S}_+$ the full forward cone over Γ. We also denote by B_+ and D_+ the intersections of B and D with $\mathrm{cl}\, S_+$ respectively, see Fig. 4

In the non-linear interaction more geometry is present. In addition to the cone over Γ we will also have to include, in a very residual way, smooth characteristic surfaces tangent to $\widetilde{S}$ at D – see Sect.3. Thus we define

$$\mathcal{R} = \{H \subset \widetilde{X} \text{ smooth hypersurface} : p|_{N^*H} = 0,\ N^*_{D\setminus\Gamma}H = N^*_{D\setminus\Gamma}\widetilde{S}\}, \tag{2.17}$$

with the first easy observation that $\bigcap_{H\in\mathcal{R}} H = D$. By the analogy with the previous notation we also write $\Lambda_H \overset{\text{def}}{=} N^*H \setminus 0$.

The intersection properties of Λ_S and $\Lambda_R, \Lambda_F, \Lambda_H$ are given in

Proposition 2.4. *The following Lagrangian submanifolds*
a) Λ_S *and* Λ_F *intersect cleanly at* $\Lambda_R \cap \Lambda_S = \Lambda_S \cap N^*B$,
b) Λ_S *and* Λ_R *are simply tangent along* $\Lambda_R \cap \Lambda_F = \Lambda_S \cap N^*B$,
c) Λ_S *and* Λ_H *intersect cleanly at* $\Lambda_S \cap N^*D$.

The restrictions of the characteristic (singular) surfaces $\widetilde{R}$ and $\widetilde{S}$ to the boundary can be related to $F_\partial = \widetilde{F} \cap \partial X$ and $\Gamma \subset \partial X$ respectively. Thus we define the following smooth Lagrangian submanifolds of $T^*\partial X \setminus 0$:

$$\Lambda_{00} = N^*\Gamma,\ \Lambda_{11} = N^*(\widetilde{F}\cap\partial X),\ \Lambda_{31} = N^*(\mathrm{cl}(\widetilde{R}\setminus\widetilde{F})\cap\partial X),\ \Lambda_{13} = N^*(H\cap\partial X), \tag{2.18}$$

and

$$\Lambda_{21}\cup\Lambda_{23} = N^*(\widetilde{S}\cap\partial X), \quad \Lambda_{11}\cap\Lambda_{23} = \emptyset, \quad \Lambda_{13}\cap\Lambda_{21} = \emptyset.$$

Using the time function restricted to the boundary, $t \restriction \partial X$, we then obtain Lagrangians with boundaries:

$$\Lambda^\pm_{ij} = \Lambda_{ij} \cap \{\pm t \geq 0\}, \quad j = 1,3,\ i = 1,2,3.$$

The relations between these Lagrangians in then given by the *billiard ball map* $\delta^\pm : \imath^*(\Sigma_\partial) \to \imath^*(\Sigma_\partial)$ (see [35, 36]) which is two valued ($\pm$) and has a square root singularity over $\mathcal{G}$. To recall the definition of δ we introduce the natural map

$$\gamma_p : \Sigma_\partial \longrightarrow \Sigma/H_p$$

which, same as $\imath^*$ above, has a simple fold. Thus we can associate to it an involution I_p exchanging the points with the same image (as I_∂ did for $\imath^*$). The two valued map $\delta^\pm$ is then defined by

$$\delta^\pm \circ \imath^* = I_p.$$

The relation between the Lagrangians with boundaries is then given by

$$\Lambda^\pm_{31} = \delta^\pm\Lambda^\pm_{11}, \quad \Lambda^\pm_{21}\cup\Lambda^\pm_{23} = \delta^\pm\Lambda_0. \tag{2.19}$$

Finally, we relate the geometry described above to the differential equation. We assume that the principal symbol of a strictly hyperbolic operator P in X, p, satisfies the glancing assumptions with respect to the boundary ∂X defined by q. On the extension $\widetilde{X}$, we introduce a strictly hyperbolic extension of P with the symbol $\tilde{p}$. We will also denote it by P keeping in mind, however, the freedom we have in its choice. The past $\widetilde{X}_- \subset \widetilde{X}$ is defined so that $\widetilde{X}$ is its domain of influence, and

$$\widetilde{F} \cap \widetilde{X}_- = F \cap \widetilde{X}_- \subset X \cap \widetilde{X}_- \stackrel{\text{def}}{=} X_-, \quad F \cap \widetilde{X}_- \cap \partial X = \emptyset.$$

3. Resolution of singularities and the conormal spaces

As described in Sect.2 the interaction geometry is quite complicated as it involves cusp and conic singularities. To define a conormal space with reasonable propagation of regularity for P, one follows the method originating from [32] and subsequently applied in [27, 34, 42, 43, 44]. Its essence is the resolution of singularities and the use of the vector fields tangent to the lifted geometry in the resolved space. The insistence on conormality is motivated by the good multiplicative properties of bounded conormal functions, as already indicated in Sect.1 and the conviction that conormal regularity excludes any hidden singularities that could produce self-spreading.

For our problem the method of resolution is similar to that used in [34, 44] and it involves a *non-homogeneous blow-up.* To describe it let us consider

$$\mathbb{R}^n = \mathbb{R}^3 \times \mathbb{R}^{n-3}, \quad z = (x, y), \quad z \in \mathbb{R}^n, x \in \mathbb{R}^3, y \in \mathbb{R}^{n-3}$$

on which we define an $\mathbb{R}_+$–action T_δ^{1-2-3}:

$$T_\delta^{1-2-3}(x, y) = (\delta x_1, \delta^2 x_2, \delta^3 x_3, y), \quad \delta \in \mathbb{R}_+ \tag{3.1}$$

We start with a definition of spaces of functions with given non-homogeneous orders of vanishing:

$$M_r^{1-2-3}(\widetilde{X}) \subset C^\infty(\widetilde{X}), \quad u \in M_r^{1-2-3}(\widetilde{X}) \Longleftrightarrow T_\delta^* u = \mathcal{O}(\delta^r), \ \delta \to 0 \tag{3.2}$$

This allows us to the define a filtration of the differential operators in terms of homogeneity. Thus

$$Q \in \mathrm{Diff}^k_{p,(1-2-3)}(\widetilde{X}) \Longleftrightarrow Q : M_r^{1-2-3}(\widetilde{X}) \longrightarrow M_{r-p}^{1-2-3}(\widetilde{X}) \quad \text{for} \quad r \geq p.$$

The homogeneous differential operator important in our discussion is Friedlander's operator in $\mathbb{R}^3$:

$$P_0 = 4D_{x_2}^2 - 9x_2 D_{x_3}^2 - 6D_{x_3} D_{x_1}.$$

Proposition 3.1. *There exist coordinates $(x, y) \in \mathbb{R}^n$, $x \in \mathbb{R}^3, y \in \mathbb{R}^{n-3}$ in $\widetilde{X}$ such that*

$$P = P_0 + Q, \quad Q \in \mathrm{Diff}^2_{3,(1-2-3)}(\widetilde{X}), \tag{3.3}$$

$$\partial X = \{(x, y) : x_2 - \frac{1}{4}x_1^2 = 0\}, \tag{3.4}$$

and with the notation of Sect.2 and any $H \in \mathcal{R}$ given by (2.17)
a) $\Gamma = \{(0, y) : y \in \mathbb{R}^{n-3}\} \cap \widetilde{X}$
b) $\widetilde{R} = \{(x, y) : x_3^2 - x_2^3 = 0, \ x \in \mathbb{R}^3, y \in \mathbb{R}^{n-3}\}$
c) $\widetilde{F} = \{(x, y) : 2x_3 - 3x_2 x_1 + x_1^3 + f_4 = 0, \ x \in \mathbb{R}^3, y \in \mathbb{R}^{n-3}\}$, $f_4 \in M_4^{1-2-3}(\mathbb{R})^\kappa$
d) $\widetilde{S} = \{(x, y) : x_1^4 + 8x_1 x_3 - 6x_1^2 x_2 - 3x_2^2 + s_5 = 0, \ x \in \mathbb{R}^3, y \in \mathbb{R}^{n-3}\}$, $s_5 \in M_5^{1-2-3}(\mathbb{R})^\kappa$.
e) $H = \{(x, y) : x_1 + h_2 = 0\}, \quad h_2 \in M_2^{1-2-3}(\mathbb{R})^\kappa$.

Proof. By Proposition 2.1, $\widetilde{R}$ is a surface with a cusp singularity and thus we can find coordinates (x, y) such that $\widetilde{R}$ is given by $b)$. The cusp locus is then $L = \{x_2 = x_3 = 0\}$ and as noted in (2.12) it is simply tangent to ∂X at Γ, $L \setminus \Gamma \subset \widetilde{X} \setminus X$. If X is given by $\rho(x, y) > 0$ near $(0, 0)$ then by (2.15) $\rho'_{x_2}(0, 0) > 0$. We conclude that

$$\rho(x, y) = x_2 - g(x_1, x_3, y), \quad \partial_{x_1} g(0, y) = 0,$$

near $(0, 0)$. We expand g into $g_0(x_3, y) + x_1 g_1(x_3, y) + x_1^2 g_2(x_1, x_3, y)$ where $g_0(0, y) = g_1(0, y) = 0$ and $g_2 > 0$. (since $L \setminus \Gamma \subset \widetilde{X} \setminus X$). Completing the square we write g as

$$\left(g_0(x_3, y) - \frac{1}{16}\frac{(g_1(x_3, y))^2}{g_2(x_1, x_3, y)}\right) + \frac{1}{4}\left(2x_1(g_2(x_1, x_3, y))^{\frac{1}{2}} + \frac{1}{4}\frac{g_1(x_3, y)}{(g_2(x_1, x_3, y))^{\frac{1}{2}}}\right)^2.$$

We then change variables by replacing x_1 by the term in the second bracket and observe that the term in the first bracket, say $\tilde{g}$, vanishes for all x_1 when $x_3 = 0$. We can now apply Theorem 4.1 of [1] to obtain a cusp preserving change of variables (x_2, x_3) depending smoothly on the parameters (x_1, y) and putting $x_2 - \tilde{g}(x_1, x_3, y)$ to x_2. Thus in the new coordinates $\rho(x, y) = x_2 - x_1^2/4$ as desired.

If $\widetilde{R} = \{x_3^2 - x_2^3 = 0\}$ is characteristic for a strictly hyperbolic operator P then one easily sees that $P = P_0 + Q$, $Q \in \mathrm{Diff}^2_{3,1-2-3}$ – see Proposition 2.1, [34]. We observe also that neglecting Q produces errors of higher homogeneity (see the proof of Proposition 2.1 [34].). Since the right hand side of $c)$ is the reflection of $\widetilde{R}$ with respect to ∂X obtained using the symbol of P_0, we conclude that

$$\widetilde{F} = \{(x, y) : f(x, y) = 0\}, \quad f(x, y) = 2x_3 - 3x_2 x_1 + x_1^3 + f_4(x_1, x_2, y), \ f_4 \in \mathcal{M}_4^{1-2-3}. \tag{3.5}$$

The same conclusion can be made about $\widetilde{S}$:

$$\widetilde{S} = \{(x, y) : q(x, y) = 0\}, \quad q(x, y) = x_1^4 + 8x_1 x_3 - 6x_1^2 x_2 - 3x_2^2 + q_5, \ q_5 \in \mathcal{M}_5^{1-2-3}. \tag{3.6}$$

The surface B is given by $\widetilde{S} \cap \widetilde{F} \cap \widetilde{R}$ which up to terms of higher homogeneity is parametrized by (t, t^2, t^3, y). The codimension two surface D consists of characteristics of P tangent to ∂X and not contained in B. Neglecting Q in P, D one would obtain $D = \{(0, 0, t, y)\}$ (which would lie in ∂X). Since that describes D up to terms of higher homogeneity we conclude that $D = \{(f_1(t, y), f_2(t, y), t, y)\, |t| < \epsilon\}$ and it easily follows from (3.6) that $\nabla q \restriction_D = t[(8, \widetilde{f}_1(t, y), \widetilde{f}_2(t, y) + O(t)]$, $\widetilde{f}_i \in C^\infty$. Since H is smooth and tangent to $\widetilde{S}$ at D, it follows that its normal at D is given by $(8, \widetilde{f}_1, \widetilde{f}_2) + O(t)$. The implicit function theorem immediately gives $e)$ with $h = h(x_2, x_3, y)$ satisfying $h(0, 0, y) = 0$, that is $h \in M_2^{1-2-3}(\mathbb{R})^\mathrm{K}$. □

We will consider the surfaces on the right-hand side as the *model geometry*. The sense in which they are models can be explained as follows. The model surface for $\widetilde{F}$ in $c)$ is characteristic for Friedlander's operator P_0 and the cusp $\widetilde{R}$ is obtained from that model surface by reflection (according to the rules of geometric optics given in Proposition 2.1) through the boundary $x_2 - \frac{1}{4}x_1^2 = 0$. Note that this surface, although microlocally diffractive near $N^*\widetilde{R}$, is not globally diffractive for P_0 : it contains the characteristic $\{x_1 = x_2 = 0\}$. Thus we see that $Q \neq 0$ and essentially it has to contain a term of the form $-cx_2 D_{x_1}^2$ which destroys the degeneracy of the characteristic $\{x_1 = x_2 = 0\}$. The surface defined by the right hand side of d) is the cone over $0 \in \mathbb{R}^3$ with respect to the characteristic flow-out by P_0.

In view of Proposition 3.1 it is natural to resolve the geometry using the 1-2-3 blow-up given by the $\mathbb{R}_+$–action (3.1). Thus we define the space

$$\widetilde{X}_1 = (\widetilde{X} \setminus \Gamma) \sqcup (\mathbb{S}^2_{1-2-3} \times \mathbb{R}^{n-3}) \simeq \mathbb{R}_+ \times \mathbb{S}^2_{1-2-3} \times \mathbb{R}^{n-3} \tag{3.7}$$

where $\mathbb{S}^2_{1-2-3}$ is a non-round sphere $\{\omega \in \mathbb{R}^3 : \sum_{1\leq i\leq 3} \omega_i^{12/i} = 1\}$ and where the C^∞ structure on $\widetilde{X}_1$ is given by the second identification (see [28]). We now have the blow-down map

$$\widetilde{X}_1 \xrightarrow{\beta_1} \widetilde{X}, \quad (r, \omega, y) \longmapsto (r\omega_1, r^2\omega_2, r^3\omega_3, y)$$

which is a diffeomorphism on $\widetilde{X}_1 \setminus \partial\widetilde{X}_1$. Thus following [28] we define the *pull-back* of Y to be

$$\beta_1^* Y = \mathrm{cl}[\beta_1^{-1}(Y \setminus \Gamma)], \quad Y \subset \widetilde{X}.$$

Propositions 3.1 imply that $\beta_1^*\widetilde{F}, \beta_1^*\widetilde{S}$ and $\beta_1^*\partial X$ are smooth hypersurfaces in $\widetilde{X}_1$ intersecting the boundary $\partial\widetilde{X}_1$ cleanly, and $\beta_1^*\widetilde{R}$ has a cusp singularity transversal to $\partial\widetilde{X}_1$. Also,

$$\begin{aligned}
\beta_1^*\widetilde{F} \cap \partial\widetilde{X}_1 &= \beta_1^*\{2x_3 - 3x_2x_1 + x_1^3 = 0\} \cap \partial\widetilde{X}_1, \\
\beta_1^*\widetilde{S} \cap \partial\widetilde{X}_1 &= \beta_1^*\{x_1^4 + 8x_1x_3 - 6x_1^2x_2 - 3x_2^2 = 0\} \cap \partial\widetilde{X}_1, \\
\beta_1^* H \cap \partial\widetilde{X}_1 &= \beta_1^*\{x_1 = 0\} \cap \partial\widetilde{X}_1.
\end{aligned}$$

The boundary of the resolved space $\partial\widetilde{X}_1$ and the above intersections are shown in Fig.4.

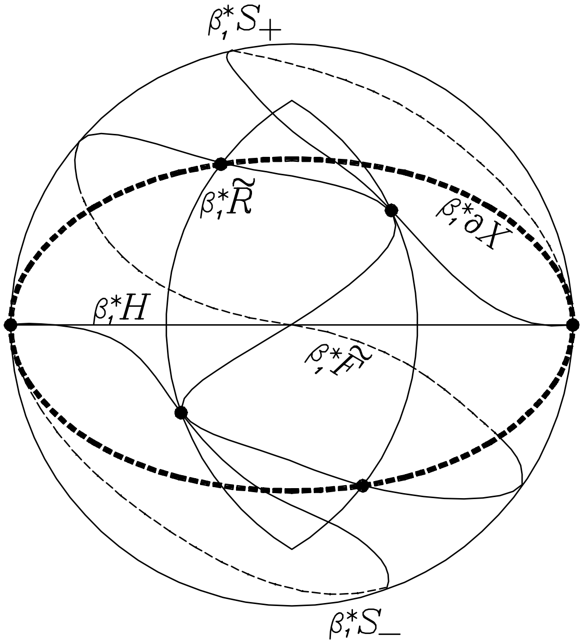

FIGURE 4. The geometry on $\partial\widetilde{X}_1$ as seen from the positive x_2 direction.

The lift of the operator P is of the form:

$$r^4 \beta_{1*} P \beta_1^* = P_1 \in \mathrm{Diff}^2_b(\widetilde{X}_1), \quad {}^b\sigma_2(P_1)\restriction_{{}^bT^*_{\partial\widetilde{X}_1}\widetilde{X}_1} = {}^b\sigma_2(r^4\beta_{1*}P_0\beta_1^*)\restriction_{{}^bT^*_{\partial\widetilde{X}_1}\widetilde{X}_1}$$

where we refer the reader to [25] (and also [14], Sect.18.3) for the definition of the totally characteristic operators, the compressed cotangent bundle and the b-symbol map.

As the blow-ups described above depend on the particular choice of coordinates, or alternatively the $\mathbb{R}_+$–actions, it is important to note certain invariance properties. To state them, let $\chi : \widetilde{X} \to \widetilde{X}$ be a local diffeomorphism such that $\chi(\Gamma) \subset \Gamma$. For simplicity we shall consider the 1-2-3 action given by $T_\delta = T_\delta^{1-2-3}$ only and require that

$$\left[Id - (T_\delta \circ \chi)^{-1} \circ (\chi \circ T_\delta)\right]^* : \mathcal{M}_r^{1-2-3} \longrightarrow \mathcal{M}_{r+1}^{1-2-3}, \quad \text{for all } \; r \in \mathbb{N}_0. \tag{3.8}$$

This *almost homogeneity* condition is now present in

Proposition 3.2. *Any diffeomorphism χ satisfying* (3.8) *lifts to a boundary preserving diffeomorphism*

$$\chi_1 : \widetilde{X}_1 \longrightarrow \widetilde{X}_1$$

Also, if a diffeomorphism preserves $\widetilde{R}$ and $\widetilde{S}$ in coordinates given by Proposition 3.1, then it satisfies (3.8).

For the proof and further geometric results needed in the study of interior propagation we refer the reader to [34]. In Sect.6 we will however need the following result:

Proposition 3.3. *If ρ is a defining function of ∂X then each of the following can be obtained with a diffeomorphism satisfying* (3.8)*:*
i) $H = \{x_1 = 0\}$, $\widetilde{S} = \{x_2^2 - 4x_1x_3 = 0\}$, $B = \{x_3 = x_2 = 0\}$, $d\rho\restriction_\Gamma = dx_2$,
ii) $\widetilde{F} = \{2x_3 - 3x_1x_2 + x_1^3 = 0\}$, $\widetilde{R} = \{x_3^2 - x_2^3 = 0\}$, $d\rho\restriction_\Gamma = dx_2 + \alpha dx_3$,

Proof. A diffeomorphism leading to $i)$ is already essentially obtained in Proposition 3.3 of [34] with the almost homogeneity (3.8) guaranteed by construction. This implies that

$$d\rho\restriction_\Gamma = \alpha dx_2 + \beta dx_3, \quad \alpha \neq 0.$$

If $\beta \neq 0$, then $D = \widetilde{S} \cap H = \{x_2 = x_1 = 0\}$ would be transversal to the boundary which contradicts the glancing assumption. Thus, $\beta = 0$ and by rescaling $\alpha = 1$.

For $ii)$ we first recall from Proposition 3.1 $b)$ that $\widetilde{F}$ is given by $2x_3 - 3x_1x_2 + x_1^3 + h_4(x_1, x_2, y) = 0$, $h_4 \in \mathcal{M}_4^{1-2-3}$. Also, $B = \mathrm{cl}(\widetilde{R} \cap \widetilde{F} \setminus X)$ is up to terms of higher homogeneity given by the normal form (t, t^2, t^3, y), that is, for small t:

$$B = \{(t, t^2(1 + t\phi(t, y))^2, t^3(1 + t\phi(t, y))^3, y)\},$$

where we used the exact form of $\widetilde{R}$. Thus changing x_1 to $x_1(1 + x_1\phi(x_1, y))$ puts B into its normal form. Since the defining function of $\widetilde{F}$ has to vanish there and its gradient has to coincide with the normal of $\widetilde{R}$ ($\widetilde{R}$ and $\widetilde{F}$ are simply tangent along B) we conclude that

$$h_4(t, t^2, y) = 0, \; \partial_{x_1} h_4(t, t^2, y) = \partial_{x_2} h_4(t, t^2, y) = 0,$$

and thus $h_4(x_1, x_2, y) = (x_2 - x_1^2)^2 \tilde{h}_4(x_1, x_2, y)$. A simple applications of the homotopy method concludes the proof. In fact, let us put $f_s = 2x_3 - 3x_2x_1 + x_1^3 + sh_4$. We want to find a cusp ($\widetilde{R}$) preserving family of diffeomorphisms such that

$$\chi_s^* f_s = f_0, \; 0 \leq s \leq 1.$$

If V_s is the family of vector fields generating χ_s, the required conditions are equivalent to

$$V_s h_s = -h_4, \quad V_s \text{ tangent to } \widetilde{R}.$$

This is obtained near Γ with

$$V_s = \frac{(x_2 - x_1^2)\tilde{h}_4}{3 + 4sz\tilde{h}_4 + s(x_2 - x_1^2)(\tilde{h}_4)'_{x_1}} \frac{\partial}{\partial x_1}.$$

Since the diffeomorphism constructed above clearly satisfies (3.8) we conclude that $\rho \equiv x_2 - x_1^2/4(\mathrm{mod} M_3^{1-2-3})$. $\square$

Remark 3.1. In case i) considered in the proposition

$$\rho(x, y) = x_2 + c_1(y)x_1^2 + c_2(y)x_2^2 + c_3(y)x_3^2 + \sum_{i<j} a_{ij}(y)x_i x_j + \mathcal{O}(|x|^3),$$

and the assumption that $\{x_1 = x_2 = 0\}$ and $\{x_2 = x_3 = 0\}$ are glancing easily implies that $c_1, c_3 > 0$.

The map $\Psi(x_1, x_2, x_3) = (2x_3 + x_1^3 - 3x_2x_1, x_2 - x_1^2, \frac{4}{3}x_1, y)$ transforms Q to $\{4x_3x_1 - x_2^2 = 0\}$, thus the cone on the right hand side of d) in Proposition 3.1 is essentially symmetric with respect to the interchange of x_1 and x_3. Roughly speaking, an additional blow-up near $\beta_1^*(Q \cap G) \cap \partial\widetilde{X}_1$ is needed to undo the asymmetry of the 1-2-3 blow-up.

To introduce it we first change coordinates (using a diffeomorphism satisfying (3.8)) so that $i)$ of Proposition 3.3 holds. Using the lift of these coordinates, we blow-up with the 2-1-1 homogeneity the codimension three submanifold $\partial\widetilde{X}_1 \cap \beta_1^*\{x_1 = x_2 = 0, x_3 > 0\} = \partial\widetilde{X}_1 \cap \beta_1^* D_+$:

$$\widetilde{X}_2 \xrightarrow{\beta_{12}} \widetilde{X}_1 \xrightarrow{\beta_1} \widetilde{X}, \quad \beta_2 = \beta_1 \circ \beta_{12}$$

$$\widetilde{X}_2 = \widetilde{X}_1 \setminus (\partial\widetilde{X}_1 \cap \beta_1^* D_+) \sqcup (\mathbb{S}^2_{2-1-1_+} \times \mathbb{R}^{n-3}),$$

where $\mathbb{S}^2_{2-1-1_+}$ is a half non-round sphere $\{\nu \in \mathbb{R}^3 : \nu_1^2 + \nu_2^4 + \nu_3^4 = 1, \nu_3 \geq 0\}$ and

$$\beta_{12}(\rho, \nu, y) = (\rho^2\nu_1, \rho\nu_2, \rho\nu_3, y),$$

with the coordinates in $\widetilde{X}_1$ near $\partial\widetilde{X}_1 \cap \beta_1^* D_+$ chosen so that

$$\beta_1(X_1, X_2, r, y) = (rX_1, r^2X_2, r^3, y) \in \widetilde{X}.$$

The manifold $\widetilde{X}_2$ has a codimension two corner and $\partial\widetilde{X}_2$ is shown in Fig.5.

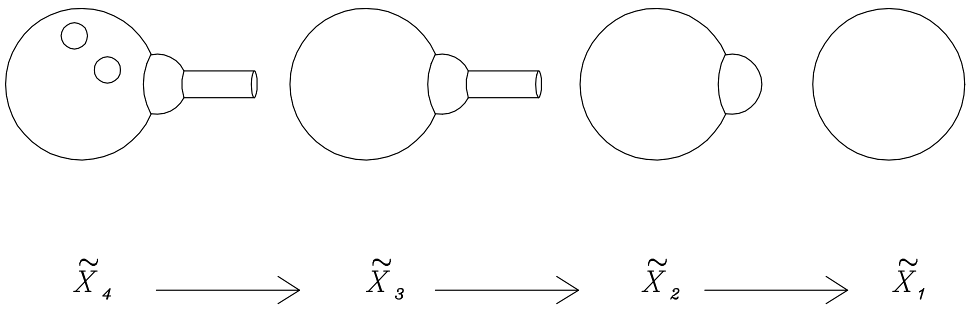

FIGURE 5. The hierarchy of blow-ups

Since $\widetilde{S}$ and H are simply tangent at D another blow-up is still needed:

$$\widetilde{X}_3 \xrightarrow{\beta_{23}} \widetilde{X}_2 \xrightarrow{\beta_{12}} \widetilde{X}_1 \xrightarrow{\beta_1} \widetilde{X}, \quad \beta_3 = \beta_1 \circ \beta_{12} \circ \beta_{23}$$

Here, the line $\beta_2^* D_+$ is blown-up with the 2-1-0 homogeneity in the coordinates where $\beta_2^* H \cap N = \{X_1 = 0\} \cap N$ and $\beta_2^* \widetilde{S} \cap N = \{4X_1 - X_2^2 = 0\} \cap N$ where N is a neighbourhood of $\beta_2^* D_+$ – see [43].

There are additional tangencies and singularities that have not yet been resolved: the tangencies described in Proposition 2.5 persist in $\widetilde{X}_1$ at $\beta_1^* B$ as does the cusp singularity of $\beta_1^* \widetilde{R}$ at $\beta_1^* L$. The former is resolved using a succession of normal blow-ups [33] (see Fig.6) and the latter using the 3-2 blow up [43], only at $\beta_1^* B_+$ and $\beta_1^* L_+$ respectively ($\beta_1^* L_+ = \beta_1^*(L \cap \{x_1 \geq 0\}$. This leads to the space $\widetilde{X}_4$:

$$\widetilde{X}_4 \xrightarrow{\beta_{34}} \widetilde{X}_3 \xrightarrow{\beta_3} \widetilde{X}, \quad \beta = \beta_4 = \beta_3 \circ \beta_{34}$$

see Fig.5.

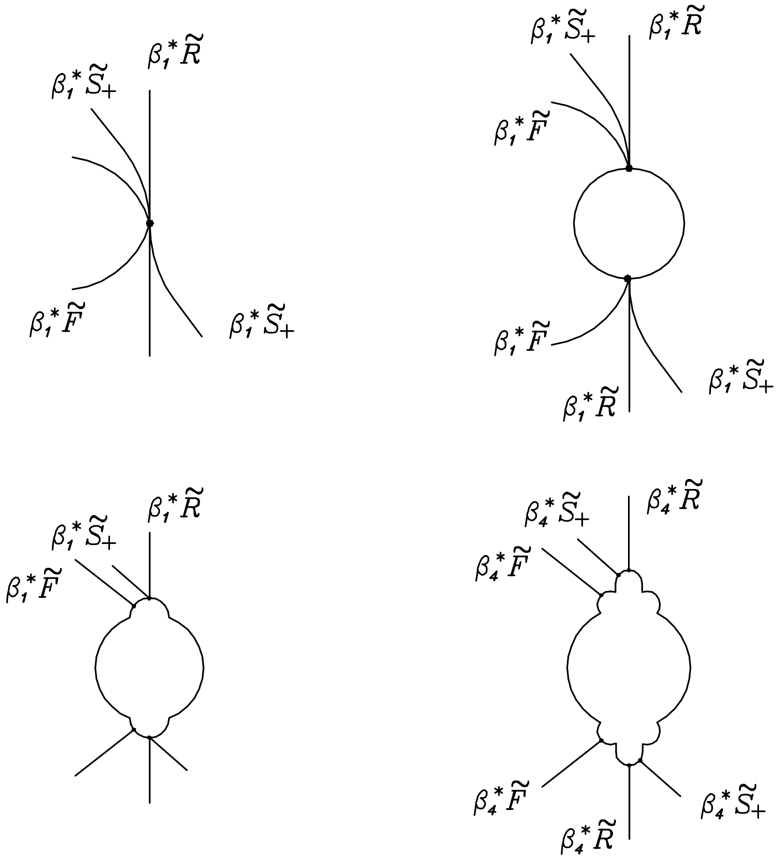

FIGURE 6. The three normal blow-ups of $\beta_1^* B$

For future reference we also define $\widetilde{X}_5$, analogously to $\widetilde{X}_4$ but obtained by applying the same blow-ups at the lifts of $D_\pm, L_\pm, B_\pm$ rather than of D_+, L_+, B_+ only:

$$\widetilde{X}_5 \xrightarrow{\beta_{35}} \widetilde{X}_3 \xrightarrow{\beta_3} \widetilde{X}, \quad \beta_5 = \beta_3 \circ \beta_{35}.$$

We shall now define the C^∞-algebra $J_k L^2_c(\widetilde{X}, H)$ associated to the geometry in the open manifold $\widetilde{X}$. In the notation we stress the dependence on the 'artificial' characteristic hypersurface $H \in \mathcal{R}$.

Let us first consider the surfaces in $\widetilde{X}_4$ obtained from the geometry in $\widetilde{X}$:

$$\beta^*\widetilde{F},\ \beta^*\widetilde{R},\ \beta^*\widetilde{S}_+,\ \beta^*(\widetilde{F}\cap\widetilde{R}\setminus B),\ \beta^*(\widetilde{S}_+\cap\widetilde{R}\setminus B),\ \beta^*H$$

where we note that the lifts of B_+, D_+ and L_+ are included in the boundary of $\widetilde{X}_4$. Let $\mathcal{S}$ be the variety obtained by taking a disjoint union of the five submanifolds above with $\partial\widetilde{X}_4$:

$$\mathcal{S} = \beta^*\widetilde{F} \sqcup \beta^*\widetilde{R} \sqcup \beta^*\widetilde{S}_+ \sqcup \beta^*(\widetilde{F}\cap\widetilde{R}\setminus B) \sqcup \beta^*(\widetilde{S}_+\cap\widetilde{R}\setminus B) \sqcup \beta^*H \sqcup \partial\widetilde{X}_4 \tag{3.9}$$

Ideally, we would want to define $J_k L^2_c(\widetilde{X}, H)$ as the β-pushforward of the conormal spaces associated to $\mathcal{S}$ which is in fact done for the interior problem. Here, however, this would be disastrous.

In Sect.5 we shall define $K_1 = K_1(\epsilon) \subset \widetilde{X}_1$ (see Fig.8) which in some sense constitutes a 'non-homogeneous' past. We can take ϵ small enough so that

$$\beta_1^*(\widetilde{F}\cap\widetilde{R}\setminus B) \subset \widetilde{X}_1 \setminus K_1(\epsilon) \Leftrightarrow \beta_1^*(\widetilde{F}\cap\partial X) \subset \widetilde{X}_1 \setminus K_1(\epsilon) \tag{3.10}$$

Since the all the higher generation blow-ups occur away from K_1 we can think of it as a subset of $\widetilde{X}_4$ (or $\beta_{14}^* K_1 = K_1$).

Definition 3.1. For $k \in \mathbb{N}_0$, we define

$$J_k L^2_c(\widetilde{X}, H) = \beta_*\{U \in I_k L^2_\nu(\widetilde{X}_4; \mathcal{V}(\widetilde{X}_4, \mathcal{S}))\ :\ U\restriction_{K_1} \in I_k L^2_\nu(\widetilde{X}_4, \beta^*\widetilde{F}\sqcup\partial\widetilde{X}_4)\restriction_{K_1}\},$$

where the variety $\mathcal{S}$ is given by (3.9) and $K_1 = K_1(\epsilon)$ is given by Definition 5.1 with ϵ such that (3.10) is satisfied. The norm is defined using the norm of the lift:

$$\|u\|_{J_k L^2(\widetilde{X},H)} = \|\beta^* u\|_{I_k L^2_\nu(\widetilde{X}_4, \mathcal{V}(\widetilde{X}_4,\mathcal{S}))} + \|\beta^* u|_{K_1}\|_{I_k L^2_\nu(\widetilde{X}_4, \beta^*\widetilde{F}\sqcup\partial\widetilde{X}_4)\restriction_{K_1}}.$$

We also define $J^1_k L^2(\widetilde{X}, H)$ by demanding that $u, Du \in J_k L^2_c(\widetilde{X}, H)$, with the obvious norm.

For non-integral values of the order of regularity we use complex interpolation and define:

$$J_{k+s} L^2(\widetilde{X}, H) = [J_k L^2(\widetilde{X}, H), J_{k+1} L^2(\widetilde{X}, H)]_s,\quad 0 < s < 1, \tag{3.11}$$

and similarly for $J^1_{k+s} L^2(\widetilde{X}, H)$.

This is a *pseudo-conormal* space as it involves an additional condition in K_1. The corresponding pseudo-conormal space for the manifold with boundary X is essentially obtained by restriction with an additional singular support condition:

Definition 3.2. For $s \geq 0$, $s \in \mathbb{R}$ we define

$$\begin{aligned} J_s L^2(X) = \quad & \{u \in L^2(X) :\ \text{for every } H \in \mathcal{R} \text{ there exists } \tilde{u} \in J_s L^2(\widetilde{X}, H) \\ & \text{with } \tilde{u}\restriction_X = u \text{ and sing supp}^{(s)} u \cap (\widetilde{F}\setminus F \cup \widetilde{R}\setminus R \cup \widetilde{S}_+\setminus S_+) = \emptyset\ \}. \end{aligned}$$

We recall that using the regularity function $s_u(x)$ (cf. [14], Sect.18.1), we define $\text{sing supp}^{(s)} u = \{x : s_u(x) \leq s\}$ which by lower semi-continuity of s_u is closed. The space $J_sL^2(X)$ is *not* a normed space and although it can be made into a Fréchet space we shall not need this fact here. The L^2_{loc} based spaces are defined in the obvious way: $u \in J_sL^2_{\text{loc}}(X)$ if and only if for any $\chi \in C_0^\infty(X)$, $\chi u \in J_sL^2(X)$.

Remark 3.2. Although the definition of the blow-up involves the choice of H, it can in fact be made independent of it. It is also true that away from Γ the spaces $J_sL^2(X)$ is the same as the space defined without including the lift of H in the defining variety. That statement is non-trivial only near D.

The complications of the definitions are now compensated by the simplicity of the proof of the following

Theorem 3.1. *The spaces $J_sL^2_{\text{loc}}(\widetilde{X}, H) \cap L^\infty_{\text{loc}}(\widetilde{X})$ and $J_sL^2_{\text{loc}}(X) \cap L^\infty_{\text{loc}}(X)$ given by Definitions 3.1 and 3.2 respectively are C^∞-modules and C^∞-algebras.*

In fact, we use the identification of the conormal spaces on the 'blown-up' side with b-Sobolev spaces (see Appendix B) and then apply the well known algebra properties of those spaces.

As in the earlier work on conormal regularity the difficult part is the propagation theorem. For the interior problem it is proved in Theorem 7, page 1026 of [34]:

Theorem 3.2. *If the variety $\mathcal{S}_5$ in $\widetilde{X}_5$ is given by (3.9) with β replaced by β_5, and*

$$Pu = f \quad in \quad \widetilde{X}, \ u \restriction_{\widetilde{X}_-} = 0, \ f \in (\beta_5)_*(I_k L^2_{\nu_5}(\widetilde{X}_5, \mathcal{S}_5)),$$

then

$$u, Du \in (\beta_5)_*(I_k L^2_{\nu_5,,loc}(\widetilde{X}_5, \mathcal{S}_5)).$$

The main result of this paper is the propagation theorem for the Dirichlet problem and the space $J_kL^2(X)$ – see Sect.8. A simpler refinement of Theorem 3.2 to $J_kL^2(\widetilde{X}, H)$ will be given in Sect.5.

4. Microlocally characterized spaces of distributions

4.1. The important notion of solutions to linear real principal type equations associated to *Lagrangian submanifolds* was introduced in [13]. That followed a rich tradition in geometric optics and semi-classical analysis already exploited in [17, 23] and generalized the notion of oscillatory solutions by recasting it in terms of propagation of singularities. The control of multiplicative properties required in the study of non-linear problems made it necessary to introduce a larger class of geometrically defined *marked Lagrangian* distributions [27, 30]. Additional motivation came also from the study of operators with double characteristics and of singular Radon transforms [12, 38]. For the basic material needed in this paper we refer the reader to the presentation in [34], Sect.9, while a proper development of the theory will appear in [30].

The purpose of this section is to extend the notion of marked Lagrangian distributions in two directions: to sub- and super-marked Lagrangian spaces, the closely related spaces associated with Lagrangian manifolds with boundaries, and to marked Lagrangian spaces on a manifold with boundary, M. In the last case we allow, unlike in [34], certain Lagrangians which are not smooth in ${}^bT^*M\backslash 0$. These two directions are rather independent at this point, with a rather special connection, however, which will be exploited in Sect.7.

To make this section self-contained we start with the general discussion of marked spaces, conducted for simplicity in the case of infinite regularity. We shall then proceed with the more detailed theory of $\frac{1}{2}$- and 2-marked spaces. For our purposes it will be sufficient to consider only markings by a single submanifold.

Let M be a C^∞ n-manifold without boundary and let $\Lambda_0, \Lambda_1 \subset T^*M\backslash 0$ be conic Lagrangian C^∞ submanifolds, intersecting cleanly. Then $K = \Lambda_0 \cap \Lambda_1$ is a C^∞ embedded conic submanifold of Λ_0, and any such K can be obtained locally as a clean intersection with a Lagrangian submanifold Λ_1. We define the following marked Lagrangian varieties associated to this geometry:

$$\mathcal{L}_0 = \Lambda_0, \qquad \mathcal{L}_1 = \{\Lambda_0, K\}, \qquad \mathcal{L}_\infty = \Lambda_0 \sqcup \Lambda_1 = \{\Lambda_0 \cup \Lambda_1 \backslash \Lambda_0 \cap \Lambda_1,\ \Lambda_0 \cap \Lambda_1\} \tag{4.1}$$

and the corresponding $\Psi^0_{\mathrm{phg}}(M)$-modules of first order pseudodifferential operators:

$$\begin{aligned}\Psi^1_{\mathrm{phg}}(M;\mathcal{L}_t) =\{&A \in \Psi'_{\mathrm{phg}}(M):\ \sigma_1(A)\big|_L = 0,\ H_{\sigma_1}(A) \\ &\text{is tangent to } L \text{ for all } L \in \mathcal{L}_t\}, \quad t = 0, 1, \infty.\end{aligned} \tag{4.2}$$

The $H_{(s)}(M)$-based spaces of (marked) Lagrangian distributions associated to the varieties (4.1) are defined as

$$\begin{aligned}IH_{(s)}(M;\mathcal{L}_t) =\{&u \in H_{(s)}(M):\ A_1 \dots A_l u \in H_{(s)}(M) \\ &\text{for } A_i \in \Psi^1_{\mathrm{phg}}(M;\mathcal{L}_t) \text{ and } l \in \mathbb{N}\}, \qquad t = 0, 1, \infty.\end{aligned} \tag{4.3}$$

The space on which the iterated regularity is based, $H_{(s)}$ can clearly be replaced by $H^{\mathrm{comp}}_{(s)}$ or $H^{\mathrm{loc}}_{(s)}$. Suppose now that $K = \Lambda_0 \cap \Lambda_1$ is assumed to be an embedded hypersurface in Λ_0. Locally, we can use the following model for the geometry (which is a special case of a more general extension of Darboux's theorem [38]): $M \subset \mathbb{R}^n$ open, $0 \in M$ and

$$\Lambda_0 = T^*_0\mathbb{R}^n\backslash 0, \qquad \Lambda_1 = N^*\{x:\ x_1 = \cdots = x_{n-1} = 0\}. \tag{4.4}$$

In this case one easily sees that $\Psi^1_{\mathrm{phg}}(M;\mathcal{L}_t)$ is the $\Psi^0_{\mathrm{phg}}(M)$-span of

$$x_i D_{x_j} \qquad i,j = 1,\dots,n \qquad t = 0 \tag{4.5}$$

$$x_i D_{x_j}, x_n D_{x_n}, x_n^2 D_{x_j} \quad i = 1,\dots,n-1, j = 1,\dots,n \qquad t = 1 \tag{4.6}$$

$$x_i D_{x_j}, x_n D_{x_n} \quad i,j = 1,\dots,n, i \neq n, \qquad t = \infty. \tag{4.7}$$

Thus

$$u \in IH_{(s)}(M;\mathcal{L}_t) \Longleftrightarrow \widehat{u}^\wedge \in SL^2(m_s, g_{(1+t)^{-1}}) \tag{4.8}$$

where $m_s = (1+|\xi|^2)^{\frac{s}{2}}$,

$$g_\alpha = \frac{|d\xi'|^2}{1+|\xi|^2} + \frac{d\xi_n^2}{(1+|\xi'|^2)^\alpha + \xi_n^2 + 1}$$

and SL^2 are the $L^2(\mathbb{R}^n)$ based symbols (replacing L^∞ by L^2 in Sect.18.4 of [14]). Since we have

$$IH_{(s)}(M;\mathcal{L}_0) \subset IH_{(s)}(M,\mathcal{L}_1) \subset IH_{(s)}(M,\mathcal{L}_\infty)$$

one expects that changing of $\alpha = (1+t)^{-1}$ in (4.8) will lead to new classes of distributions which we will call sub-marked Lagrangian for $0 < t < 1$ and super-marked Lagrangian distributions for $1 < t < \infty$. Since, unlike (4.2) and (4.3), the definitions using the model metrics g_α are *not* a priori invariant we might have to allow for more geometric information. This will indeed be the case for the super marked distributions.

To start we need to review the definition of the k-jet bundle of a manifold Y. Using the identification of Y with the diagonal in $Y \times Y$,

$$Y \simeq \Delta_Y = \{(y, y) :\ y \in Y\} \subset Y \times Y$$

we define for $k > 0$ the following vector bundle over Y :

$$J^k(Y)^* = \mathcal{J}(Y)/\mathcal{J}(Y)^{k+1}, \qquad \mathcal{J}(Y) \subset C^\infty(Y \times Y) \tag{4.9}$$

$$\mathcal{J}(Y) = \{f \in C^\infty(Y \times Y) :\ f|_{\Delta_Y} = 0\}. \tag{4.10}$$

Then $J^k(Y) = (J^k(Y)^*)^* \longrightarrow Y$ is the *k-jet bundle.* If $Y \subset X$ is an embedded submanifold of a C^∞ manifold X, then using the pull back of $\iota :\ Y \times Y \hookrightarrow X \times X$ we obtain a natural mapping $J^k(X)^* \longrightarrow J^k(Y)^*$ and consequently $J^k(Y) \longrightarrow J^k(X)$. Restricting $J^k(X)$ to Y we obtain $J^k_Y(X)$ the jet of X at Y, with the natural inclusion $J^k(Y) \subset J^k_Y(X)$. The inclusion map can be replaced by a more general $f :\ X_1 \longrightarrow X_2$ which then induces

$$J^k f :\ J^k(X_1) \longrightarrow J^k(X_2).$$

If $X = X_1 = X_2$ and $Y_1 \subset Y \subset X$ are embedded submanifolds, we say that f preserves the k-th jet of Y at Y_1 if

$$J^k f(J^k_{Y_1}(Y)) \subset J^k_{Y_1}(Y),$$

where $J^k_{Y_1}(Y)$ is naturally included in $J^k_{Y_1}(X)$.

We now extend the notion of marked Lagrangian varieties of the type shown in (4.1) to the *jet Lagrangian varieties*:

$$\mathcal{L}^0_t = \{\Lambda_0,\ J^{-([-t]+1)}_{\Lambda_0 \cap \Lambda_1} \Lambda_1, t\} \tag{4.11}$$

where we include the *index* $t, 0 \le t < \infty$ in the variety and allow the convention $J^0(Y) \simeq Y$.

Definition 4.1. For $0 \le t < \infty$ and Λ_0, Λ_1 given by (4) we define the t-marked Lagrangian spaces associated to the jet Lagrangian varieties $\mathcal{L}^0_t$ given by (8) as

$$IH_{(s)}(M, \mathcal{L}^0_t) = \{u \in H_{(s)}(M) :\ (\phi u)^\wedge \in SL^2(m_s, g_{(1+t)^{-1}}) \text{ for } \phi \in C^\infty_0(M)\}.$$

For the harder finite regularity case and $t = 2^{-1}, 1, 2$, the invariance properties will be given in Proposition 4.1. In Proposition 4.3 we shall give, for the same t's and finite regularity, the proof of the following identity, quite easy in the case considered now

$$IH_{(s)}(M, \mathcal{L}_\infty) = IH_{(s)}(M, \mathcal{L}^1_t) + IH_{(s)}(M, \mathcal{L}^0_{\frac{1}{t}}),\ 0 \le t \le \infty. \tag{4.12}$$

where $\mathcal{L}^1_t$ is the variety obtained from $\mathcal{L}_t$ in (8) by exchanging Λ_0 and Λ_1. This reflects the main rôle of the sub- and super-marked Lagrangian spaces, which is in providing decompositions of spaces associated to pairs of intersecting Lagrangians into terms which can be treated individually.

We conclude this introductory discussion by defining (marked) Lagrangian distributions associated to Lagrangians with boundaries, again only in the model case and with infinite order of regularity.

Thus, in addition to the models given by (4.4) we consider

$$\Lambda^\pm_0 = \Lambda_0 \cap \{\pm\xi_n \ge 0\}, \quad \partial\Lambda^\pm_0 = \{\xi_n = 0\} \cap \Lambda_0 = \Lambda_0 \cap \Lambda_1. \tag{4.13}$$

The marking will now indicate the microlocalization on one side of the boundary: the lower the marking the more localized the singularities are to $\Lambda_0^{\pm}$. As we shall see in Proposition 4.2, for the case $t = 2$, the invariance requires an additional marking in the transversal direction as can be seen from the uncertainty principle. As before, supermarking requires additional geometric information and we introduce the following jet Lagrangian varieties:

$$\mathcal{L}_t^{\pm} = \{\Lambda_0^{\pm}, t\},\ t \geq 1;\ \mathcal{L}_t^{\pm} = \{\Lambda_0^{\pm}, J_{\partial\Lambda_0^{\pm}}^{-([-t^{-1}]+1)}\Lambda_1, t\}, \qquad 0 < t \leq 1. \tag{4.14}$$

For $t = 0$ we could consider $\Lambda_0^{\pm} \sqcup \Lambda_1$ obtaining a slightly larger space than the one considered [38].

Definition 4.2. For $\Lambda_0^{\pm}$ given by (4.13) and the jet varieties in (4.14) we define for $0 < t < \infty$

$$\begin{aligned} IH_{(s)}(M, \mathcal{L}_t^{\pm}) = &\{u \in IH_{(s)}(M, \mathcal{L}_{t^{-1}}) : \chi(|\xi_1|)\chi(-|\xi_1|^{-\frac{t}{1+t}}\xi_n)(|\xi_1|^{-\frac{t}{1+t}}\xi_n)^N(\phi u)^\wedge \\ &\in SL^2(m_s, g_{(\frac{t}{1+t})}),\ \text{for } N \geq 0 \text{ and } \phi \in C_0^\infty(M)\} \end{aligned} \tag{4.15}$$

and where $\mathcal{L}_t, m_s, g_\alpha$ are given by (4.14) and (4.8), $\chi \in C^\infty(\mathbb{R})$,

$$\chi(t) = \begin{cases} 0, & t < 1 \\ 1, & t > 2. \end{cases} \tag{4.16}$$

The decomposition (4.10) can now be refined to

$$IH_{(s)}(M, \mathcal{L}_0) = IH_{(s)}(M, \mathcal{L}_t^1) + \sum_{\pm} IH_{(s)}(M, \mathcal{L}_t^{\pm}), 0 < t < \infty. \tag{4.17}$$

4.2. We shall now develop carefully the ideas presented in the first part of this section for the cases $t = \frac{1}{2}, 1, 2$ and finite regularity. We recall that the marked Lagrangian distributions of [30] constitute the case $t = 1$.

For the applications in Sect.7 it is convenient to state the definitions for a slightly different model:

$$\Lambda_0 = N^*\{x_1 = x_2 = 0\} \subset T^*\mathbb{R}^n\backslash 0, \qquad \Lambda_1 = N^*\{x_1 = 0\} \subset T^*\mathbb{R}^n\backslash 0, \tag{4.18}$$

with the coordinates $x \in \mathbb{R}^n$ written as (x_1, x_2, x'), $x' \in \mathbb{R}^{n-2}$. For $M \subset \mathbb{R}^n$, open, $0 \in M$, we give

Definition 4.3. Let Λ_0 and Λ_1 be given by (4.18). Then

$$\begin{aligned} I_{2k}H_{(s)}(M; \Lambda_0, \Lambda_0 \cap \Lambda_1, \tfrac{1}{2}) = &\{u \in H_{(s)}(M);\ D_{x'}^{\alpha'}(x_1 D_{x_1})^{k_1}(x_1 D_{x_2})^{k_2}(x_2 D_{x_2})^{k_3} x_2^{k_4} u \\ &\in H_{(s+\frac{2}{3}k_4)}(M), \quad |\alpha'| + k_1 + k_2 + k_3 + \frac{2}{3}k_4 \leq 2k\}, \end{aligned}$$

$$\begin{aligned} &I_{2k+1}H_{(s)}(M; \Lambda_0, \Lambda_0 \cap \Lambda_1, \tfrac{1}{2}) = \\ &\qquad [I_{2(k+1)}H_{(s)}(M; \Lambda_0, \Lambda_0 \cap \Lambda_1, \tfrac{1}{2}), I_{2k}H_{(s)}(M; \Lambda_0, \Lambda_0 \cap \Lambda_1, \tfrac{1}{2})]_{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} I_k H_{(s)}(M; \Lambda_0, J^1_{\Lambda_0\cap\Lambda_1}\Lambda_1, 2) = &\{u \in H_{(s)}(M):\ D_{x'}^{\alpha'}(x_1 D_{x_1})^{k_1}(x_1 D_{x_2})^{k_2}(x_2 D_{x_2})^{k_3} x_2^{k_4} u \\ &\in H_{(s+\frac{1}{3}k_4)}(M), |\alpha'| + k_1 + k_2 + k_3 + \frac{1}{3}k_4 \leq k.\} \end{aligned}$$

One should observe that for $k=\infty$ these definitions coincide with Definition 4.1. In this case, as before, we could use the variety notation $\mathcal{L}_t$, $t=\frac{1}{2},2$. In both sub- and super-marked cases above, it is crucial that the operators appear with variable *weights* in the stability conditions. Thus we say that $\langle D_x\rangle^{\frac{2}{3}}x_2$ has weight $\frac{2}{3}$, $\langle D_x\rangle^{\frac{1}{3}}x_2$, weight $\frac{1}{3}$. The correct filtration is based on operators of order 1 and that explains the need for fractional weights for operators of different orders.

We now want to verify the invariance of the t-marked spaces. Thus, let $\Gamma\subset T^*\mathbb{R}^n\backslash 0$ be a connected open conic neighbourhood of $m_0=(0;(1,0,\ldots,0))$ and let us consider a canonical transformation

$$\chi:\ \Gamma\longrightarrow T^*\mathbb{R}^n\backslash 0,\ \chi(m_0)=m_0 \tag{4.19}$$

preserving the model geometry (4.18):

$$\chi(\Lambda_0\cap\Gamma)\subset\Lambda_0,\ J^l\chi(J^l_{\Lambda_0\cap\Lambda_1}\Lambda_1\cap J^l_\Gamma(T^*\mathbb{R}^n\backslash 0))\subset J^l_{\Lambda_0\cap\Lambda_1}\Lambda_1. \tag{4.20}$$

For such χ we have

Proposition 4.1. *Let F be a Fourier Integral Operator of order* 0 *associated to a canonical transformation χ satisfying* (4.20) *with $l=-([-t]+1)$. Then, if $t=\frac{1}{2},1,2$*

$$F:\ I_kH_{(s)}(M,\Lambda_0,J^{-([-t]+1)}_{\Lambda_0\cap\Lambda_1}\Lambda_1,t)\longrightarrow I_kH_{(s)}(M,\Lambda_0,J^{-[-t]+1}_{\Lambda_0\cap\Lambda_1}\Lambda_1,t).$$

Proof. We will give the proof in the case $t=2$. When $t=1$, the invariance is clear from the symplectically invariant definitions (4.2), (4.3) and the case $t=\frac{1}{2}$ is similar and simpler, as in particular it does not involve the jet bundle. It can also be derived by the methods of [30] or by using the calculus of Lagrangian distributions of class $I_{\frac{2}{3}}$ in [14], Sect.25.1.

For $t=2$, the argument is easier in the model given by (4.4) and it is clear that by composition of Fourier Integral Operators we will obtain the invariance in the general case and in particular in the model given by (4.18). This means that, by Egorov's theorem, we change the operators in Definition 4.3 to

$$\begin{aligned}&(x')^{\alpha'}D^{|\alpha'|}_{x_1}(x_1D_{x_1})^{k_1}(x_1D_{x_2})^{k_2}(x_2D_{x_2})^{k_3}x_2^{k_4}u\in H_{(s+\frac{1}{3}k_4)}(M),\\&|\alpha'|+k_1+k_2+k_3+\frac{1}{3}k_4\le k,\quad WF^{(k+s)}(u)\subset\Gamma.\end{aligned} \tag{4.21}$$

Without the loss of generality, we can also assume that $s=0$. As in Sect.9 of [30] we first observe that if $\chi^{-1}(y,\eta)=(X(y,\eta),\Xi(y,\eta))$ then

$$Fu(y)=\frac{1}{(2\pi)^n}\int e^{i\langle y,\eta\rangle}b(y,\eta)\hat{u}(\Xi(y,\eta))d\eta,\ b\in S^0(\mathbb{R}^n;\mathbb{R}^n)$$

and we start by specifying the properties of $\Xi=(\Xi_1,\Xi_2,\Xi')$, $\Xi'\in\mathbb{R}^{n-2}$, implied by the jet condition (4.20):

$$\begin{aligned}&\Xi_1(y,\eta)=\eta_1h_1(y,\eta),\ h_1(m_0)\neq 0,h_1\text{ homogeneous of degree }0\\&\Xi_2(y,\eta)=\eta_2h_2(y,\eta)+y_1g_1(y,\eta)+y_2^2g_2(y,\eta)+\langle y',g(y,\eta)\rangle,\\&h_2(y,\eta)=h_{20}(\eta)+y_2h_{22}(\eta,y),\ h_2(m_0)\neq 0,\\&h_i\text{ homogeneous of degree }0,\ g_i,g\text{ homogeneous of degree }1,\\&|\Xi'(y,\eta)|\le C|\eta_1|\text{ for }(y,\eta)\in\widetilde{\Gamma},\end{aligned} \tag{4.22}$$

where $\widetilde{\Gamma}\subset\Gamma$ is another conic neighbourhood of m_0, supp$b\subset\widetilde{\Gamma}$.

To illustrate the simple idea of the proof, let us consider the following integral:

$$I(y_1, y_2) = \frac{1}{(2\pi)^2} \int e^{i\langle y,\eta\rangle} a(\eta_1, \eta_2 + y_2^k \eta_1) d\eta, \quad a \in SL^2(1, g_{\frac{1}{3}}).$$

By taking the Fourier transform in the y_1 variable and making a substitution

$$\lambda = \eta_1, \; y = \lambda^{\frac{1}{3}} y_2, \; \eta = \lambda^{\frac{1}{3}} \eta_2$$

(that is, introducing a nonhomogeneous blow-up on the Fourier transform side, see Sect.7), we obtain

$$I_1(\lambda, y) = \frac{1}{2\pi} \int e^{iy\eta} a_1\big(\lambda, \eta + \lambda^{1-\frac{1}{3}(k+1)} y^k\big) \lambda^{\frac{1}{3}} d\eta,$$

with a_1 stable under λD_λ, ηD_η, D_η with weights $1, 1, \frac{1}{3}$ respectively. If $1-(k+1)/3 \le 0$, that is $k \ge 2$, one easily checks the stability of I_1 under the same operators and with the same weights.

Returning to the actual situation, we observe that in view of (4.21) it suffices to check that

$$\prod_{i \neq 2} (y_i D_{y_1})^{k_i} (y_2 D_{y_2})^{k_2} y_2^l Fu(y) \in H_{(\frac{1}{3}l)}(M) \tag{4.23}$$

if

$$\prod_{i \neq 2} (\eta_1 D_{\eta_i})^{k_i} (\eta_2 D_{\eta_2})^{k_2} (\eta_1^{\frac{1}{3}} D_{\eta_2})^l \chi(\eta_1) D_\eta^\alpha \hat{u}(\eta) \in L^2(\mathbb{R}^n), \quad u \in \mathcal{E}'(M) \tag{4.24}$$

where in both cases $\Sigma k_i + \frac{1}{3} l \le k$ and α is arbitrary. Using (4.22) we write

$$Fu(y) = \frac{1}{(2\pi)^n} \int e^{i\langle y,\eta\rangle} b(y,\eta) \hat{u}(\eta_1 h_1, \eta_2 h_{20} + \eta_2 y_2 h_{22} + y_1 g_1 + y_2^2 g_2 + \langle y', g\rangle, \Xi'(y,\eta)) d\eta \tag{4.25}$$

where $h_1 \neq 0$ and $h_{20} \neq 0$ in $\widetilde{\Gamma}$. Thus, we can replace $\hat{u}$ by $a(\eta_1, \eta_2 + \eta_2 y_2 \bar{h}_{22} + y_1 \bar{g}_1 + y_2^2 \bar{g}_2 + \langle y', \bar{g}\rangle, \Xi'(y,\eta))$, with $a(\eta)$ satisfying (4.24). We denote the integral obtained this way by $F(a,b)$ and we show that the application of the operators in (4.23) changes $F(a,b)$ to $F(\tilde{a}, \tilde{b})$ with $\tilde{b} \in S^0(\mathbb{R}^n, \mathbb{R}^n)$, or $S^{-\frac{1}{3}}(\mathbb{R}^n, \mathbb{R}^n)$ supp$\tilde{b} \subset \widetilde{\Gamma}$ and $\tilde{a}$ satisfying (4.24) with k decreased depending on the weight of the operator applied (1 or $\frac{1}{3}$). We start with

$$y_2 F(a,b)(y) = \frac{1}{(2\pi)^n} \int e^{iy\eta} (-D_{\eta_2})(ba) d\eta \equiv$$
$$\frac{-1}{(2\pi)^2} \int e^{i\langle y,\eta\rangle} (1 + y_2 \bar{h}_{22} + \eta_2 y_2 (\bar{h}_{22})'_{\eta_2} + y_1 (\overline{g}_1)'_{\eta_2} + y_2^2 (\overline{g}_2)'_{\eta_2} + \langle y', (\overline{g})'_{\eta_2}\rangle)(D_{\eta_2}\tilde{a}) b \; d\eta,$$

where we omitted the terms $F(a, b_1), b_1 \in S^0$. The term in brackets can be absorbed into b, while a can be replaced by $\tilde{a} = \eta_1^{\frac{1}{3}} D_{\eta_2} a$. Thus we obtain $F(\tilde{a}, \tilde{b})$ with $\tilde{b} \in S^{-\frac{1}{3}}(\mathbb{R}^n, \mathbb{R}^n)$ and $\tilde{a}$ satisfying (4.24) with k replaced by $k - \frac{1}{3}$. Boundedness properties on Sobolev spaces for Fourier Integral Operators show that $F(\tilde{a}, \tilde{b}) \in H_{(s+\frac{1}{3})}(M)$ if $F(a,b) \in H_{(s)}(M)$.

We now consider

$$(y_2 D_{y_2})Fu = \frac{1}{(2\pi)^n}\int e^{iy\eta} y_2 \cdot \eta_2 (ba) d\eta +$$
$$\frac{1}{(2\pi)^n}\int e^{iy\eta} y_2 \Big[\Big(\eta_2 y_2 (\bar h_{22})'_{y_2} + \eta_2 \bar h_{22} + y_1 (g_1)'_{y_2} + \langle y', g_{y_2}\rangle\Big)(D_{\eta_2} a) + \langle \Xi'_{y_2}, D_{\eta'}\rangle a\Big] b\, d\eta \qquad (4.26)$$

Since the needed estimate is local near $y_1 = y_2 = 0$, we can, without the loss of generality, shrink the support of b, $\widetilde\Gamma$, so that $|y_2 h_{22}(\eta)| < \frac{1}{2}$ in $\widetilde\Gamma$. Thus by changing b to $(1 + y_2 h_{22}(\eta))^{-1} b$, we can introduce the factor $(1 + y_2 h_{22}(\eta))$ in the integrand on the right hand side of (4.26). Neglecting the terms of the form $F(a, b_1)$, $b_1 \in S^0$, the first term in the right hand side of (4.26) can then be written as

$$\begin{aligned} &-\frac{1}{(2\pi)^n}\int e^{i\langle y,\eta\rangle}(\eta_2 + \eta_2 y_2 \bar h_{22} + y_1 \bar g_1 + y_2^2 \bar g_2 + \langle y', \bar g\rangle)(D_{\eta_2} a)\cdot b d\eta \\ &+\frac{1}{2\pi)^n}\int e^{i\langle y,\eta\rangle}(y_1 y_2 \bar g_1 + y_2^3 \bar g_2 + \langle y', \bar g\rangle) ab\, d\eta. \end{aligned} \qquad (4.27)$$

The first integral is of the form $F(\tilde a, b)$ with $\tilde a = \eta_2 D_{\eta_2} a$ (η_2 becomes the first term in brackets), which satisfies (4.24) with k decreased by 1.

In the second term we integrate by parts using the stability under $\eta_1 D_{\eta_1}$, $\eta_1 D^3_{\eta_2}$ and $\eta_1 D^\alpha_{\eta'}$, $|\alpha| = 1$ in (4.24) for $y_1(y_2 \bar g_1)$, $y_2^3(\bar g_2)$ and $\langle y', \bar g\rangle$ respectively. In fact, let us verify this for $\eta_1 D_{\eta_1}$:

$$\begin{aligned} &(\eta_1 D_{\eta_1})\left[a(\eta_1, \eta_2 + \eta_2 y_2 h_{22} + y_1 g_1 + y_2^2 g_2 + \langle y', g\rangle, \Xi')\right] \equiv \\ &\left[y_2 \eta_2 (\eta_1 (\bar h_{22})'_{\eta_1}) + \eta_1 y_1 (\bar g_1)'_{\eta_1} + \eta_1 y_2^2 (\bar g_2)'_{\eta_1} + \langle y', \bar g'_{\eta_1}\rangle\right] D_{\eta_2} a \end{aligned} \qquad (4.28)$$

where we omitted the terms $\eta_1 D_{\eta_1} a$ and $\eta_1 D_{\eta'} a$ as then we can use the operators in (4.24). Since we have the stability under $\eta_1 D^3_{\eta_2}$ with weight 1 in (4.24), the goal is to bring the number of η_2 derivatives falling on a to three. For the first term in (4.28) we use (4.27) again, with b in the first term replaced by $y_2(\eta_1 (h_{22})'_{\eta_1}) b \in S^0(\Gamma)$ and a in the second term by $y_2 D_{\eta_2} a$. For the remaining terms in (4.28), we use $y_i \exp(i\langle y, \eta\rangle) = D_{\eta_i} \exp(i\langle y, \eta\rangle)$ and integration by parts. In each case there is a gain in the number of η_2 derivatives, so after at most three applications we obtain $\eta_1 D^3_{\eta_2} a$.

Since $\eta_1^{-1} g_i, \eta_1^{-1}\bar g \in S^0(\Gamma)$, the second integral is again of the form $F(\tilde a, b)$, with $b \in S^0$ and $\tilde a$ satisfying (4.24) with k decreased by 1. The analysis of the second term on the right hand side of (4.26) and the verification of the stability under the remaining operators in (4.23) are similar and are left to the reader. □

We shall now present the finite regularity analogue of Definition 4.2. In that we restrict ourselves to the case relevant in our applications, $t = 2$.

Definition 4.4. Let Λ_0 and Λ_1 be given by (4.18) and let $M \subset \mathbb{R}^n$ be a bounded open set, $0 \in M$. If $\Lambda_0^\pm = \Lambda_0 \cap \{\pm\xi_n \ge 0\}$, then

$$\begin{aligned} I_{2k} H_{(s)}(M, \Lambda_0^\pm, 2) = \{ & u \in H_{(s)}(M) : D^{\alpha'}_{x'} (x_1 D_{x_1})^{k_1} (x_1 D_{x_2})^{k_2} (x_2 D_{x_2})^{k_3} x_2^{k_4} \\ & (\chi(|\xi_1|)\chi(\mp|\xi_1|^{-\frac{2}{3}}\xi_2)(|\xi_1|^{-\frac{2}{3}}\xi_2)^{k_0}(\phi u)^\wedge)^\vee \in H_{(s+\frac{2}{3}k_4)}(M) \\ & \text{for } \phi \in C_0^\infty(M),\ |\alpha'| + k_1 + k_2 + k_3 + \frac{2}{3}k_4 + \frac{1}{3}k_0 \le 2k\} \end{aligned}$$

with $\chi \in C^\infty(\mathbb{R})$ satisfying (4.16) and $(\bullet)^\vee$ denoting the inverse Fourier transform. For odd orders of regularity $2k+1$, $I_{2k+1}H_{(s)}(M, \Lambda_0^\pm, 2)$ is defined by complex interpolation as in Definition 4.3.

To study the invariance properties, let Γ and χ be as in (4.19) with (4.20) replaced by

$$\chi(\Lambda_0^\pm \cap \Gamma) \subset \Lambda_0^\pm. \tag{4.29}$$

The analogue of Proposition 4.1 is:

Proposition 4.2. *Let F be a Fourier Integral Operator of order* 0 *associated to the canonical transformation χ which satisfies* (4.29)*. Then*

$$F : I_k H_{(s)}(M, \Lambda_0^\pm, 2) \longrightarrow I_k H_{(s)}(M, \Lambda_0^\pm, 2).$$

Proof. We can consider only Λ_0^+ and assume that k is even as the odd order case will follow by complex interpolation. We shall use the same model (4.4) as in the proof of Proposition 4.1 with $\Lambda_0^\pm$ given by (4.13) As in (4.21), Egorov's theorem provides the defining operators (see (4.34) below) and we can assume that

$$WF^{(k)}(u) \subset \widetilde{\Gamma},\ WF^{(k)}(Fu) \subset \widetilde{\Gamma},$$

where $\widetilde{\Gamma} \subset \Gamma$ is a small conic neighbourhood of m_0. It is convenient however to use the representation of F involving the generating function of the canonical transformation:

$$\begin{aligned} (y,\eta) = \chi(x,\xi) &\Longleftrightarrow (y,\eta;x,\xi) = (y,\phi'_y;\phi'_\xi,\xi), \\ Fu(y) &= \frac{1}{(2\pi)^n}\int e^{i\phi(y,\xi)} b(y,\xi)\hat{u}(\xi)\,d\xi. \end{aligned} \tag{4.30}$$

The assumption (4.29) implies that

$$\phi(y,\xi) = y_1\xi_1 h_1(y,\xi) + y_2\xi_2 h_2(y,\xi) + \langle y', g(y,\xi)\rangle, \tag{4.31}$$

where h_1, h_2 are homogeneous of degree 0 and positive in Γ, g is homogeneous of degree 1 with $g'_{\xi'}$ of rank $n-2$ in Γ.

We shall, as in the proof of Proposition 4.1, consider more general integrals, starting with the Fourier transform of (4.30):

$$G(a,b)(\eta) = \frac{1}{(2\pi)^n}\int e^{i\phi(y,\xi) - i\langle y,\xi\rangle}\psi(\xi_1^{-1}\eta_1) b(y;\xi,\eta)a(\xi)\,d\xi\,dy \tag{4.32}$$

where $b \in S^{-l}_{\frac{2}{3},0}(\mathbb{R}^n, \mathbb{R}^{2n})$, supp $b \subset \Gamma_1$ with Γ_1 a small conic neighbourhood of $(0;(1,0,\ldots,0),(1,0,\ldots 0))$ (with respect to the $\mathbb{R}_+$-action in the last $2n$ coordinates). In fact b is assumed to satisfy a stronger estimate:

$$\begin{aligned} &|D_\xi^\alpha D_\eta^\beta b| \le C_{\alpha\beta}(1+\xi_1+\eta_1)^{-(|\alpha''|+|\beta''|+\frac{2}{3}(\alpha_2+\beta_2))-l} \\ &l = l(b),\ \alpha'' = (\alpha_1,\alpha_3,\ldots,\alpha_n),\ \beta'' = (\beta_1,\beta_3,\ldots,\beta_n). \end{aligned} \tag{4.33}$$

The cut-off $\psi \in C_0^\infty(\mathbb{R}\backslash 0)$, is chosen so that $\psi(t) = 1$ for $\frac{1}{C} < t < C$ for some C. We assume also that a satisfies

$$\begin{aligned} &\chi(-\xi_1^{-\frac{2}{3}}\xi_2)(\xi_1^{-\frac{2}{3}}\xi_2)^{k_0}(\xi_1 D_{\xi_1})^{k_1}\xi_1^{|\alpha'|}D_{\xi'}^{\alpha'}(\xi_2 D_{\xi_2})^{k_2}(\xi_1^{\frac{2}{3}}D_{\xi_2})^{k_3} a \in (1+|\xi_1|)^{\max(0,-p)}L^2(\mathbb{R}^n), \\ &\tfrac{1}{3}k_0 + k_1 + k_2 + \frac{2}{3}k_3 + |\alpha'| \le \max(0,p), \quad p = p(a) \in \tfrac{1}{3}\mathbb{Z}. \end{aligned} \tag{4.34}$$

We observe that the boundedness of Fourier Integral Operators in the class $I^{-l}_{\frac{2}{3}}$, implies that

$$G(a,b) \in H_{(l(b)+\max(0,-p(a)))}(M). \tag{4.35}$$

The cut-off $\psi(\xi_1^{-1}\eta_1)$ in (4.32) can be inserted, at the expense of smooth error terms, in the integrand on the right hand side of

$$\widehat{Fu}(\eta) = \frac{1}{(2\pi)^n}\int e^{i\phi(y,\xi)-i\langle y,\eta\rangle} b(y,\xi)\hat{u}(\xi)d\xi\ dy \tag{4.36}$$

since

$$\int e^{i\phi(y,\xi)-i\langle y,\eta\rangle} b(y,\xi)(1-\psi(\xi_1^{-1}\eta_1))dy = \mathcal{O}((1+|\xi_1|+|\eta_1|)^{-N}) \tag{4.37}$$

for any $N>0$. In fact since $h_1(y,\xi)>0$ in $\widetilde{\Gamma}$, $h_1(0,(1,0,\ldots,0))=1$, we have $(\phi(y,\xi)-\langle y,\eta\rangle)'_{y_1} > c(1+|\xi_1|+|\eta_1|)$ in the support of $(1-\psi)(\xi_1^{-1}\eta_1)$ if C and c are appropriately chosen. Thus, standard oscillatory integral estimates (see Theorem 7.7.1 of [14], as applied in Sect.25.1 there) give (4.22). We conclude that $\widehat{Fu}$ is essentially of the form (4.32) with $p(a)=k$, and we want to show that it also satisfies the estimate (4.34) with $p=k$.

If we apply the first operator in (4.34) to (4.32) we obtain

$$\chi(-\eta_1^{-\frac{2}{3}}\eta_2)(\eta_1^{-\frac{2}{3}}\eta_2)G(a,b) = I_1 + I_2, \tag{4.38}$$

where the decomposition was obtained by inserting $1=\chi(\xi_1^{-\frac{2}{3}}\xi_2)+(1-\chi)(\xi_1^{-\frac{2}{3}}\xi_2)$ in the integrand:

$$I_1 = \frac{1}{(2\pi)^n}\int e^{i\phi(y,\xi)-i\langle y,\eta\rangle}\eta_1^{-\frac{2}{3}}\eta_2 b(y;\xi,\eta)\chi(-\eta_1^{-\frac{1}{3}}\eta_2)\chi(\xi_1^{-\frac{1}{3}}\xi_2)a(\xi)d\xi\ dy. \tag{4.39}$$

We shall now integrate by parts to put I_1 in the form (4.32) with new a and b. Strictly speaking, we should also introduce a cutoff in ξ'/ξ_1 similar to ψ, reducing the integration in ξ'/ξ_1 to a compact set. For this we observe that since $g'_{\xi'}$ in (4.31) has rank $n-2$, y' can be expressed in terms of $\langle y', g'_{\xi'}\rangle \in \mathbb{R}^{n-2}$ and there exist differential operators $Q_1(y,\xi,D_{\xi'})$ and $Q_2(y,\xi,D_{\xi'})$ of order 1 with coefficients homogeneous of degree 1 in ξ such that

$$\begin{aligned}e^{i\phi(y,\xi)-i\langle y,\eta\rangle} &= (\eta_2-\xi_2(h_2+y_2(h_2)'_{y_2}))^{-1}\left[(-D_{y_2}+Q_2(y,\xi,D_{\xi'}))-(h_1)'_{y_2}\times\right.\\ &\left.(1-(\xi_2/\xi_1)\xi_1^2(h_2)'_{\xi_1}(h_1)'_{\xi_2})^{-1}\left(\xi_1D_{\xi_1}-\xi_1(h_2)'_{\xi_1}\xi_2D_{\xi_2}+Q_1(y,\xi,D_{\xi'})\right)\right]e^{i\phi(y,\xi)-i\langle y,\eta\rangle}\end{aligned} \tag{4.40}$$

The last term is obtained by writing

$$\begin{aligned}y_1e^{i\phi(y,\xi)-i\langle y,\eta\rangle} &= (D_{\xi_1}-y_2\xi_2(h_2)'_{\xi_1}-\langle y',\bar{g}'_{\xi_1}\rangle)e^{i\phi(y,\xi)-i\langle y,\eta\rangle} =\\ &\left[D_{\xi_1}-\xi_2(h_2)'_{\xi_2}\left(D_{\xi_2}-\xi_1y_1(h_1)_{\xi_2}-\langle y',\bar{g}'_{\xi_2}\rangle\right)-\langle y',\bar{g}'_{\xi_1}\rangle\right]e^{i\phi(y,\xi)-i\langle y,\eta\rangle}\end{aligned} \tag{4.41}$$

and observing that if $\widetilde{\Gamma}$ is small enough then $|(\xi_2/\xi_1)\xi_1^2(h_2)'_{\xi_1}(h_1)'_{\xi_2}|<\frac{1}{2}$.

Since $|\eta_2-\xi_2(h_2+y_2(h_2)'_{y_2})| > c\max\{\xi_1^{\frac{2}{3}}, |\xi_2|+|\eta_2|\}$ in the support of the integrand, we obtain $I_1=G(a_1,b_1)$ where $b_1\in S^{-l_1}_{\frac{2}{3},0}(\mathbb{R}^n;\mathbb{R}^{2n})$, $l_1=l(b)+\frac{2}{3}$ (with the stronger estimate (4.33) valid) and a_1 satisfies (4.34) with $p(a_1)=p(a)-1$. Note that the gain of regularity in b_1, compensates the decrease of p, so that the weight of the operator on a is $\frac{1}{3}$, same as that of $\chi(-\eta_1^{-\frac{2}{3}}\eta_2)\eta_1^{-\frac{2}{3}}\eta_2$ — see (4.34) and (4.35).

The analysis of I_2 is similar but we now invoke the assumption that a is stable under

$$(1-\chi)(\xi_1^{-\frac{2}{3}}\xi_2)\xi_1^{-\frac{2}{3}}\xi_2 \tag{4.42}$$

with weight $\frac{1}{3}$. To do that we again use (4.40) to write

$$\begin{aligned} I_2 \;=\; & \frac{1}{(2\pi)^n}\int (\xi_1/\eta_1)^{\frac{2}{3}}\xi_1^{-\frac{2}{3}}\left[\xi_2(h_2+y_2(h_2)'_{y_2}) - D_{y_2} + Q_2(y,\xi,D_{\xi'}) - \right. \\ & (h_1)'_{y_2}(1-(\xi_2/\xi_1)\xi_1^2(h_2)'_{\xi_1}(h_1)'_{\xi_2})^{-1}(\xi_1 D_{\xi_1} - \xi_1(h_2)'_{\xi_1}\xi_2 D_{\xi_2} + \\ & \left. Q_1(y,\xi,D_{\xi'}))\right] e^{i\phi(y,\xi)-i\langle y,\eta\rangle} b(y;\xi,\eta)(1-\chi)(\xi_1^{-\frac{2}{3}}\xi_2)\chi(-\eta_1^{\frac{2}{3}}\eta_2)a(\xi)\,d\xi\,dy, \end{aligned}$$

and then integrate by parts. Thus $I_2 = G(a_2,b_2)+G(a_3,b_3)$, where a_2 satisfies (4.34) with $p(a_2)=p(a)-\frac{1}{3}, b_2$ satisfies (4.33) with $l(b_2)=l(b)$ and a_3 satisfies (4.34) with $p(a_3)=p(a)-1$, b_3 satisfies (4.33) with $l(b_3)=l(b)+\frac{2}{3}$. In both cases the effective weights of operators are preserved.

The action of the remaining operators is similar and simpler. Thus we obtain

$$\begin{aligned} & \chi(-\eta_1^{-\frac{2}{3}}\eta_2)(-\eta_1^{-\frac{2}{3}}\eta_2)^{k_0}(\eta_1 D_{\xi_1})^{k_1}\eta_1^{|\alpha'|}D_{\eta'}^{\alpha'}(\eta_2 D_{\eta_2})^{k_2}(\eta_1^{\frac{2}{3}}D_{\eta_2})^{k_3}G(a,b) \\ & = \sum_{i=1}^{N} G(a_i,b_i), \qquad \frac{1}{3}k_0+k_1+k_2+\frac{2}{3}k_3+|\alpha'| \le p(a) \end{aligned} \tag{4.43}$$

with a_i satisfying (4.34) with $p(a_i)\le p(a)$ and $b_i\in S^{-l}_{\frac{2}{3},0}(\mathbb{R}^n;\mathbb{R}^{2n})$, $l=\max(0,-p(a))$. Using (4.35) we conclude that $\widehat{Fu}$ satisfies (4.34) with $p=k$. $\square$

Having established the invariance we can now give the general definition. Let M be a C^∞ manifold of dimension n and let $\Lambda,\widetilde{\Lambda}\subset T^*M\backslash 0$ be conic Lagrangian submanifolds intersecting cleanly at $\Lambda\cap\widetilde{\Lambda}$, a hypersurface in Λ. For $t=\frac{1}{2},1,2$ we define the *jet Lagrangian variety*

$$\mathcal{L}_t = \{\Lambda, J^{-([-t]+1)}_{\Lambda\cap\widetilde{\Lambda}}\widetilde{\Lambda}, t\}. \tag{4.44}$$

We also consider $\Lambda^+\subset\Lambda\subset T^*M\backslash 0$, a Lagrangian submanifold with boundary, $\partial\Lambda^+ = \Lambda\cap\widetilde{\Lambda}$ and the *variety*

$$\mathcal{L}^+ = \{\Lambda^+, 2\}. \tag{4.45}$$

We should remark that subsequently we may use either the notation $\mathcal{L}_t$ or the explicit description of the variety as given by the right hand side of (4.44).

Definition 4.5. For $\Lambda,\widetilde{\Lambda},\mathcal{L}_t$ above and $t=\frac{1}{2},1,2$ we define the space

$$I_k H_{(s)}(M;\mathcal{L}_t)$$

as consisting of $u\in\mathcal{D}'(M)$ such that there exists $u_0\in I_kH_{(s)}(M;\Lambda\backslash\Lambda\cap\widetilde{\Lambda})$, a countable covering of $\Lambda\cap\widetilde{\Lambda}$ by parametrizations

$$\chi_j:\ \Gamma_j\longrightarrow T^*\mathbb{R}^n\backslash 0, \qquad \Lambda\cap\widetilde{\Lambda}\subset\bigcup_j\Gamma_j, \tag{4.46}$$

$$\chi_j(\Gamma_j\cap\Lambda)\subset\Lambda_0,\ J^{-([-t]+1)}\chi_j : J^{-([-t]+1)}_{\Lambda\cap\widetilde{\Lambda}}\widetilde{\Lambda}\longrightarrow J^{-([-t]+1)}_{\Lambda_0\cap\Lambda_1}\Lambda_1,$$

where Λ_0 and Λ_1 are given by (4.18),

$$F_j \in I^0(\mathbb{R}^n, M; (\chi_j^{-1})') \qquad \text{and} \qquad v_j \in I_k H_{(s)}(\mathbb{R}^n; \Lambda_0, J_{\Lambda\cap\Lambda_1}^{-([-t]+1)}\Lambda_1, t),$$

where the last space was given in Definition 4.3. The distributions $F_j v_j$ are assumed to have locally finite supports such that

$$u - u_0 - \sum_j F_j v_j \in H_{(k)}(M).$$

If $\mathcal{L}^+$ is given by (4.45), the space

$$I_k H_{(s)}(M; \mathcal{L}^+)$$

is defined in a similar way using $I_k H_{(s)}(\mathbb{R}^n; \Lambda_0^+, 2)$ given in Definition 4.4.

We now proceed with the finite regularity case of the decompositions (4.12) and (4.17). For the marked Lagrangian distributions ($t = 1$) the proposition below was already established in [30]. Its proof here illustrates, in a computationally simple case, the general philosophy of relating microlocal and conormal spaces (see [25, 34, 42]), on which we shall rely heavily in Sect.6 and 7. For the variety $\Lambda \sqcup \widetilde{\Lambda} = \{\Lambda \cup \widetilde{\Lambda} \backslash \Lambda \cap \widetilde{\Lambda}, \Lambda \cap \widetilde{\Lambda}\}$ we define $I_k H_{(s)}(M, \Lambda \sqcup \widetilde{\Lambda})$ by (4.3) with the obvious finite order modification.

Proposition 4.3. *Let Λ and $\widetilde{\Lambda}$ be as in Definition 4.5. Then for $t = \frac{1}{2}, 1, 2$ we have*

$$I_k L^2_{\text{loc}}(M; \Lambda \sqcup \widetilde{\Lambda}) = I_k L^2_{\text{loc}}(M; \mathcal{L}_t) + I_k L^2_{\text{loc}}(M; \widetilde{\mathcal{L}}_{\frac{1}{t}}), \tag{4.47}$$

where $\mathcal{L}_t$ is given by (4.24) and $\widetilde{\mathcal{L}}_t$ is obtained by exchanging Λ and $\widetilde{\Lambda}$ there. In addition, if $\Lambda = \cup\Lambda_\pm$, $\Lambda_+ \cap \Lambda_- = \partial\Lambda^\pm = \Lambda \cap \widetilde{\Lambda}$, then

$$I_k L^2_{\text{loc}}(M, \Lambda \sqcup \widetilde{\Lambda}) = I_k L^2_{\text{loc}}(M; \widetilde{\mathcal{L}}_2) + \sum_\pm I_k L^2_{\text{loc}}(M; \mathcal{L}^\pm), \tag{4.48}$$

with $\mathcal{L}^\pm$ given by (4.45).

In the proof we shall only consider the case $t = \frac{1}{2}$ as $t = 2$ is symmetric to it, while $t = 1$ is discussed in Remark 4.2. The proof is based on a lifting of the right hand side of (4.47) to a conormal space. As the result is local and in view of the invariance, we take M to be a neighbourhood of 0 in $\mathbb{R}^n$ and $\Lambda = \Lambda_1$, $\widetilde{\Lambda} = \Lambda_0$ given by (4.18).

With $M = \mathbb{R}^n$ we consider the following successive blow-ups of $\{x_1 = x_2 = 0\}$

$$M_2 \xrightarrow{\beta_{1-2}} M_1 \xrightarrow{\beta_{3-1}} M, \qquad \beta = \beta_{1-2} \circ \beta_{3-1}, \tag{4.49}$$

where M_1 is a manifold with boundary and M_2 a manifold with corners:

$$M_1 = M\backslash\{0\} \sqcup (\mathbb{S}^1_{1-3} \times \mathbb{R}^{n-2}) \simeq \mathbb{R}_+ \times \mathbb{S}^1_{1-3} \times \mathbb{R}^{n-2}, \mathbb{S}^1_{1-3} = \{\omega \in \mathbb{R}^2 : \omega_1^4 + \omega_2^{12} = 1\},$$

with the C^∞ structure given by the second identification. The blow-down map β_{3-1} : $M_1 \longrightarrow M$ is given by

$$\beta_{3-1} : (r, \omega, x') \longmapsto (r^3\omega_1, r\omega_2, x'). \tag{4.50}$$

The second resolution M_2 is obtained by the blow-up of $\beta^*_{3-1}\{x_2 = 0\} \cap \partial M_1$:

$$M_2 = M_1\backslash(\beta^*_{3-1}\{x_2 = 0\} \cap \partial M_1) \sqcup (\mathbb{S}^1_{1_--2} \times \mathbb{R}^{n-2}) \sqcup (\mathbb{S}^1_{1_+-2} \times \mathbb{R}^{n-2}),$$

where now $\mathbb{S}^1_{1_\pm-2} = \{(\theta_1, \theta_2) : \pm\theta_1 \geq 0, \theta_1^4 + \theta_2^2 = 1\}$ (see Fig.7) with the usual C^∞ structure (see [28]). The intermediate blow-down map $\beta_{1-2} : M_2 \longrightarrow M_1$ is given by $\beta_{1-2}(\rho,\theta,x') = (\pm\rho\theta_1, \rho^2\theta_2, x')$, where $(\rho,\theta,x') \in \mathbb{R}_+ \times \mathbb{S}^1_{1+-2} \times \mathbb{R}^{n-2}$, with coordinates near $\beta^*_{3-1}\{x_2 = 0, \pm x_1 > 0\} \cap \partial M_1$, (r, X_2, x') chosen so that $\beta_{3-1}(r, X_2, x') = (\pm r^3, rX_2, x')$.

Using the definition (4.3) we easily see that $u \in I_k L^2_{\mathrm{loc}}(M; \Lambda_0 \sqcup \Lambda_1)$ if and only if

$$(x_1 D_{x_1})^{k_1} (x_2 D_{x_2})^{k_2} (x_1 D_{x_2})^{k_3} D^{\alpha'}_{x'} u \in L^2_{\mathrm{loc}}(M), \qquad \Sigma k_i + |\alpha'| \le k.$$

This condition, Definition 4.3 of $I_k L^2_{\mathrm{loc}}(M; \Lambda_0, J^1_{\Lambda_0\cap\Lambda_1}, 2)$ and the lifting of the vector fields in projective coordinates, give

Lemma 4.1. *For Λ_0, Λ_1 and M_2, M above*

$$I_k L^2_{\mathrm{loc}}(M; \Lambda_0 \sqcup \Lambda_1) \xleftrightarrow{\beta_*} I_k L^2_{\nu,\mathrm{loc}}(M_2, \mathcal{V}(\partial M_2 \sqcup \beta^*\{x_1 = 0\}))$$

$$I_k L^2_{\mathrm{loc}}(M; \Lambda_0, J^1_{\Lambda_0\cap\Lambda_1}\Lambda_1, 2) \xleftrightarrow{\beta_*} I_k L^2_{\nu,\mathrm{loc}}(M_2, \mathcal{V}(\partial M_2)), \tag{4.51}$$

where $\beta_*\nu = dx$.

Proof. We only need to check that the left hand side of (4.51) is defined by stability under vector fields – that is clear for the space in the preceding line and the remaining portion of the proof is a straightforward computation. In view of Definition 4.4 we first need to show that

$$x_2^l u \in H_{(\frac{1}{3}l)}(M),\ \frac{1}{3} l \le k \Longleftrightarrow u, \left(x_2^3 D_{x_i}\right)^m u \in L^2(M),\ m \le k. \tag{4.52}$$

The right hand side is equivalent to $\left(x_2^3 \langle D_x\rangle\right)^m u \in L^2(M)$ and the left hand side to $\left(x_2^3 \langle D_x\rangle^{\frac{1}{3}}\right)^l u \in L^2(M)$. Since $[x_2, \langle D_x\rangle^{\frac{1}{3}}] \in \Psi^{-\frac{2}{3}}(M)$, the left to right implication follows. For the opposite direction we observe that if $u \in C_0^\infty(M)$ then

$$\begin{aligned} \|x_2\langle D_x\rangle^{\frac{1}{3}} u\|^2_{L^2(M)} &= \int_M x_2^2 \langle D_x\rangle^{2/3} u\, \bar{u} dx + \int_M [\langle D_x\rangle^{1/3}, x_2^2] \langle D_x\rangle^{\frac{1}{3}} u\, \bar{u} dx \\ &\le \|x_2^2 \langle D_x\rangle^{2/3} u\|_{L^2(M)} \|u\|_{L^2(M)} + \|u\|^2_{L^2(M)} \end{aligned}$$

and similarly

$$\|x_2^2 \langle D_x\rangle^{\frac{2}{3}} u\|^2_{L^2(M)} \le \|x_2^3 \langle D_x\rangle u\|_{L^2(M)} \|x_2 \langle D_x\rangle^{\frac{1}{3}} u\|_{L^2(M)} + \|u\|_{L^2(M)} \|x_2 \langle D_x\rangle^{\frac{1}{3}} u\|_{L^2(M)}.$$

Hence using $2ab \le \epsilon a^2 + \epsilon^{-1} b^2$ we conclude that for $u \in C_0^\infty(M)$

$$\left\| \left(x_2 \langle D_x\rangle^{\frac{1}{3}} \right)^l u \right\|_{L^2(M)} \le C \sum_{m \le k} \left\| \left(x_2^3 \langle D_x\rangle u\right)^m \right\|_{L^2(M)},$$

if $l \le 3k$. A density argument and commutation with the remaining vector fields in the definition of $I_k L^2(M; \Lambda_0, J^1_{\Lambda_0\cap\Lambda_1}\Lambda_1, 2)$ conclude the proof. $\square$

We should remark that the last space on the right hand side is equal to $H^b_{(k)}(M_2)$ with the measure ν and thus the supermarked space on the left is an interpolation space in k. For every order $k = 2l$ the space $I_{2l} L^2_{\mathrm{loc}}(M; \Lambda_1, \Lambda_1 \cap \Lambda_0, \frac{1}{2})$ is easily seen to be characterized by the condition

$$(x_1 D_{x_1})^{k_1} (x_2 D_{x_2})^{k_2} (x_1 D_{x_2})^{k_3} D^{k_4}_{x_2} D^{\alpha'}_{x'} u \in H_{(-\frac{1}{3}k_4)}(M), \tag{4.53}$$

$\sum_{i<4} k_i + \frac{2}{3} k_4 + |\alpha'| \le k = 2l$. For $I_{2l} L^2_{\mathrm{loc}}(M, \Lambda_1^\pm, 2)$, $\Lambda_1^\pm = \Lambda_1 \cap \{\pm x_2 \ge 0\}$, we need to add the stability under the operator $\chi(|D_{x_1}|)\chi(\mp|D_{x_1}|^{\frac{1}{3}} x_2)|D_{x_1}|^{\frac{1}{3}} x_2$ with weight $\frac{1}{3}$. The following Lemma constitutes the harder part of the proof of Proposition 4.3 and it will be crucial in Sect.7:

Lemma 4.2. *If*

$$v \in I_k L^2_{\nu,\mathrm{loc}}(M_2, \mathcal{V}(\beta^*\{x_1 = 0, \pm x_2 \geq 0\} \sqcup \partial M_2))$$

then

$$\beta_* v \in I_k L^2_{\mathrm{loc}}(M; \Lambda_1^{\pm}, 2) + \beta_* H^b_{(k)}(M_2).$$

Proof. We observe first that the two conormal spaces in the Lemma are interpolation spaces in k. As the Lagrangian space, $I_k L^2_{\mathrm{loc}}(M; \Lambda_1^{\pm}, 2)$, was defined by complex interpolation for k odd, we only need to prove the lemma for $k = 2l$, $l \in \mathbb{Z}_+$. It is also sufficient to consider the $+$ case alone.

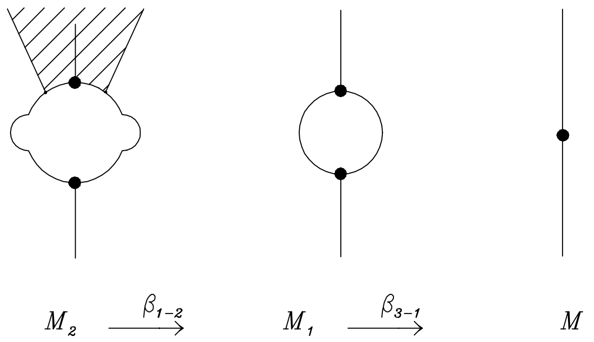

FIGURE 7. The manifold M_2 and the support of v

We can assume that $\mathrm{supp} v \subset \beta^*\{|x_1| < \epsilon x_2^3\}$, as on the remaining part of the support the function is in $H^b_{(k)}(M_2)$, see Fig.7. Thus we can consider $\mathrm{supp} v \subset M_1 \cap \beta^*_{3-1}\{|x_1| < \epsilon x_2^3\}$ and use coordinates (X, r), $\beta_{3-1}(X, r) = (r^3 X, r)$, where we drop the insignificant coordinate x'. We also have

$${}^bT^*_{\mathrm{supp}\, v} M_2 \simeq \{(X, r; \Xi, \lambda) : \; X, \Xi, \lambda \in \mathbb{R}, r \in \bar{\mathbb{R}}_+\},$$

and we can assume that

$$WF(v) \subset \{(X, r; \Xi, \lambda) : \; |\lambda| < |\Xi|\}. \tag{4.54}$$

In fact, using a Ψ^0_b-partition of unity, we obtain $v = v_1 + v_2$, $WF(v_1) \subset \{|\lambda| < |\Xi|\}$ and $WF(v_2) \subset \{|\lambda| > \frac{1}{2}|\Xi|\}$. Since $(rD_r)^p v_i \in L^2_\nu(M_2)$, $p \leq k$ we obtain $v_2 \in H^b_{(k)}(M_2)$, so it can be neglected.

Using the ellipticity of D_X in $WF(v)$ we can then write

$$v = D_X^m \tilde{v} + v^\#, \;\; \tilde{v} \in I_k H^b_{(m)}(M_2, \mathcal{V}(\beta^*\{x_1 = 0, x_2 > 0\} \sqcup \partial M_1)), \quad v^\# \in H^b_{(k)}(M_2) \tag{4.55}$$

Replacing v by $v - v^\#$, we now claim that $\beta_* v \in I_k L^2_{\mathrm{loc}}(M; \Lambda_1^+, 2)$. Since all the other operators in (4.53) lift we only need to investigate $D^p_{x_2}$ and

$$\chi(|D_{x_1}|)\chi(\mp |D_{x_1}|^{\frac{1}{3}} x_2)|D_{x_1}|^{\frac{1}{3}} x_2.$$

Thus, we start with $p = 3m \leq 2l = k$ and

$$D_{x_2}^{3m}\beta_* v = \beta_*(\{\frac{1}{r}(rD_r - 3XD_X)\}^{3m} D_X^m \tilde{v}) =$$
$$= D_{x_1}^m \beta_*(r^{3m}\{\frac{1}{r}(rD_r - 3XD_x)\}^{3m}\tilde{v}) + \beta_*([r^{3m}\{\frac{1}{r}(rD_r - 3XD_X)\}^{3m}, r^{-3m}D_X^m]\tilde{v}), \tag{4.56}$$

where we used $D_{x_1}\beta_* = \beta_* r^{-3}D_X$. Since $[r^{-1}(rD_r - 3XD_X), D_X] = r^{-1}3iD_X$, the expansion of the commutator gives

$$D_{x_2}^{3m}\beta_* v = D_{x_1}^m \beta_* v_1,\ v_1 \in I_{2k-2m}L^2_{\nu,\mathrm{loc}}(M_2, \mathcal{V}(\beta^*\{x_1 = 0, x_2 > 0\} \sqcup \partial M_2)),$$

so that $D_{x_2}^{3m}\beta_* v \in H_{(-m)}(M)$ if $2m \leq 2l = k$, that is $\frac{2}{3}p \leq k$, $p = 3m$. Thus the weight of $D_{x_2}^p$ is in agreement with the weights in $I_k L^2_{\mathrm{loc}}(M, \Lambda_1^+, 2)$. For p not divisible by 3 we use

$$\begin{aligned} \|D_{x_2}u\|^2_{H_{(-\frac{1}{3})}(M)} &\leq \frac{1}{3}\|D_{x_2}^3 u\|^2_{H_{(-1)}(M)} + \frac{2}{3}\|u\|^2_{L^2(M)} \\ \|D_{x_2}^2 u\|^2_{H_{(-\frac{2}{3})}(M)} &\leq \frac{2}{3}\|D_{x_2}^3 u\|^2_{H_{(-1)}(M)} + \frac{1}{3}\|u\|^2_{L^2(M)},\ u \in C_0^\infty(M). \end{aligned} \tag{4.57}$$

In (4.53), D_{x_2} is the only operator with a fractional weight $(\frac{2}{3})$, so that if $D_{x_2}^2$ occurs we can use the stability under $D_{x_2}^3$. If $VD_{x_2}A$ occurs, where A is a product of operators and V is an operator different from D_{x_2}, then we can use (4.57), integration by parts and stability under V^2A, $D_{x_2}^3A$, $(D_{x_2}^3A \in H_{(-1)}(M))$. This proves (4.53) for $\beta^* v$ and we still need to consider the remaining multiplier. That however is easy, since for v, after all the reductions above (mod $H^b_{(k)}(M_2)$),

$$\chi(|D_{x_1}|)\chi(-|D_{x_1}|^{\frac{1}{3}}x_2)|D_{x_1}|^{\frac{1}{3}}x_2\beta_* v = 0.$$

In fact, let us consider $H \in L^\infty(M_2)$ defined by

$$H|_{M_2\setminus\partial M_2} = (\beta|_{M_2\setminus\partial M_2})^*((x_2)^0_-).$$

Then $\beta_* H = (x_2)^0_-$ and $(x_2)^0_-\beta_* v = \beta_* Hv = 0$, if as assumed, $\operatorname{supp} v \subset \beta^*\{|x_1| < \epsilon x_2^3\}$. We then observe that

$$\chi(|D_{x_1}|)\chi(-|D_{x_1}|^{\frac{1}{3}}x_2)|D_{x_1}|^{\frac{1}{3}}x_2(x_2)^0_- = \chi(|D_{x_1}|)\chi(-|D_{x_1}|^{\frac{1}{3}}x_2)|D_{x_1}|^{\frac{1}{3}}x_2$$

concluding the proof. $\square$

Remark 4.1. A converse of Lemma 4.2 is also true:

$$\beta^* I_k L^2_{\mathrm{loc}}(M; \Lambda_1^\pm, 2) \subset I_k L^2_{\nu,\mathrm{loc}}(M_2, \mathcal{V}(\beta^*\{x_1 = 0, \pm x_2 \geq 0\} \sqcup \partial M_2)),$$

and although we do not need it, this fact is implicitly present in the Dirichlet estimates. Its failure (see Remark 7.1) in one case considered in Sect.7 explains to some extent the complications of the space $J_k L^2(X)$.

Proof of Proposition 4.3. It is easy to see that the right hand sides in (4.47) and (4.48) are contained in $I_k L^2_{\mathrm{loc}}(M; \Lambda \sqcup \widetilde{\Lambda})$. Since $I_k L^2(M; \mathcal{L}^\pm) \subset I_k L^2(M, \mathcal{L}_{\frac{1}{2}})$, it remains to verify (4.48) and that easily follows from Lemmas 4.1 and 4.2 as

$$\begin{aligned} &I_k L^2_{\nu,\mathrm{loc}}(M_2, \mathcal{V}(\partial M_2 \sqcup \beta^*\{x_1 = 0\})) = \\ &H^b_{(k)}(M_2) + \sum_{\pm} I_k L^2_{\nu,\mathrm{loc}}(M_2, \mathcal{V}(\partial M_2 \sqcup \beta^*\{x_1 = 0, \pm x_2 > 0\})), \end{aligned}$$

which concludes the proof. $\square$

Remark 4.2. The same proof can be used for $t = 1$, with the $3-1$ and $1-2$ blow-up replaced by a $2-1$ and $1-1$ blow-ups of the same submanifolds (with the obvious modification in the latter case). The last $(1-1)$ blow-up is not strictly speaking necessary but it is useful in characterizing $\beta^* I_k(M; \Lambda_0 \sqcup \Lambda_1)$ as a b-Sobolev space.

4.3. Let M be a C^∞ manifold with a C^∞ boundary ∂M. The b-Lagrangian distributions in Sect.7 of [34] were defined for C^∞ homogeneous Lagrangian submanifolds $\Lambda \subset T^*M\backslash 0$ such that $\jmath\Lambda \subset {}^bT^*M\backslash 0$ is smooth,

$$\jmath : T^*M\backslash 0 \longrightarrow {}^bT^*M\backslash 0.$$

They were defined using (4.2) and (4.3) with totally characteristic operators, $\Psi^1_b(M)$ employed in place of Ψ^1. We need to generalize this notion to allow cases when $\jmath\Lambda \subset {}^bT^*M\backslash 0$ is singular in a controlled way. A C^∞ submanifold of ${}^bT^*M\backslash 0$, ${}^b\Lambda$, is called a homogeneous (b-)Lagrangian if and only if $\dim {}^b\Lambda = \dim M$, ${}^b\Lambda \cap {}^bT^*_{\partial M}M$ is a homogeneous C^∞ Lagrangian submanifold of $T^*\partial M$,

$${}^b\omega|_{T_m{}^b\Lambda} = 0 \qquad \forall\ m \in {}^b\Lambda \cap {}^bT^*_{M^\circ}M\backslash 0,$$

and ${}^b\Lambda$ is homogeneous with respect to the natural $\mathbb{R}_+$-action on ${}^bT^*M\backslash 0$. The last two conditions simply state that ${}^b\Lambda \cap {}^bT^*_{M^\circ}M\backslash 0 \subset T^*M^\circ\backslash 0$ is a homogeneous Lagrangian submanifold of $T^*M^\circ\backslash 0$. Locally ${}^b\Lambda$ is given by the zeros of $n = \dim M$ functions in $C^\infty({}^bT^*M\backslash 0)$:

$${}^b\Lambda \cap \Gamma = \{m \in \Gamma :\ f_1(m) = \cdots = f_n(m) = 0\},$$
$$f_i \in C^\infty({}^bT^*M\backslash 0), \qquad {}^b\{f_i, f_j\} = 0,$$

where $\Gamma \subset {}^bT^*M\backslash 0$ is a conic neighbourhood of $\bar{m} \in {}^b\Lambda$. To pass to the global situation we consider the ideal

$$\mathcal{I}_{{}^b\Lambda} = \{f \in C^\infty({}^bT^*M\backslash 0) : f|_{{}^b\Lambda = 0}\}$$

which is locally finitely generated. Since ${}^b\Lambda$ is homogeneous we introduce for every $k \in \mathbb{Z}$

$${}^b\mathcal{M}_k({}^b\Lambda) = \mathcal{I}_{\Lambda^b} \cap S^k_{\rm hg}({}^bT^*M\backslash 0)$$

and observe that

$${}^b\Lambda = \{m :\ f(m) = 0,\ \forall\ f \in {}^b\mathcal{M}_k({}^b\Lambda)\}.$$

Example 4.1. If $\Lambda_0 \subset T^*\partial M\backslash 0$ is a homogeneous Lagrangian submanifold and $M \simeq \partial M \times [0,1)$, then $\Lambda = \{(x, y; \xi, \eta) : (y, \eta) \in \Lambda_0, \xi = 0\} \subset T^*M\backslash 0$ is a Lagrangian submanifold *and* $\jmath\Lambda \subset {}^bT^*M\backslash 0$ is a C^∞ b-Lagrangian submanifold.

Example 4.2 Let us define $\tilde\Lambda \subset T^*\mathbb{R}^2\backslash 0$ by $\tilde\Lambda = N^*\{x+y^2 = 0\}$. If $M = \bar{\mathbb{R}}^2_+ = \{x \geq 0\}$ and $\Lambda = \tilde\Lambda|_{T^*M\backslash 0}$ then ${}^bT^*M\backslash 0 \supset \jmath\Lambda = \{(x, y; \lambda, \eta) :\ x + y^2 = \eta x - 2y\lambda = 0,\ (\eta, \lambda) \neq (0, 0)\}$ is not smooth.

We wish to consider Lagrangian submanifolds similar to those in Example 4.2 in the sense of having defining functions which are *polynomials in* ξ at the boundary in a way which is invariant under b-canonical transformations – see [26], part III.

We use x as the defining function of the boundary and $\lambda = {}^b\sigma_1(xD_x)$ as its b-dual variable in ${}^bT^*M\backslash 0$. Then we consider

$$S^{m,k}_{\rm hg}({}^bT^*M\backslash 0) = \left\{ \sum_{0\leq p\leq k} \frac{\lambda^p}{x^p} a_p :\ a_p \in S^{m-p}_{\rm hg}({}^bT^*M\backslash 0)\right\}, \tag{4.58}$$

where $S^l_{\mathrm{hg}}({}^bT^*M\backslash 0)$ are homogeneous symbols $a \in C^\infty({}^bT^*M\backslash 0)$. These are the symbols satisfying $a(T_r m) = r^l a(m)$, where T_r is the $\mathbb{R}_+$ action generated by $\rho_{\partial M} + \lambda\partial_\lambda$, $\rho_{\partial M}$ the radial vector field on $T^*\partial M\backslash 0$.

The right-hand side of (4.58) involves a choice of coordinates but the space $S^{m,k}_{\mathrm{hg}}$ is nevertheless invariantly defined. In fact, for $k = 0$, $S^{m,k}_{\mathrm{hg}} = S^m_{\mathrm{hg}}$ and $S^{m,1}_{\mathrm{hg}}$ can be characterized by demanding that $\tilde{x} \cdot S^{m,1}_{\mathrm{hg}} \subset S^m_{\mathrm{hg}}$ and that $\tilde{x} \cdot S^{m-1}_{\mathrm{hg}}|_{{}^bT^*_{\partial M}M}$ vanishes at $\tilde{\lambda} = {}^b\sigma_1(\tilde{x}D_{\tilde{x}}) = 0$, for any defining function of ∂M. Since $\tilde{\lambda}$ is well defined on ${}^bT^*_{\partial M}M$ this gives an invariant definition. We then see that

$$S^{m,k}_{\mathrm{hg}} = \left\{ \sum_{0\le p\le k} b_{l_1}\dots b_{l_p} :\ b_{l_i} \in S^{m,1}_{\mathrm{hg}} \right\}$$

is also invariantly defined. We should also remark that

$$S^{m,k}_{\mathrm{hg}}({}^bT^*M\backslash 0)|_{T^*M^\circ\backslash 0} = \left\{ \sum_{0\le o\le k} \xi^p \iota^* a_p|_{T^*M^\circ\backslash 0} : a_p \in S^{m-p}_{\mathrm{hg}}({}^bT^*M\backslash 0) \right\}, \tag{4.59}$$

where $\xi = \sigma_1(D_x)$. This characterizes $S^{m,k}_{\mathrm{hg}}$ as well. Clearly, the homogeneous symbols (S_{hg}) could be replaced by the polyhomogeneous ones (S_{phg}), or by the usual symbols.

Definition 4.6. A homogeneous C^∞ Lagrangian submanifold $\Lambda \subset T^*M\backslash 0$ is called *b-polynomially* defined if and only if there exists an ideal $\mathcal{I} \subset \bigcup_{m,k} S^{m,k}_{\mathrm{phg}}({}^bT^*M\backslash 0)$ such that

$$\Lambda = \{m \in T^*M\backslash 0 :\ \jmath^* f(m) = 0,\ \ \forall\ f \in \mathcal{I}\}.$$

Putting

$${}^b\mathcal{M}^{\#}_p(\Lambda) = \mathcal{I} \cap \bigcup_{k\ge 0} S^{p,k}_{\mathrm{phg}}({}^bT^*M\backslash 0) \tag{4.60}$$

and using elliptic elements of $S^0_{\mathrm{phg}}({}^bT^*M\backslash 0)$ we see that a b-polynomially defined Lagrangian Λ is given as

$$\Lambda = \{m \in T^*M\backslash 0 : \jmath^* a(m) = 0,\ \forall\ a \in {}^b\mathcal{M}^{\#}_p(\Lambda)\}. \tag{4.61}$$

We can define ${}^b\mathcal{M}^{\#}_p(\Lambda)$ for an arbitrary $\Lambda \subset T^*M\backslash 0$ as the set of $a \in \bigcup_{k\ge 0} S^{p,k}_{\mathrm{phg}}({}^bT^*M\backslash 0)$ such that $\jmath^* a|_\Lambda = 0$. Then for b-polynomially defined Lagrangians (4.61) holds. One easily observes that if $\jmath\Lambda = {}^b\Lambda$ is a smooth b-Lagrangian, then ${}^b\mathcal{M}^{\#}_m(\Lambda) \supset {}^b\mathcal{M}_1(\Lambda)$ with strict inclusion in general (see for instance Example 4.1 above; equality occurs for $\Lambda = N^*\partial M$).

The property of being b-polynomially defined is local near each fiber of $T^*_{\partial M}M$. Thus if $m \in T^*_{\partial M}M$, we say a C^∞ Lagrangian Λ is b-polynomially defined near m if for an open cone $\Gamma \subset T^*M\backslash 0$, $m \in \Gamma$

$$\Lambda \cap \Gamma = \{m \in \Gamma :\ \jmath^* f(m),\ \forall\ f \in \mathcal{L}_\Gamma\} \tag{4.62}$$

where $\mathcal{I}_\Gamma$ is an ideal in $\bigcup_{p,k} S^{p,k}_{\mathrm{phg}}({}^bT^*M\backslash 0)$. If Γ is everywhere locally b-polynomially defined, a partition of unity argument shows that Λ is b-polynomially defined. We could rephrase the local definition by saying that Λ is defined by ${}^b\mathcal{M}^{\#}_p(\Lambda)$ near m (or in Γ).

Example 4.3. We give an example of a Lagrangian submanifold which is *not b*-polynomially defined and to do that we start with the non-homogeneous case. Thus we define $\Lambda \subset N \subset T^*\bar{\mathbb{R}}^2_+$, for a small neighbourhood of $(0,0)$, N, as the set of zeros of

$$f_1(x,y;\xi,\eta) = f(\xi) + y(\xi+1),$$
$$f_2(x,y;\xi,\eta) = x + \eta(\xi+1)^{-1}(y + f'(\xi))$$

where $f \in C^\infty(\mathbb{R})$, $f^{(k)}(0) = 0$ for all k and $f(t) \neq 0$ if $t \neq 0$. We easily check that $df_1(0;0)$ and $df_2(0;0)$ are linearly independent and that $\{f_1, f_2\} \equiv 0$. Since $\partial_\xi^k f_1(0;0) = 0$ for all k, $f_1(0,y;\xi,\eta)$ cannot be written as a polynomial in ξ. By introducing an additional variable z with the dual ζ, a homogeneous example is obtained by taking $\bar{f}_i(x,y,z;\xi,\eta,\zeta) = f_i(x,y;\xi/\zeta,\eta/\zeta)$, $i = 1,2$ and choosing a homogeneous function $\bar{f}_3$, so that $\{\bar{f}_i, \bar{f}_3\} = 0$, as we may by Darboux's theorem. The vanishing of $\bar{f}_i$'s defines a smooth homogeneous Lagrangian submanifold in a conic neighbourhood of $(0;(0,0,1)) \in T^*\mathbb{R}^3 \setminus 0$.

If $\Gamma^0 = \Gamma \cap T^*M^\circ\backslash 0$, where Γ is the open cone above, we can consider a canonical transformation

$$\chi : \Gamma^\circ \longrightarrow T^*M^\circ\backslash 0, \tag{4.63}$$

which we assume extends to a b-canonical transformation [25]

$${}^b\chi : {}^b\Gamma \longrightarrow {}^bT^*M\backslash 0, \qquad \jmath(\Gamma) \subset {}^b\Gamma \tag{4.64}$$

such that

$${}^b\chi(\jmath(m)) = \jmath(m_1), \qquad m_1 \in T^*_{\partial M}M\backslash 0. \tag{4.65}$$

The basic invariance property is now given in

Lemma 4.3. *Let us assume that $\Lambda, \Lambda_1 \subset T^*M\backslash 0$ are homogeneous C^∞ Lagrangian submanifolds and χ with the properties* (4.63), (4.64) *and* (4.65) *satisfies also*

$$\chi(\Lambda \cap \Gamma^0) \subset \Lambda_1, \qquad N^*\partial M \cap \Lambda = \emptyset. \tag{4.66}$$

If Λ_1 is locally b-polynomially defined near m_1, then Λ is locally b-polynomially defined near m.

Proof. The second condition in (4.66) shows that $\Lambda = \mathrm{cl}(\Lambda \cap T^*M^0\backslash 0)$ which in view of the first condition holds also for Λ_1. Thus Λ_1 is defined by ${}^b\mathcal{M}_1^\#(\Lambda_1)|_{T^*M^\circ\backslash 0}$ near m_1. Since ${}^b\mathcal{M}_1^\#(\Lambda_1)|_{T^*M^\circ\backslash 0} \subset S^1(T^*M^\circ\backslash 0)$, Λ is defined by $\chi^*({}^b\mathcal{M}_1^\#(\Lambda_1)|_{T^*M^\circ\backslash 0})$ near m. Recalling that ${}^b\chi^* x = ax$ and ${}^b\chi^*\lambda = b\lambda + cx$, $a, b \neq 0$, and that ${}^b\chi^*$ preserves $S^m_{\mathrm{phg}}({}^bT^*M\backslash)$, we easily see that

$${}^b\chi^* : \ S^{m,k}_{\mathrm{phg}}({}^bT^*M\backslash 0) \longrightarrow S^{m,k}_{\mathrm{phg}}({}^b\Gamma).$$

Thus $\chi^*({}^b\mathcal{M}_1^\#(\Lambda_1)|_{T^*M^0\backslash 0})$ extends to $\bigcup\limits_{k\geq 0} S^{m,k}_{\mathrm{phg}}({}^b\Gamma)$, and this extension defines Λ near m. ∎

We next consider marked Lagrangians. Let $K \subset \Lambda$ be a homogeneous submanifold of codimension 1 satisfying the condition

$$K \text{ is tangent to } T^*_{\partial X}X \text{ to a \textit{fixed} finite order.} \tag{4.67}$$

Note that we do not demand that $K = \{m \in \Lambda : \ a(m) = 0\}$, $a \in S^{m,k}_{\mathrm{phg}}({}^bT^*M\backslash 0)$ even though this is the case in all applications. For Λ and K satisfying (4.67) we define,

$$ {}^b\mathcal{M}_1^{\#}(\Lambda, K) = \{a \in {}^b\mathcal{M}_1^{\#}(\Lambda) : \ H_{(\jmath^* a|_{T^*M^\circ\backslash 0})} \text{ is tangent to } K \cap T^*M^\circ\backslash 0\}. $$

The b-canonical transformations preserve b-Hamilton vector fields which satisfy

$$ {}^bH_f|_{({}^bT^*M\backslash 0)^\circ} = H_{\jmath^* f|_{T^*M^\circ\backslash 0}}. $$

Thus, because of (4.64), the definition above is invariant and we obtain the following analogue of Lemma 4.3:

Lemma 4.4. *Let* $\chi, \Lambda, \Lambda, \Lambda_1$ *be as in Lemma 4.3 and assume that homogeneous hypersurfaces* $K \subset \Lambda$, $K_1 \subset \Lambda$ *satisfy* (4.67) *and*

$$ \chi(K \cap \Gamma) \subset K_1. \tag{4.68} $$

Then

$$ f \in {}^b\mathcal{M}_1^{\#}(\Lambda_1, K_1), \ \operatorname{supp} f \subset {}^b\chi({}^b\Gamma) \Longrightarrow {}^b\chi_1^* f \in \mathcal{M}_1^{\#}(\Lambda, K). $$

To quantize these geometric notions we recall the definition of a class of operators from [25], Sect.III.5:

$$ \Psi_b^{m,k}(M) = \left\{ \sum_{\text{finite}} P_i A_i + B_i Q_i : \quad A_i, B_i \in \Psi_b^{m-m_i}(M), P_i, Q_i \in \mathrm{Diff}^{m_i}(M), \ m_i \leq k \right\}. $$

By restricting to the interior M°, and using (4.59) one easily obtains a surjective symbol map

$$ {}^b\sigma_{m,k} : \ \Psi_b^{m,k}(M) \to S^{m,k}({}^bT^*M\backslash 0), \quad {}^b\sigma_{m,k}(PA) = \sigma_{\mathrm{ord}(P)}(P) \cdot {}^b\sigma_{m-\mathrm{ord}(P)}(A), $$

if P is a homogeneous differential operator of $\mathrm{ord}(P) \leq k$. These leads us to define

$$ \Psi_b^{m,k}(M, \Lambda) = \{C \in \Psi_b^{m,k}(M) : \ {}^b\sigma_{m,k}(C) \in {}^b\mathcal{M}_m^{\#}(\Lambda)\} \tag{4.69} $$

and similarly

$$ \Psi_b^{m,k}(M; \Lambda, K) = \{C \in \Psi_b^{m,k}(M) : \ {}^b\sigma_{m,k}(C) \in {}^b\mathcal{M}_m^{\#}(\Lambda, K)\}. $$

We can also consider a disjoint union of two cleanly intersecting Lagrangians Λ_1, Λ_2, $\Lambda_1 \sqcup \Lambda_2$ and define ${}^b\mathcal{M}_m^{\#}(\Lambda_1 \sqcup \Lambda_2)$ and $\Psi_b^{m,k}(M, \Lambda_1 \sqcup \Lambda_2)$ with the analogue of Lemma 4.4 easily available. We note that

$$ \begin{aligned} &{}^b\mathcal{M}_m^{\#}(\Lambda_1 \sqcup \Lambda_2) \subset {}^b\mathcal{M}_m^{\#}(\Lambda_i), \qquad i = 1, 2 \\ &\Psi_b^{m,k}(\Lambda_1 \sqcup \Lambda_2) \subset \Psi_b^{m,k}(\Lambda_i). \end{aligned} \tag{4.70} $$

Let us consider the following Lagrangian varieties

$$ \mathcal{L} = \Lambda, \quad \{\Lambda, K\} \ \text{ or } \ \Lambda_1 \sqcup \Lambda_2 \tag{4.71} $$

with Λ, K, Λ_i above. We say that $\mathcal{L}$ is b-polynomially defined if the Lagrangians in $\mathcal{L}$ are b-polynomially defined and K satisfies (4.67). We then have

Definition 4.7. For a Lagrangian variety $\mathcal{L}$, $l \in \mathbb{N}_0$ and $s \in \mathbb{R}$, we define the space of distributions

$$ \begin{aligned} I_l^b \bar{H}_{(s)}(M; \mathcal{L}) = \{u \in \bar{H}_{(s)}(M) : \ & C_1 \dots C_{l'} u \in \bar{H}_{(s)}(M) \\ \text{if } \ & C_i \in \Psi_b^{1,k}(M, \mathcal{L}) \text{ for some } k, \text{ and any } l' \leq l\}. \end{aligned} \tag{4.72} $$

To study the invariance we recall from [25], Sect.III.5, that if F is an elliptic b-Fourier Integral Operator of order 0 and G its parametrix then

$$F\Psi_b^{m,k}(M)G \subset \Psi_b^{m,k}(M). \tag{4.73}$$

If F is associated to a b-canonical transformation satisfying (4.63) and (4.64) we rewrite (4.66) and (4.68) as

$$\chi(\mathcal{L}\cap\Gamma) \subset \mathcal{L}_1 \tag{4.74}$$

with an obvious modification for $\mathcal{L} = \Lambda_1 \sqcup \Lambda_2$. Let $C \in \Psi_b^{m,k}(M)$ and $C_1 = F\circ C\circ G$. By (4.73), $C_1 \in \Psi_b^{m,k}(M)$ and we want to compute its symbol. Let F^0, G^0, C^0 and C_1^0 be restriction of the operators above to M°. Then C^0 and C_1^0 are pseudodifferential operators of order m on compact subsets of M°. Egorov's Theorem (see Theorem 25.3.5 of [14]) shows that $\chi^*\sigma_m(F^0\circ C^0\circ G^0) = \sigma_m(C^0)$ on the subsets of Γ which have compact projections to M°. It follows from the definition of ${}^b\sigma_{m,k}$ that

$${}^b\sigma_{m,k}(C_1) = {}^b\chi^{*b}\sigma_{m,k}(F\circ C_1\circ G) = {}^b\sigma_{m,k}(C) \quad \text{in } {}^b\Gamma.$$

Combining this discussion with Lemmas 4.3 and 4.4 we obtain

Proposition 4.4. *If χ satisfies (4.63), (4.64) and (4.74) and F is an elliptic b-Fourier Integral Operator of order 0 associated to ${}^b\chi$, then for $\mathcal{L}$ locally b-polynomially defined near Γ*

$$u \in I_l^b\bar{H}_{(s)}(M,\mathcal{L}) \quad \textit{and} \quad WF_b(u) \subset {}^b\Gamma \Longrightarrow Fu \in I_l^b\bar{H}_{(s)}(M,\mathcal{L}).$$

The main application will be to Lagrangians which are actually *polynomially defined.* That property does not carry the necessary invariance but it is very useful for computations (see Sect.6).

Proposition 4.5. *Suppose that $\mathcal{L}$ is b-polynomially defined and that ${}^b\mathcal{M}_1^\#(\mathcal{L})$ is spanned over $\Psi_b^0(M)$ by $\{A_jQ_j\}_{j\in J}$, $Q_j \in \mathrm{Diff}^{m_j}(M)$, $A_j \in \Psi_b^{1-m_j}(M)$, A_j elliptic (in the totally characteristic sense). Then*

$$u \in I_k\bar{H}_{(s)}(M,\mathcal{L}) \Longrightarrow Q_{j_1}\dots Q_{j_l}u \in \bar{H}_{(s+l-\sum_{p\leq l} m_{j_p})}(M), \ l\leq k, \quad j_p\in J. \tag{4.75}$$

Proof. Definition 4.7 implies that $A_{j_1}Q_{j_1}\dots A_{j_l}Q_{j_l}u \in \bar{H}_{(s)}(M)$. We recall the commutation relation:

$$\begin{aligned} &Q\in\Psi_b^{m,k}(M),\ A\in\Psi_b^l(M) \Longrightarrow AQ = QA_1 + Q',\\ &A_1\in\Psi_b^l(m),\ {}^b\sigma_l(A) = {}^b\sigma_l(A_1),\ Q'\in\Psi_b^{m+l-1,k}\end{aligned} \tag{4.76}$$

which follows from iterating

$$AV = V(xAx^{-1}) + [Vx, A]x^{-1},\ V\in\mathrm{Diff}^1(M), \quad [Vx,A]x^{-1}\in\Psi_b^{l-1}(M).$$

We also observe that

$$\Psi_b^{m,k}(M)\cdot I_q\bar{H}_{(r)}(M,\partial M) \subset I_{q-m+k}\bar{H}_{(r-k)}(M,\partial M),\ q,\ q-m+k\geq 0.$$

Thus by successive applications of (4.76), we obtain that for an elliptic operator $A\in\Psi_b^L(M)$, $L = l-\sum_{p\leq l} m_{j_p}$, $AQ_{j_1}\dots Q_{j_l}u \in I_{-L}\bar{H}_{(s+L)}(M,\partial M)$. Let B be the parametrix of A in $\Psi_b^{-L}(M)$. Applying it to the previous expression we obtain (4.75). □

5. Refined estimates in the past

In the space of distributions defined in Sect.3 only the forward cone and part of the reflected front (the cusp) appear. Since the only data in the past is the incident front this is natural in the case of forward propagation. However, in principle some singularities could appear in the past on the boundary of the resolved space $\widetilde{X}_4$ and the purpose of this section is to present some refined conormal estimates which show the absence of such a behaviour. The essential component here is the use of weighted L^2-estimates 'in the past' similar to those in [32].

We will use the coordinates introduced in Propositions 3.1 and 3.2 and define the *non-homogeneous past*, K, using the model operator P_0. To start, we define $q_0 \in C^\infty(\widetilde{X}_1)$, independent of r and y as

$$q_0(\omega) = r^{-4}(\beta^* q)(r,\omega), \quad q(x) = x_1^4 + 8x_1x_2 - 6x_1^2x_2 - 3x_2^2$$

Thus $\beta^* Q = \{(r,\omega,y) : q_0(\omega) = 0\}$ is the model cone (see Proposition 3.1). Throughout this section we will write $\beta \equiv \beta_1$. We then consider

$$\widetilde{X}_1 \setminus \{ \ (r,\omega,y) \ : \ q_0(\omega) = -\epsilon \ \}, \quad \epsilon > 0 \tag{5.1}$$

which for small ϵ has three components.

Definition 5.1. The past in $\widetilde{X}_1$, K_1, is defined as the component of (5.1) which contains $\beta^* Q_-$, where Q_- is the retarded model cone over Γ. We then define

$$K = \beta_* K_1, \tag{5.2}$$

The intersection of K_1 and $\partial\widetilde{X}_1$ is shown in Fig.8. It is actually convenient to consider also a 1-2-3 homogeneous change of variables (cf. Proposition 3.3) which allows us to write

$$q(x) = 4x_1x_3 - x_2^2, \quad P_0 = D_{x_1}D_{x_3} - D_{x_2}^2, \tag{5.3}$$

and

$$P = P_0 + Q, \quad Q \in \mathrm{Diff}_{3,(1-2-3)}(\widetilde{X}). \tag{5.4}$$

The surface ∂K is smooth and spacelike for P away from Γ (if $\widetilde{X}$ is sufficiently small). More precisely:

Proposition 5.1. *If* $r(x,y) = (\sum_{1\le i\le 3} x_i^{12/i})^{1/12}$ *is the 1-2-3 radial variable and* $\mathbf{S}_z^-$, $z \in \widetilde{X}$, *is the retarded solid* P*-cone over* z *then*
a) $\mathbf{S}_z^- \cap \widetilde{X} \subset K \cap \widetilde{X}$ *if* $z \in K \cap \widetilde{X}$.
b) $\inf\{r(z') : z' \in \mathbf{S}_z^-\} \ge C^{-1} r(z)$ *if* $z \in K \cap \widetilde{X}$.

Proof. Since the singular set of ∂K is of codimension 2, and thus the smooth part, $\partial K \setminus \Gamma$, is connected, it suffices for part a) to show that $\partial K \setminus \Gamma$ is space like, i.e. that for $m \in \partial K \setminus \Gamma$ and f, a defining function of ∂K near m, $p(df(m)) > 0$. As the function f we can take $\beta_* q_0 + \epsilon$ which is smooth in a sufficiently small neighbourhood of m. Thus we want

$$p(d(\beta_* q_0)) = r^{-4} p_\beta({}^b dq_0) > cr^{-4}, \quad p_\beta = {}^b\sigma_2(r^4 \beta^* P \beta_*), \quad c > 0.$$

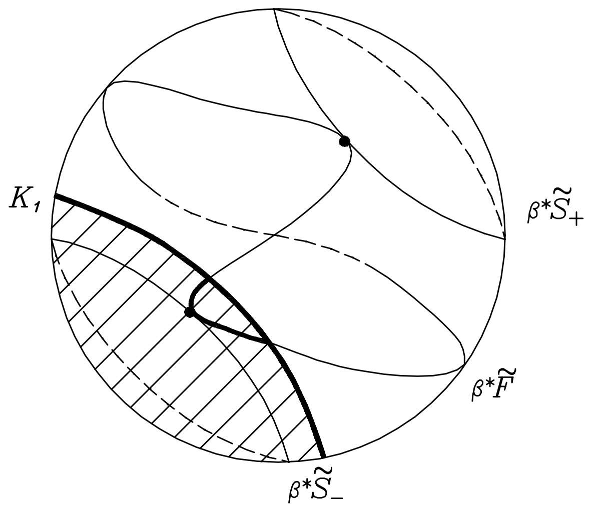

FIGURE 8. $K_1 \cap \partial\widetilde{X}_1$, $\beta^*\widetilde{F} \cap \partial\widetilde{X}_1$ and $\beta^* Q_\pm \cap \partial\widetilde{X}_1$

From (5.4) we see that $\beta^* P \beta_* = \beta^* P_0 \beta_* + r Q_1$, $Q_1 \in \operatorname{Diff}^2_b(\widetilde{X}_1)$. Hence, by shrinking the domain, it suffices to show that

$$p_{0\beta}(\,{}^b dq_0) > c, \quad p_{0\beta} = {}^b\sigma_2(r^4 \beta^* P_0 \beta_*)$$

for $q_0 = -\epsilon$. Since q_0 is independent of r and y, the left hand side can computed in local coordinates on $\partial\widetilde{X}_1$, yielding c proportional to ϵ.

To establish b) let us consider the retarded cone S^-_x, $x \in \mathbb{R}^3$, over $\{(x,y) : y \in \mathbb{R}^{n-3}\} \cap N$ where N is a small neighbourhood of $\Gamma \in \mathbb{R}^n$. It is defined as the projection to N of the union of the maximally extended retarded bicharacteristics starting at $N^*(\{(x,y) : y \in \mathbb{R}^{n-3}\} \cap N) \cap p^{-1}(0)$. By the analogy with $\mathbf{S}^-_z$ we define the solid retarded cone over $\{(x,y) : y \in \mathbb{R}^{n-3}\} \cap N$, $\mathbf{S}^-_x$. We then claim that for N small enough

$$\mathbf{S}^-_{(x,y)} \cap N \subset \mathbf{S}^-_x \cap N \tag{5.5}$$

for which it is enough to show that the tangent cone of $S^-_{(x,y)}$, $T_{(x,y)} S^-_{(x,y)}$ is contained in a component of $\mathbb{R}^n \setminus T_{(x,y)} S^-_x$, where $T_{(x,y)} S^-_x$ is the tangent cone of S^-_x. Since S^-_x is characteristic, $T_{(x,y)} S^-_x$ is tangent to $T_{(x,y)} S^-_{(x,y)}$ and the conclusion follows from the general fact about quadratic forms: let g be a Lorentz quadratic form (i.e. of indices of inertia $1, n-1$). If $C_\pm$ are the connected components of the cone $C = \{X : g(X) > 0\}$ and Π is a plane tangent to C then Π separates C_+ and C_-.

It now follows from part a) that $\Gamma \cap \mathbf{S}^{-\,\circ}_z = \emptyset$ if $z \in K \cap \widetilde{X}$, as otherwise a neighbourhood of a point in Γ would be contained in $\mathbf{S}^{-\,\circ}_z$ ($\Gamma \subset \partial K$).

Thus the minimal value of $r(z')$, $z' \in \mathbf{S}^-_z$, $z \in K \cap \widetilde{X}$ is attained on $\partial \mathbf{S}^-_z = S^-_z$. We also note that (5.5) implies

$$\inf\nolimits_{z \in S^-_{(x,y)} \cap N} r(z) \geq \inf\nolimits_{z \in S^-_x \cap N} r(z), \tag{5.6}$$

so we would like the bound for the right hand side.

We will first obtain it for Q^-_x, that is, for $P = P_0$, the operator with constant coefficients and homogeneous of degree -4. In that case $Q_x = \{(x', y') : q(x' - x) = 0\}$ for a 1-2-3 homogeneous polynomial of degree 4, q. If $r(x,y) = 1$ we easily see that

$0 \notin Q_x^-$. In fact, we would then have $(x,y) \in Q_0^+$ which contradicts $\beta^* K \cap \beta^* Q_0^+ = \emptyset$ if $(x,y) \in K$. Thus, for $(x,y) \in K$, $r(x,y) = 1$, we have $r(x',y') \geq C^{-1}$ if $(x',y') \in Q_x^-$.

For any $(x,y) = z \in K$ we obtain $(\tilde{x},\tilde{y}) = \tilde{z} = T_{r(z)^{-1}} z \in K$ with $r(\tilde{z}) = 1$. Since q is homogeneous we then have $Q_x^- = (T_{r(z)}) Q_{\tilde{x}}^-$ and for $z' \in Q_x^-$,

$$r(z') \geq r(z) \inf\{r(v) : v \in Q_{\tilde{x}}^-\} \geq C^{-1} r(z).$$

The general case follows from a perturbation argument based on (5.5) and (5.6).

Let us first observe that

$$\inf_{z \in S_{(x,y)}^-} r(z) = \inf_{\gamma_{(x,y)}^-} \inf_{z \in \gamma_{(x,y)}^-} r(z),$$

where $\gamma_{(x,y)}^-$ runs through all the retarded characteristics starting at (x,y). For $\gamma = \gamma_{(x,y)}^-$, let us define

$$I_\gamma = \inf\{r(\gamma(t)) : 0 < t < \inf\{t : r(\gamma(t)) > 2r(x,y)\}\}$$

and it is enough to show that for some $c > 0$ and $(x,y) \in K \cap N$, $I_{\gamma_{(x,y)}^-} > cr(x,y)$. In fact,

$$\begin{gathered}\inf\{r(\gamma(t)) : 0 < t < t_\infty, \gamma(t) \in N\} = \inf_k \inf\{r(\gamma(t)) : t_k^1 < t < t_k^2\} = \\ \inf_k \inf_{\tilde{\gamma}_k = \gamma_{\gamma(t_k^1)}^-} \inf\{r(\tilde{\gamma}_k(t)) : 0 < t < t_k^2 - t_k^1\} \geq \inf_k I_{\tilde{\gamma}_k} > cr(x,y),\end{gathered}$$

if we choose $t_0^1 = 0$ and

$$t_k^2 = \inf\{t > t_k^1 : r(\gamma(t)) > 2r(\gamma(t_k^1))\}, \quad t_{k+1}^1 = \sup\{t > t_k^2 : r(\gamma(t)) > r(\gamma(t_k^2))\}.$$

Part a) of the proposition was used here in asserting that $\gamma(t_k^1) \in K$.

In view of (5.5) we obtain

$$I_{\gamma_{(x,y)}^-} \geq \inf_y \inf_{\tilde{\gamma} = \tilde{\gamma}_{(x,y)}^-} I_{\tilde{\gamma}},$$

where $\tilde{\gamma}$ runs through the retarded characteristics lying in S_x^-. Consequently, we only need to show that $I_{\tilde{\gamma}} > cr(x)$ for any such $\tilde{\gamma}$.

Let us make a change of variables

$$z = T_{\delta^{-1}} z', \quad z = (x,y), \; z' = (x',y')$$

so that

$$P(z', D_{z'}) = \delta^{-4}(P_0(x, D_x) + \delta P_1(\delta, z, D_z))$$

and we introduce the operator $P_\delta(z, D_z) = P_0 + \delta P_1$. Using its principal symbol p_δ, we can define $\tilde{\gamma}_{(x,y),\delta}^-$ and $S_{x,\delta}^\pm$ as we did for p. It follows that

$$\tilde{\gamma}_{T_\delta(x,y)}^- = T_\delta \tilde{\gamma}_{(x,y),\delta}^-, \quad S_{T_\delta x}^- = T_\delta S_{x,\delta}^-, \tag{5.7}$$

To describe $\tilde{\gamma}_{(x,y),\delta}^-$ we can use the parametrization

$$\tilde{\gamma}_{(x,y),\delta}^-(t) = \pi\left(\exp(tH_{p_\delta}(x,y;\xi,0))\right), \quad |\xi| = 1, t > 0,$$

with $(x,y;\xi,0)$ in the retarded component of

$$p_\delta^{-1}(0) \cap N_{(x,y)}^*\{(x,\tilde{y}) : \tilde{y} \in \mathbb{R}^{n-3}\}.$$

Since $P_0(x, D_x)$ is strictly hyperbolic in $\mathbb{R}^3$ (or by a direct computation) there exists T independent of $0 < \delta < \delta_0$ such that for $T < t < 2T$, $r(\tilde{\gamma}_{(x,y),\delta}^-(t)) > 2$. This is obtained by first arranging that $d(\Gamma, \tilde{\gamma}(t)) > C$, $T < t < 2T$ where d is the euclidean distance and then observing that $r(z) > C'\min(d(z,\Gamma), d(z,\Gamma)^{\frac{1}{3}})$. Hence for $r(x) = 1$,

$$I_{\tilde{\gamma}^-_{T_\delta(x,y)}} \geq \inf_{0<t<T} r(T_\delta \widetilde{\gamma}^-_{(x,y),\delta}(t)),$$

and it is enough to show that there exists $\delta_0 > 0$ such that for $\delta < \delta_0$ and $r(x) = 1$, $\inf_{0<t<T} r(\gamma_{(x,y),\delta}(t)) > c$. Since

$$d(p_\delta^{-1}(0) \cap \{\eta = 0\} \cap S^*N, p_0^{-1}(0) \cap \{\eta = 0\} \cap S^*N) \longrightarrow 0, \ \delta \longrightarrow 0,$$

the continuous dependence on parameters and initial data for solution of ordinary differential equations implies that

$$\sup_{0<t<T} d(\tilde{\gamma}^-_{(x,y),0}(t), \tilde{\gamma}^-_{(x,y),\gamma}(t)) \longrightarrow 0, \ \delta \longrightarrow 0.$$

On the other hand $Q_x^- = S_{x,0}^-$ and hence

$$\inf r(\tilde{\gamma}_{(x,y),0}(t)) \geq \inf_{z \in Q_x^-} r(z) \geq C^{-1}.$$

By taking δ small enough we get $c = 1/2C$. □

If $\phi(z)$ is a time function for P (in view of Proposition 3.1, we can take $\phi(x,y) = x_1 + x_3$) once we arrange in Lemma 5.2 below that $\sigma_2(Q)|_{N^*\Gamma} = 0$ we can assume that $\{z : |\phi(z)| < \delta\} \subset \widetilde{X} \subset \{z : |\phi(z)| < 2\delta\}$ for some small δ. We then define $\widetilde{X}_- = \{z \in \widetilde{X} : \phi(z) < -\delta/2\}$ and denoting $K \cap \widetilde{X}$ by K,

$$L^2_-(K) = \{\ u \in L^2(K) \ : \ u \equiv 0 \ \text{ in } \ K \cap \widetilde{X}_- \ \}.$$

If A is the forward fundamental solution of P we form the operators

$$B_j = \begin{cases} D_{x_j} A & 1 \leq j \leq 3 \\ D_{y_{j-3}} A & 4 \leq j \leq n \end{cases} \tag{5.8}$$

which in view of part a) of Proposition 5.1 and the energy estimate, have the mapping property

$$B_j : L^2_-(K) \longrightarrow L^2_-(K), \quad 1 \leq j \leq n.$$

We can now state

Lemma 5.1. *If A is the forward fundamental solution for P and $K, \widetilde{X}$ and B_j are defined above then*

$$B_j : r^{-p} L^2_-(K) \longrightarrow r^{-p} L^2_-(K) \tag{5.9}$$

for all $p \geq 1$ and $1 \leq j \leq n$.

Proof. By Proposition 5.1, $Af \in \mathcal{D}'(K)$ is well defined for $f \in L^2_-(K)$ and the energy inequality gives (5.9) for $p = 0$. If b_j is the Schwartz kernel of B_j, then, since $\operatorname{supp} b_j(z, \bullet) \subset \mathbf{S}^-_z$, part b) of Proposition 5.1 gives

$$b_j(z, z') = 0 \quad \text{if} \quad r(z') < cr(z), \ z, z' \in K. \tag{5.10}$$

One can now apply the dyadic decomposition argument as in [32]:

$$K = \bigcup K_j^\rho, \quad K_j^\rho = K \cap \{z : 2^{-j-1}\rho \leq r(z) \leq 2^{-j}\rho\}, \quad K^\rho_{(j)} = \bigcup_{k \leq j} K_k^\rho.$$

Let us define

$$u^\rho_{(j)}(z) = \begin{cases} u(z), & z \in K^\rho_{(j)} \\ 0 & z \in K \setminus K^\rho_{(j)}. \end{cases}$$

It is easy to see that with $L^2(K)$ norms and $p > 0$,

$$\|r^p u\|^2 \leq \rho^p \sum_j 2^{-2jp} \|u^\rho_{(j)}\|^2 \leq \frac{2^{4p}}{2^{2p}-1} \|r^p u\|^2. \tag{5.11}$$

It now follows from (5.10) that $(Bu)^1_{(j)} = (B(u_{(j)})^c)^1_{(j)}$ and from (5.11) that

$$\|r^p Bu\|^2 \leq c^{-p} \frac{2^{4p}}{2^{2p}-1} \|r^p u\|^2$$

which proves the lemma. □

We now show that if the only singular data for the free propagation in the past is the incident front, no new singularities appear in the past on the resolved level as well (see Fig.8). In particular, the retarded cone is not present at all.

Since, away from Γ, $\widetilde{F}$ and $\widetilde{S}$ are simply tangent along B we find that in coordinates in which (5.3) holds

$$\widetilde{F} = \{(x,y) : x_3 = x_2^2 a(x_1, x_2, y)\}, \quad a \in C^\infty.$$

The map

$$(x_1, x_2, x_3, y) \mapsto (x_1, x_2(1 - x_1 a)^{\frac{1}{2}}, x_3 - x_2^2 a, y)$$

preserves $\widetilde{S}$, B and maps $\widetilde{F}$ into

$$\widetilde{F} = \{(x,y) : x_3 = 0\}.$$

It is also convenient to assume, without any loss of generality, that the coefficient of $D^2_{x_2}$ in P is 1.

Proposition 5.2. *If the operators B_j are defined by (5.8) and K_1 is given by Definition 5.1 in the definition of K, $K_1 = \beta^* K$, then*

$$B_j : \beta_* I_k L^2_\nu(K_1, \partial\widetilde{X}_1 \sqcup \beta^*\widetilde{F}) \cap L^2_-(K) \longrightarrow \beta_* I_k L^2_\nu(K_1, \partial\widetilde{X}_1 \sqcup \beta^*\widetilde{F})$$

Before proceeding with the proof we shall establish

Lemma 5.2. *Let us define the following vector fields:*

$$V_{00} = \sum_{j=1}^{3} j x_j D_{x_j}, \ V_{0j} = D_{y_j}, \ 1 \leq j \leq n-3, \ V_{11} = D_{x_1}, \ V_{21} = D_{x_2}.$$

Then, after a possible change of coordinates satisfying (3.8),

$$[P, V_{ik}] = -4i\delta_{k0} P + \sum_{j,l} W_{ikjl} V_{jl} + Z_{ik}$$

where $W_{ikjl}, Z_{ik} \in$ $\mathrm{Diff}^1(X_1)$ and the coefficients of W_{ikjl} are $\mathcal{O}(r^{\max(0,j-i)})$.

Proof. We first want to arrange that $\tau_2(Q)|_{N^*\Gamma} = 0$ while preserving the form of $\widetilde{F}$ (with a change of coordinates satisfying (3.8)). That however is quite easy as the only terms in Q with homogeneous symbols possibly non–zero on $N^*\Gamma$ are $aD^2_{x_1} + bD_{x_1}D_{x_2}$. Thus replace a and b by $a(0,y)$ and $b(0,y)$ and make the change of variables

$$x_1 \longmapsto x_1 - \frac{b}{2} x_2 + (a - \frac{1}{4} b^2) x_3, \quad x_i \longmapsto x_i, \quad i \neq 1.$$

Since then $D_{x_1} \mapsto D_{x_1}$, $D_{x_2} + \frac{b}{2} D_{x_1} \mapsto D_{x_2}$ and $D_{x_3} - (a - \frac{b^2}{4}) D_{x_1} \mapsto D_{x_3}$, the reduction is complete as $D_{x_2}^2 - D_{x_1} D_{x_3} + a D_{x_1}^2 + b D_{x_1} D_{x_2} \mapsto D_{x_2}^2 - D_{x_1} D_{x_3}$.

Since $[P_0, V_{ik}]$ can be explicitly calculated and $\{x_3 = 0\}$ is characteristic for P, it is easy to see that we only need to check this for $[Q, V_{ik}]$, Q as in (5.4). Thus $W_{0k11} V_{11}$ contains only terms $D_{x_1} D_{x_2}$, $D_{x_1}^2$, $D_{x_3} D_{x_1}$, as the terms $D_{y_j} D_{x_1}$ can be put in $W_{0k0j} V_{0j}$. Since the symbols of Q and V_{0k} vanish on $N^*\Gamma$, the coefficients have at least one factor of x_i and thus are $\mathcal{O}(r)$.

The term $W_{0k21} V_{21}$ can only have $D_{x_3} D_{x_2}$, as $D_{x_2}^2$ does not occur ($D_{x_2}^2$ has coefficient 1 in P and $D_{x_1} D_{x_2}$, $D_{y_j} D_{x_2}$ can be included in $W_{0k11} V_{11}$ and $W_{0k0j} V_{0j}$ respectively). Since $D_{x_3} D_{x_2}$ is homogeneous of degree -5 it has to have a coefficient in $\mathcal{O}(r^2)$.

Finally we consider $W_{1121} V_{21}$ which has terms of homogeneity -4 or higher. Again, it has to have coefficients in $\mathcal{O}(r)$. $\square$

Proof of Proposition 5.2. We start by characterizing the push-forward of the conormal spaces in the proposition:

$$g \in \beta_* I_k L^2_\nu(\beta^* K, \partial \widetilde{X}_1 \sqcup \beta^* \widetilde{F}) \cap L^2_-(K) \Longleftrightarrow \prod_{i,k} V_{ik}^{\alpha_{ik}} g \in r^{-2\alpha_{21} - \alpha_{11}} L^2_-(K), \tag{5.12}$$

$|\alpha| \leq k$, which follows easily from the definition of the conormal space in $\widetilde{X}_1$. If $u = Af$, $f \in \beta_* I_k L^2_\nu(\beta^* K, \partial \widetilde{X}_1 \sqcup \beta^* \widetilde{F}) \cap L^2_-(K)$, we want $g = D_{x_i} u, D_{y_j} u$ to satisfy the condition on the right hand side of (5.12). To obtain that we shall use a more refined version of the usual 'system' argument (see [32], Sect.6). For that we will define

$$(u)_{p,m,l} \in L^2(\widetilde{X}; \mathbb{C}^N), \quad N = N(p,m,l) \in \mathbb{N}, \tag{5.13}$$

by an inductive procedure using

$$\begin{aligned} (u)^\sharp_{p,m,l} \ &= \ \{V_{0k}(u)^\sharp_{p-1,l,m}; V_{ik}(u)^\sharp_{p,m,l-1}; V_{jk}(u)^\sharp_{p,m-1,l} : \\ &\qquad i = 0,1; j = 0,1,2; k = 0, \cdots, n-3\}, \ (u)^\sharp_{0,0,0} = u, \end{aligned}$$

where the vector fields are applied to each component of $(u)^\sharp_\bullet$, assumed to be 0 if any of the indices is less than 0. Note that we allow repetitions and order the components of $(u)^\sharp_\bullet$ using the lexicographic ordering of products of vector fields. To define (5.13) we start by setting $(u)_{0,0,0} = u$ and continue with

$$\begin{aligned} (u)_{0,0,l+1} &= \{(u)_{0,0,l}; (u)^\sharp_{0,0,l+1}\}, \\ (u)_{0,m+1,l} &= \{(u)_{0,m,l}; (u)^\sharp_{0,m+1,l}\}, \\ (u)_{p+1,m,l} &= \{(u)_{p,m,l}; (u)^\sharp_{p+1,m,l}.\} \end{aligned}$$

where to identify with $\mathbb{C}^N$, $N = N(p,m,l)$ we again use the lexicographic ordering of products of vector fields. We again adopt a convention that $(u)_{p,m,l} = 0$ if any of the indices is less than 0.

The characterization (5.12) can be rephrased in terms of $(g)_{p,m,l}$ as follows:

$$g \in \beta_* I_k L^2_\nu(\beta^* K, \partial \widetilde{X}_1 \sqcup \beta^* \widetilde{F}) \cap L^2_-(K) \Longleftrightarrow (g)_{p,m,l} \in r^{-m-2l} L^2_-(K; \mathbb{C}^{N(p,m,l)}), \tag{5.14}$$

$p+m+l \le k$. If $Pu = f$, we will obtain the following system

$$\begin{aligned} \mathcal{P}_{p,m,l}(u)_{p,m,l} &= \mathcal{W}^{20}_{p,m,l}(u)_{p-1,m,l+1} + \mathcal{W}^{10}_{p,m,l}(u)_{p-1,m+1,l} + \\ & \quad \mathcal{W}^{11}_{p,m,l}(u)_{p,m-1,l+1} + \mathcal{F}_{p,m,l}(f)_{m,l,p} \end{aligned} \tag{5.15}$$

where

$$\mathcal{P}_{p,m,l} \in \mathrm{Diff}^2(\widetilde{X}; \mathbb{C}^{N(p,m,l)}, \mathbb{C}^{N(p,m,l)}), \quad \sigma_2(\mathcal{P}_{p,m,l}) = \sigma_2(P) Id_{\mathbb{C}^{N(p,m,l)}}, \tag{5.16}$$

$$\begin{aligned} &\mathcal{W}^{20}_{p,m,l} \in \mathrm{Diff}^1(\widetilde{X}; \mathbb{C}^{N(p,m,l)}, \mathbb{C}^{N(p-1,m,l+1)}), \\ &\mathcal{W}^{10}_{p,m,l} \in \mathrm{Diff}^1(\widetilde{X}; \mathbb{C}^{N(p,m,l)}, \mathbb{C}^{N(p-1,m+1,l)}), \\ &\mathcal{W}^{11}_{p,m,l} \in \mathrm{Diff}^1(\widetilde{X}; \mathbb{C}^{N(p,m,l)}, \mathbb{C}^{N(p,m-1,l+1)}), \end{aligned}$$

with the coefficients of $\mathcal{W}^{ij}_{p,m,l}$ in $\mathcal{O}(r^i)$, and $\mathcal{F}_{p,m,l} \in C^\infty(\widetilde{X}; \mathbb{C}^{N(p,m,l)} \otimes \mathbb{C}^{N(p,m,l)})$. If (5.16) holds then Lemma 5.2 is still available:

$$\begin{aligned} &[\mathcal{P}_{p,m,l}, V_{ik} Id_{\mathbb{C}^{N(p,m,l)}}] = -4i\delta_{k0}\mathcal{P}_{p,m,l} + \\ &\sum_{r,s} W_{ikrs} V_{rs} Id_{\mathbb{C}^{N(p,m,l)}} + \mathcal{Z}^{p,m,l}_{ik}, \ \mathcal{Z}^{p,m,l}_{ik} \in \mathrm{Diff}^1(\widetilde{X}, \mathbb{C}^{N(p,m,l)}, \mathbb{C}^{N(p,m,l)}), \end{aligned} \tag{5.17}$$

The operator in the system is defined by successive inductions based on (5.17) and the inductive definition of $(u)_{p,m,l}$:

$$\begin{aligned} \mathcal{P}_{0,0,l+1}(u)_{0,0,l+1} &= \{\mathcal{P}_{0,0,l}(u)_{0,0,l}; \mathcal{P}_{0,0,l}(V_{ik}(u)^\sharp_{0,0,l}) - \sum_{r,s} W_{ikrs}(V_{rs}(u)^\sharp_{0,0,l}) - \\ & \quad \mathcal{Z}^{0,0,l}_{ik}(u)^\sharp_{0,0,l}, i \neq 2\}, \end{aligned}$$

where we identify $V_{jk}(u)^\sharp_{0,0,l}$ with components of $(u)^\sharp_{0,0,l+1}$ but use $\mathcal{P}_{0,0,l}$ on $\{0; V_{ik}(u)^\sharp_{0,0,l}\} \in \mathcal{D}'(\widetilde{X}; \mathbb{C}^{N(0,0,l)})$. The terms on the right hand side of the system equation are $\mathcal{W}^{ij}_{0,0,l} \equiv 0$ and

$$\mathcal{F}_{0,0,l+1}(f)_{0,0,l+1} = \{\mathcal{F}_{0,0,l}(f)_{0,0,l}; V_{00}\mathcal{F}^\sharp_{0,0,l}(f)_{0,0,l} - 4i\mathcal{F}^\sharp_{0,0,l}(f)_{0,0,l}, V_{ik}(\mathcal{F}^\sharp_{0,0,l}(f)_{0,0,l}), i > 0\},$$

where $\mathcal{F}^\sharp_{0,0,l}(f)_{0,0,l} \overset{\text{def}}{=} \mathcal{P}_{0,0,l}(\{0, (u_{0,0,l})^\sharp\})$.

The next induction will introduce $\mathcal{W}^{11}_{0,p,l}$:

$$\begin{aligned} \mathcal{P}_{0,m+1,l}(u)_{0,m+1,l} = \{&\mathcal{P}_{0,m,l}(u)_{0,m,l}; \mathcal{P}_{0,m,l}(V_{jk}(u)^\sharp_{0,m,l}) - \sum_{s,r\neq 2} W_{jkrs}(V_{rs}(u)^\sharp_{0,m,l}) - \\ & \mathcal{Z}^{0,m,l}_{jk}(u)^\sharp_{0,m,l}; \mathcal{P}_{0,m+1,l-1}(V_{ik}(u)^\sharp_{0,m+1,l-1}) - \sum_{r,s} W_{ikrs}(V_{rs}(u)^\sharp_{0,m+1,l-1}) - \\ & \mathcal{Z}^{0,m-1,l+1}_{ik}(u)^\sharp_{0,m+1,l-1} : j = 0,1; i = 0,1,2; k = 0, \cdots, n-3\}, \end{aligned}$$

with the same convention as before. Thus $\mathcal{P}_{0,m+1,l}$ is constructed from $\mathcal{P}_{0,m,l}$ and $\mathcal{P}_{0,m+1,l-1}$. Now $\mathcal{W}^{i0}_{0,m+1,l} \equiv 0$ but

$$\mathcal{W}^{11}_{0,m+1,l}(u)_{0,m,l+1} = \{\mathcal{W}^{11}_{0,m,l}(u)_{0,m-1,l+1}; W_{jk21}(V_{21}(u)^\sharp_{0,m,l}); 0; j \neq 2\},$$

where we consider $V_{21}(u)^\sharp_{0,m,l}$ as a component of $(u)^\sharp_{0,m,l+1}$. We also have

$$\begin{aligned}\mathcal{F}_{0,m+1,l}(f)_{0,m+1,l} = \{&\mathcal{F}_{0,m,l}(f)_{0,m,l}; V_{00}(f)^\sharp_{0,m,l} - 4i\mathcal{F}^\sharp_{0,m,l}(f)_{0,m,l}, V_{1k}\mathcal{F}^\sharp_{0,m,l}(f)_{0,m,l};\\ &V_{00}\left(\mathcal{F}^\sharp_{0,m+1,l-1}(f)_{0,m+1,l-1}\right) - 4i(\mathcal{F}^\sharp_{0,m+1,l-1}(f)_{0,m+1,l-1}),\\ &V_{ik}\left(\mathcal{F}^\sharp_{0,m+1,l-1}(f)_{0,m+1,l-1}\right); \quad i=1,2; k=0,\cdots,n-3\}\end{aligned}$$

where $\mathcal{F}^\sharp_{0,m,l}(f)_{0,m,l} \overset{\text{def}}{=} \mathcal{P}_{0,m,l}(\{0,(u)^\sharp_{0,m,l}\})$.

Finally, we define

$$\begin{aligned}\mathcal{P}_{p+1,m,l}(u)_{p+1,m,l} = \{&\mathcal{P}_{p,m,l}(u)_{p,m,l}; \mathcal{P}_{p,m,l}(V_{0k}(u)^\sharp_{p,m,l}) - \sum_s W_{0k0s}(V_{0s}(u)^\sharp_{p,m,l}) -\\ &\mathcal{Z}^{p,m,l}_{0k}(u)^\sharp_{p,m,l}; \mathcal{P}_{p+1,m-1,l}(V_{jk}(u)^\sharp_{p+1,m-1,l}) - \sum_{s,r\neq 2} W_{jkrs}(V_{rs}(u)^\sharp_{p+1,m-1,l}) -\\ &\mathcal{Z}^{p+1,m-1,l}_{jk}(u)^\sharp_{p+1,m-1,l}; \mathcal{P}_{p+1,m,l-1}(V_{ik}(u)^\sharp_{p+1,m,l-1}) - \sum_{r,s} W_{ikrs}(V_{rs}(u)^\sharp_{p+1,m,l-1}) -\\ &\mathcal{Z}^{p+1,m,l+1}_{ik}(u)^\sharp_{p+1,m,l-1} : j=0,1; i=0,1,2; k=0,\cdots,n-3\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}^{10}_{p+1,m,l}(u)_{p,m+1,l} &= \{\mathcal{W}^{10}_{p,m,l}(u)_{p-1,m+1,l}; W_{0k11}(V_{11}(u)^\sharp_{p,m,l}); 0; 0; k=0,\cdots,n-3\},\\ \mathcal{W}^{20}_{p+1,m,l}(u)_{p,m,l+1} &= \{\mathcal{W}^{20}_{p,m,l}(u)_{p-1,m,l+1}; W_{0k21}(V_{21}(u)^\sharp_{p,m,l}); 0; 0; k=0,\cdots,n-3\},\\ \mathcal{W}^{11}_{p+1,m,l}(u)_{p+1,m-1,l+1} &= \{\mathcal{W}^{11}_{p,m,l}(u)_{p,m-1,l+1}; 0; W_{jk21}(V_{21}(u)^\sharp_{p+1,m-1,l}); 0;\\ &\qquad j=0,1; k=0,\cdots,n-3\},\end{aligned}$$

where again we use the convention that, for instance, $V_{11}(u)^\sharp_{p,m,l}$ is a component of $(u)^\sharp_{p,m+1,l}$. The definition of $\mathcal{F}_{p+1,m,l}$ is analogous to those in the previous cases.

Since the principal symbol of $\mathcal{P}_{p,m,l}$ is $PId_{\mathbb{C}^{N(p,m,l)}}$, Lemma 5.1 can be applied in this setting and thus

$$\begin{aligned}u = Af,\ (f)_{0,0,l} &\in r^{-2l}L^2(K,\mathbb{C}^{N(0,0,l)}) \Longrightarrow\\ (g)_{0,0,l} &\in r^{-2l}L^2(K,\mathbb{C}^{N(0,0,l)}),\ g = D_{x_i}u, D_{y_j}u.\end{aligned} \tag{5.18}$$

Using the right hand side of (5.18) and (5.15) we can now prove by induction that

$$(g)_{0,m,l} \in r^{-m-2l}L^2(K,\mathbb{C}^{N(0,m,l)}), \quad g = D_{x_i}u, D_{y_j}u. \tag{5.19}$$

In fact, $(f)_{0,m,l} \in r^{-m-2l}L^2(K,\mathbb{C}^{N(0,m,l)})$ in view of the characterization (5.14), and

$$\begin{aligned}\mathcal{W}^{01}_{0,m,l}(u)_{0,m-2,l+1} &\in r^{-(m-2)-2(l+1)}L^2 = r^{-m-2l}L^2,\\ \mathcal{W}^{11}_{0,m,l}(u)_{0,m-1,l+1} &\in \mathcal{O}(r)r^{-(m-1)-2(l+1)}L^2 \subset r^{-m-2l}L^2,\end{aligned}$$

by the induction hypothesis. Thus another application of Lemma 5.1 gives (5.19). Using that as the starting point of an induction on p concludes the proof. □

The immediate consequence is the following refinement of Theorem 3.2:

Theorem 5.1. *If $J_sL^2(\widetilde{X},H)$ is given by Definition 3.1 and*

$$Pu = f \quad \text{in } \widetilde{X}, \quad u|_{\widetilde{X}_-} = 0, \quad f \in J_sL^2_c(\widetilde{X},H), \quad f|_{\widetilde{X}_-} = 0,$$

then
$$u \in J^1_s L^2_{\rm loc}(\widetilde{X}, H).$$

Proof. Let us first take $s = k \in \mathbb{N}_0$. Then clearly
$$\|v\|_{J_k L^2(\tilde{X},H)} \leq \|\beta_5^* v\|_{I_k L^2_{\nu_5}(\tilde{X}_5,\mathcal{S}_5)} + 2\|\beta^* v\|_{I_k L^2_{\nu_1}(K_1,\partial X \sqcup \beta^* F)}.$$

If $v = \chi u$ or $v = D_j \chi u$, where $\chi \in C_0^\infty(\widetilde{X})$, then by Theorem 3.2 and Proposition 5.2 the right hand side above is bounded by
$$\|\beta_5^* f\|_{I_k L^2_{\nu_5}(\tilde{X}_5,\mathcal{S}_5)} + 2\|\beta^* f\|_{I_k L^2(K_1,\partial X \sqcup \beta^* F)} \leq 2\|f\|_{J_k L^2(\tilde{X},H)}.$$

Thus,
$$\|\chi u\|_{J^1_k L^2(\tilde{X},H)} \leq 2\|f\|_{J_k L^2(\tilde{X},H)},$$

for any $\chi \in C_0^\infty(\widetilde{X})$, so that the general case is immediate by interpolation. □

We would now like to have an analogue of Proposition 5.2 for the Dirichlet problem. It is convenient to find appropriate coordinate functions.

Lemma 5.3. *There exists a diffeomorphism satisfying 3.8 such that the coordinates given by Proposition 3.1 to transform to coordinates* (x, y) *near* $\Gamma = \{x = 0\}$ *in which*
$$\begin{gathered} P = P_0 + Q, \quad P_0 = D^2_{x_2} - x_2 D^2_{x_1} - D_{x_1} D_{x_3}, \\ Q \in \mathrm{Diff}^2_{3,1-2-3}, \quad \sigma_2(Q) \restriction_{N^*\Gamma} = 0. \end{gathered} \tag{5.20}$$

and
$$X = \{x_2 \geq 0\}, \quad \partial X = \{x_2 = 0\}. \tag{5.21}$$

Proof. Starting with the coordinates given by Proposition 3.1 we take an inverse near 0 of the transformation
$$x_1 \longmapsto x_1 \tag{5.22}$$
$$x_2 \longmapsto (16)^{\frac{1}{3}}(3x_2 - x_1^2) \tag{5.23}$$
$$x_3 \longmapsto 27x_3 + 9x_1x_2 + 5x_1^3 \tag{5.24}$$

so that the cusp is given by $\{(27x_3 + 9x_1x_2 + 5x_1^3)^2 - 16(3x_2 - x_1^2)^3 = 0\}$. The operator for which the conormal bundle of this cusp is characteristic has to be of the form $P_0 + Q$, $Q \in \mathrm{Diff}^2_{3,1-2-3}(\mathbb{R}^n)$ where we rescale x_1 if necessary – this can be seen directly or by transforming the statement of the proof in Proposition 3.1 (the motivation for this computation and the ones below is given by the explicit expressions for $\widetilde{\Lambda}_m$ in Sect.7).

As in the proof of Proposition 3.1 we can preserve the cusp and take the boundary to be $\{x_2 = 0\}$. Then also
$$\widetilde{F} = \{(x, y) : x_3 - x_1x_2 + \frac{1}{3}x_1^3 + f_4 = 0\}, \quad f_4 \in \mathcal{M}_\triangle^{\infty-\in-\ni}(\mathbb{R}^\backslash).$$

We claim that we can preserve $\partial X = \{X_2 = 0\}$ and map $\widetilde{F}$ to $\{x_3 - x_1x_2 + \frac{1}{3}x_1^3 = 0\}$ by a transformation satisfying (3.8). In fact, proceeding as in the proof of *iii)* in Proposition 3.3 we need a family of vector fields V_s tangent to ∂X such that
$$V_s(x_3 - x_2x_1 + \frac{1}{3}x_1^3 + sf_4) = -f_4.$$

and for that we can simply take, near $(0,y)$,

$$V_s = \frac{-f_4}{1+s\partial_{x_3}f_4}\partial_{x_3}, \quad \partial_{x_3}f_4(0,y)=0.$$

The resulting diffeomorphism clearly satisfies (3.8). It remains to check that

$$\sigma_2(Q)\restriction_{N^*\Gamma}=0.$$

Since $\{x_3 - x_1x_2 + \frac{1}{3}x_1^3 = 0\}$ is characteristic for P and P_0 it has to be characteristic for Q. The only terms with symbols nonvanishing on $N^*\Gamma$ are $aD_{x_1}^2 + bD_{x_1}D_{x_2}$ so that leads to the condition $a(x_1^2 - x_2) = bx_1$ implying $b(0,y) = a(0,y) = 0$. This concludes the proof of the lemma. □

We shall need a number of preliminary facts and we start by considering the following problem. Let V be a fixed vector field of the form

$$V = aD_{x_3} + bD_{x_1} + \langle c, D_y\rangle, \quad a,b\in \bar{C}^\infty(X,\mathbb{R}),\ c\in\bar{C}^\infty(X,\mathbb{R}^{n-3}).$$

If ϕ denotes a time function, let us consider the solution of

$$Pu = Vf, \quad u|_{\partial X} = g, \quad u\equiv 0 \ \text{ for } \ \phi(z) < -\delta \tag{5.25}$$

where $f\in C_0^\infty(\bar{X}_\delta)$, $g\in C_0^\infty(\bar{X}_\delta\cap\partial X)$ with X_δ, a neighbourhood of 0 in X such that $\phi(z) > -\delta$ in $\bar{X}_\delta$. We then define the map

$$T:(f,g)\longmapsto u$$

allowing also the notation $T_1 g = T(0,g)$ and $T_2 f = T(f,0)$ so that $T(f,g) = T_1g + T_2f$. Before proceeding with the analogue of Lemma 5.1 for T_1, T_2 we need the following

Lemma 5.4. *If $f\in C_0^\infty(\bar{X}_\delta)$ and $g\in C_0^\infty(\bar{X}_\delta\cap\partial X)$ with X_δ sufficiently small, bicharacteristically convex neighbourhood of* 0 *in X, then the solution of* (5.25) *satisfies*

$$\|u\|_{L^2(X_\delta)} \le C\delta^2\|f\|_{L^2(X_\delta)} + C\delta\|g\|_{L^2(\partial X\cap\bar{X}_\delta)}. \tag{5.26}$$

Proof. By translating $0\in X$ we shall assume that

$$X_\delta \subset X_\delta^\sharp = \{z\in X : -\delta < \phi(z) < 0\},$$

where, as we may, we take $\phi(x,y) = x_3$, (only the tangency of the time function to the boundary is important). Let us recall the energy inequality, [14], (24.1.4), (compare also (24.1.6) there):

$$\begin{aligned}\delta^{-1}\int_Q (|h'|^2 + |h|^2)dv \ &\le \ C\delta\int_Q |Ph|^2 dv + \\ &C\int_{\bar{Q}\cap\partial X}(|h|^2 + |D_{x_1}h|^2 + |D_{x_3}h|^2 + \sum_{i=1}^{n-3}|D_{y_i}h|^2)dS\end{aligned} \tag{5.27}$$

where $Q = \{z : x_2 > 0, x_3\le 0\}$ and $h\in\mathcal{S}(\mathbb{R}^n)$, $h=0$ if $\phi < -\delta$. Let $\chi\in C_0^\infty(\mathbb{R}^n)$ be supported in a sufficiently large ball around 0, so that $\chi(z)u(z) = u(z)$ if $-\delta<\phi(z)<0$. The existence of χ follows from the finite speed of propagation (see [8], Sect.VII.8). Consequently, it suffices to establish (5.26) with u replaced by χu.

We now follow, in a slightly modified way, the proof of Lemma 24.1.5 in [14] by first introducing

$$E_s(D')^* = ((1 + D_{x_1}^2 + \sum_{i=1}^{n-3} D_{y_j}^2)^{1/2} + iD_{x_3})^s$$

and then applying (5.27) to $h = E_{-1}(D')^*\chi u$. It is important here that $h = 0$ in $\phi(z) < -\delta$, since $u = 0$ there. We also recall that by Theorem B.2.4 of [14],

$$\|E_1^*(D')h\|_{L^2(X_\delta^\sharp)} \leq C(\|D_{x_1}h\|^2_{L^2(X_\delta^\sharp)} + \|D_{x_3}h\|^2_{L^2(X_\delta^\sharp)} + \sum_{i=1}^{n-3}\|D_{y_i}h\|^2_{L^2(X_\delta^\sharp)} + \|h\|^2_{L^2(X_\delta^\sharp)})^{\frac{1}{2}}$$

and

$$E_{-1}(D')^*[P, E_1^*] = R_{x_1}(z, D'_z)D_{x_1} + R_{x_3}(z, D'_z)D_{x_3} + \sum_{i=1}^{n-3} R_i(z, D'_z)D_{y_i} + R_0(z, D'_z),$$

where $R_{x_j}, R_i \in S^0(\mathbb{R}^n, \mathbb{R}^{n-1})$, $D'_z = (D_{x_1}, D_{x_3}, D_y)$, and $R_\bullet(z, D'_z)D_\bullet w = R_\bullet(z, D'_z)w_\bullet$, if $w_\bullet = D_\bullet w$ for $\phi(v) < 0$ and $w_\bullet = 0$ elsewhere.

Thus,

$$\begin{aligned}\|\chi u\|_{L^2(X_\delta^\sharp)} &= \|E_1(D')^*h\|_{L^2(X_\delta^\sharp)} \\ &\leq C\delta^2\|E_{-1}(D')^*Pu\|_{L^2(X_\delta^\sharp)} + C\delta\|E_{-1}(D')^*g\|_{H_{(1)}(\bar{X}_\delta^\sharp\cap\partial X)},\end{aligned}$$

since $\chi g = g$. We conclude the proof by observing that

$$\|E_{-1}(D')^*Pu\|_{L^2(X_\delta^\sharp)} \leq C\|Pu\|_{H_{(0,-1)}(X_\delta)} \leq C\|f\|_{L^2(X_\delta)} \tag{5.28}$$

and

$$\|E_{-1}(D')^*g\|_{H_{(1)}(\bar{X}_\delta^\sharp\cap\partial X)} \leq \|E_{-1}(D')^*g\|_{H_{(1)}(\partial X)} \leq \|g\|_{L^2(\partial X)} = \|g\|_{L^2(X_\delta\cap\partial X)}.$$

□

By following the proof of Lemma 24.1.6 in [14] and using (5.28) we also obtain the mapping property:

$$T : L^2(X_\delta) \times L^2(\bar{X}_\delta \cap \partial X) \longrightarrow L^2(X_\delta) \tag{5.29}$$

with small norms $\mathcal{O}(\delta^2, \delta)$, in the first and second factors, respectively.

Let us now modify the previous notation to the boundary value problem case. Thus we consider

$$K_\delta = K \cap X_\delta,$$

where K was defined by (5.2) and X_δ was given in Lemma 5.4. Similarly, we define

$$K_\delta^\partial = K_\delta \cap \partial X,$$

and $L^2_-(K_\delta)$, $L^2_-(K_\delta^\partial)$, by analogy with $L^2_-(K)$. With this notation we have

Lemma 5.5. *Let $K_\delta, K_\delta^\partial$ be as above. If $g \in L^2_-(K_\delta^\partial)$ then*

$$\|r^pT_1g\|_{L^2(K_\delta)} \leq C\delta\|r^pg\|_{L^2(K_\delta^\partial)}; \tag{5.30}$$

and if $f \in L^2_-(K_\delta)$ then

$$\|r^pT_2f\|_{L^2(K_\delta)} \leq C\delta^2\|r^pf\|_{L^2(K_\delta)}, \tag{5.31}$$

where, $T(f, g) = T_2f + T_1g$ is the solution operator for (5.25).

Proof. The support property required in the proof of Lemma 5.1 holds for the mixed problems as well (see [8], Sect.VII.8). Since the existence is guaranteed for f and g in L^2 we can again proceed using the dyadic decomposition. The details of the proof are the same. □

As in the free case we shall now consider the resolved space $X_1 = \mathrm{cl}(\beta^* X)$ which is a manifold with a codimension 2 corner $\partial\tilde{X}_1 \cap \beta^*\partial X$. Thus we have

$$\beta : X_1 \longrightarrow X, \quad \partial X_1 = \beta^*\partial X \cap (\partial\widetilde{X}_1 \cap X_1).$$

We also define (compare Sect.7 below)

$$\beta^\partial = \beta|_{\beta^*\partial X}, \quad \beta^\partial : \beta^*\partial X \longrightarrow \partial X.$$

The following estimate will be useful later:

Lemma 5.6. *Let $N \subset X$ be such that $\beta^* N \subset X_1$ is open with smooth boundary and let $u \in C^\infty(\widetilde{X})$ satisfy $Pu = 0$ in N. Then*

$$\sum_{|\alpha|+|\beta|\le l} \|r^{3\alpha_3+2\alpha_2+\alpha_1} D_x^\alpha D_y^\beta u\|_{L^2(N)} \le C_l \sum_{|\alpha|+|\beta|\le l} \|r^{3\alpha_3+\alpha_1} D_{x_1}^{\alpha_1} D_{x_3}^{\alpha_3} D_y^\beta u\|_{L^2(N)}, \tag{5.32}$$

for any $l \ge 0$.

Proof. Let us define

$$\begin{aligned} H^b_{(0,l)}(X_1) \;\; = \;\; & \{w \in L^2_\nu(X_1) : w = w_1 + w_2, \ \mathrm{supp}\ w_2 \cap \beta_1^*\partial X = \emptyset, \ w_2 \in H^b_{(l)}(\widetilde{X}_1), \\ & w_1 \text{ supported near } \beta_1^*\partial X, \ W^{\gamma_1}(rD_r)^{\gamma_2} D_y^{\gamma'} \in L^2_\nu(X_1), \\ & |\gamma| \le l, \ W = r^i \beta_1^* D_{x_i} \beta_*, \ i = 1, 3\}, \end{aligned}$$

which is simply the mixed b-Sobolev space based on $\widetilde{X}_1 \supset X_1$ and $\beta^*\partial X$. Clearly, X_1 can be replaced by $\beta^* N$ in this definition. Thus (5.32) can be rewritten as

$$\|\beta^* u\|_{H^b_{(l)}(\beta^* N)} \le C_l \|\beta^* u\|_{H^b_{(0,l)}(\beta^* N)}, \quad P_1 \beta_1^* u = 0 \ \text{ in } \ \beta^* N,$$

where $P_1 = r^4 \beta^* P \beta_*$. Since P_1 is b-noncharacteristic with respect to $\beta^*\partial X$, the estimate follows from an easy modification of the proof of Theorem B.2.9 of [14]- see Appendix B. □

We also need a modification of Lemma 5.2 to help us in this setting:

Lemma 5.7. *Let us define the following vector fields:*

$$V_{i1} = D_{x_i}, \ i = 1, 2, 3, \ V_{0j} = D_{y_j}, \ j = 1, \cdots, n-3.$$

If P and X are of the form (5.20) and (5.21) and the coefficient of $D_{x_2}^2$ in P is 1, then for $i \ne 2$

$$[P, V_{0k}] = \sum_{j\ne 2,3} \widetilde{W}_{0kjl} V_{jl} + (a_{0k} V_{13} + b_{0k} D_{x_2}) V_{13} + \widetilde{Z}_{0k}, \ a_{0k} = \mathcal{O}(r^3), b_{0k} = \mathcal{O}(r^2),$$

$$[P, V_{11}] = \sum_{j\ne 2,3} \widetilde{W}_{11jl} V_{jl} + (a_{11} V_{13} + b_{11} D_{x_2}) V_{13} + \widetilde{Z}_{11}, \ a_{1k} = \mathcal{O}(r^2), b_{0k} = \mathcal{O}(r),$$

$$[P, V_{31}] = \sum_{j\ne 2} \widetilde{W}_{31jl} V_{jl} + \widetilde{Z}_{31},$$

where $\widetilde{W}_{ikjl}, \widetilde{Z}_{ik} \in \mathrm{Diff}^1(\widetilde{X})$ and the coefficients of $\widetilde{W}_{0k11}$ are in $\mathcal{O}(r)$.

Proof. The argument used in the proof of Lemma 5.2 together with Lemma 5.3 easily yields the desired statement. □

The next lemma provides the crucial *a priori* estimate:

Lemma 5.8. *Let K_δ, K_δ^∂ be as in Lemma 5.5 and let $f \in C_0^\infty(\partial X)$ satisfy $f|_{K_\delta^\partial} \in L^2_-(K_\delta^\partial)$. If u is the solution of*

$$Pu = 0 \quad in \quad X \ , u|_{\phi(z)<-\delta} = 0, \ u|_{\partial X} = f,$$

then, for δ sufficiently small

$$\|\beta^* u\|_{I_k L^2_\nu(\beta^* K_\delta; \partial\widetilde{X}_1 \cap \beta^* K_\delta)} \leq C \|\beta_{\partial *} f\|_{I_k L^2_{\nu_\partial}(\beta^* K_\delta^\partial; \partial \widetilde{X}_1 \cap \beta^* K_\delta^\partial)}. \tag{5.33}$$

Proof. To the extent that it is possible we shall follow the proof of Proposition 5.2. Thus, the boundedness of the norm on the right hand side of (5.33) is equivalent to

$$\prod_{i,k,i\neq 2} V_{ik}^{\alpha_{ik}} f \in r^{-3\alpha_{31}-\alpha_{11}} L^2_-(K_\delta^\partial), \ |\alpha| \leq k, \tag{5.34}$$

The required estimate for the solution now takes the form:

$$\prod_{i,k} V_{ik}^{\alpha_{ik}} u \in r^{-3\alpha_{31}-2\alpha_{21}-\alpha_{11}} L^2_-(K_\delta), \ |\alpha| \leq k. \tag{5.35}$$

To obtain the system we define $(u)_{p,m,l}$ as before using V_{jk}, $j \neq 2$ and introduce also

$$(u)_{p,m,l,1} \in L^2(\widetilde{X}; \mathbb{C}^{N(p,m,l)}), \ (u)_{p,m,l,1} = D_{x_2}(u)_{p,m,l},$$

with D_{x_2} applied to all the components of $(u)_{p,m,l}$ and the order preserved, so that, for instance,

$$(u)_{p+1,m,l,1} = \{(u)_{p,m,l,1}; D_{x_2}(u)^\sharp_{p+1,m,l}\}.$$

As in the proof of Proposition 5.2 we obtain the following system, where now Lemma 5.7 is used:

$$\begin{aligned} \mathcal{P}_{p,m,l}(u)_{p,m,l} \quad &= \quad \mathcal{A}^{30}_{p,m,l}(u)_{p-1,m,l+1} + \mathcal{A}^{10}_{p,m,l}(u)_{p-1,m+1,l} + \mathcal{A}^{21}_{p,m,l}(u)_{p,m-1,l+1} \\ &\qquad \mathcal{B}^{20}_{p,m,l}(u)_{p-1,m+1,l,1} + \mathcal{B}^{11}_{p,m,l}(u)_{p,m-1,l,1} \end{aligned}$$

with boundary and initial conditions

$$(u)_{p,m,l}|_{\phi(z)<-\delta} = 0, \quad (u)_{p,m,l}|_{\partial X} = (f)_{p,m,l},$$

and where

$$\mathcal{P}_{p,m,l} \in \operatorname{Diff}^2(\widetilde{X}; \mathbb{C}^{N(p,m,l)}, \mathbb{C}^{N(p,m,l)}), \quad \sigma_2(\mathcal{P}_{p,m,l}) = \sigma_2(P) Id_{\mathbb{C}^{N(p,m,l)}},$$

and $\mathcal{A}^{ij}_{p,m,l}(x, D_{x_1}, D_{x_3}), \mathcal{B}^{ij}_{p,m,l}(x, D_{x_1}, D_{x_3}) \in \operatorname{Diff}^1(\widetilde{X})$ with coefficients in $\mathcal{O}(r^i)$. Since the construction is analogous to that in Proposition 5.2 we will only describe the last, most involved, inductive definition:

$$\begin{aligned} &\mathcal{P}_{p+1,m,l}(u)_{p+1,m,l} = \{\mathcal{P}_{p,m,l}(u)_{p,m,l}; \mathcal{P}_{p,m,l}(V_{0k}(u)^\sharp_{p,m,l}) - \sum_s \widetilde{W}_{0k0s}(V_{0s}(u)^\sharp_{p,m,l}) - \\ &\widetilde{\mathcal{Z}}^{p,m,l}_{0k}(u)^\sharp_{p,m,l}; \mathcal{P}_{p+1,m-1,l}(V_{jk}(u)^\sharp_{p+1,m-1,l}) - \sum_{s,r\neq 2,3} \widetilde{W}_{jkrs}(V_{rs}(u)^\sharp_{p+1,m-1,l}) - \\ &\widetilde{\mathcal{Z}}^{p+1,m-1,l}_{jk}(u)^\sharp_{p+1,m-1,l}; \mathcal{P}_{p+1,m,l-1}(V_{ik}(u)^\sharp_{p+1,m,l-1}) - \sum_{s,r\neq 2} \widetilde{W}_{ikrs}(V_{rs}(u)^\sharp_{p+1,m,l-1}) - \\ &\widetilde{\mathcal{Z}}^{p+1,m,l+1}_{ik}(u)^\sharp_{p+1,m,l-1} : j = 0,1; i = 0,1,3; k = 0, \cdots, n-3\}, \end{aligned}$$

with

$$\begin{aligned}
&\mathcal{B}^{11}_{p+1,m,l}(u)_{p+1,m-1,l,1} = \\
&\{\mathcal{B}^{11}_{p,m-1,l,1}(u)_{p,m-1,l,1}; 0; b_{0k}V_{31}(u)^{\sharp}_{p+1,m-1,l,1}, b_{11}V_{31}(u)^{\sharp}_{p+1,m-1,l,1}; 0\}, \\
&\mathcal{B}^{20}_{p+1,m,l}(u)_{p,m,l,1} = \{\mathcal{B}^{20}_{p,m,l}(u)_{p-1,m,l,1}; b_{0k}V_{13}(u)^{\sharp}_{p,m,l,1}; 0; 0\}, \\
&\mathcal{A}^{30}_{p+1,m,l}(u)_{p,m,l+1} = \{\mathcal{A}^{30}_{p,m,l}; a_{0k}V_{13}(V_{13}(u)^{\sharp}_{p,m,l}); 0; 0\}, \\
&\mathcal{A}^{10}_{p+1,m,l}(u)_{p,m+1,l} = \{\mathcal{A}^{20}_{p,m,l}(u)_{p-1,m+1,l}; \widetilde{W}_{0k11}(V_{11}(u)^{\sharp}_{p,m,l}); 0; 0\}, \\
&\mathcal{A}^{21}_{p+1,m,l}(u)_{p+1,m-1,l+1} = \\
&\{\mathcal{A}^{21}_{p,m,l}(u)_{p,m-1,l+1}; 0; a_{0k}V_{31}(V_{31}(u)^{\sharp}_{p,m-1,l}), a_{11}V_{31}(V_{31}(u)^{\sharp}_{p,m-1,l}); 0\},
\end{aligned}$$

We now use the estimates (5.30) and (5.31) of Lemma 5.5 and proceed by induction starting with

$$(u)_{0,0,l} \in r^{-3l}L^2_-(K_\delta; \mathbb{C}^{N(p,m,l)}).$$

Since $u \in C^\infty(X)$, we can assume that the norms of $(u)_{p,m,l,1}$ in $r^{-q}L^2_-(K_\delta; \mathbb{C}^{N(p,m,l)})$ are bounded and consequently we obtain

$$\begin{aligned}
\|r^{3l+m}(u)_{p,m,l}\|_{L^2_-(K_\delta)} \;\leq\; & C\delta^2(\|r^{3l+m}(u)_{p-1,m,l,1}\|_{L^2_-(K_\delta)} + \|r^{3l+m}(u)_{p,m_1,l,1}\|_{L^2_-(K_\delta)} + \\
& \|r^{3l+m}(u)_{p,m-2,l,1}\|_{L^2_-(K_\delta)}) + C\delta\|r^{3l+m}(f)_{p,m,l}\|_{L^2_-(K^\partial_\delta)}. \quad (5.36)
\end{aligned}$$

Summing (5.36) in p, m, l, $p+l+m \leq k$ we obtain

$$\begin{aligned}
\sum_{|\alpha|\leq k} \|r^{3\alpha_{31}+\alpha_{11}} D^{\alpha_{11}}_{x_1} D^{\alpha_{31}}_{x_3} D^{\alpha'}_y u\|_{L^2_-(K_\delta)} \;\leq\; & C\delta^2 \sum_{|\alpha|+|\beta|\leq k} \|r^{3\alpha_{31}+2\alpha_{21}+\alpha_{11}} D^\alpha_x D^\beta_y u\|_{L^2_-(K_\delta)} \\
& + C\delta \sum_{|\alpha|\leq k} \|r^{3\alpha_{31}+\alpha_{11}} D^{\alpha_{11}}_{x_1} D^{\alpha_{31}}_{x_3} D^{\alpha'}_y f\|_{L^2_-(K^\partial_\delta)}.
\end{aligned}$$

If δ is small enough we can apply Lemma 5.6 with $N = K_\delta$ to obtain the desired *a priori* estimate. □

For $f \in L^2_-(\partial X)$ we consider the Poisson operator T_1,

$$PT_1 f = 0 \ \text{ in } \ X, \quad T_1 f|_{\partial X} = f, \quad T_1 f|_{\phi(z)<-\delta} \equiv 0.$$

We can use Lemma 5.8 to deduce

Proposition 5.3. *The Poisson operator T_1 given above has the mapping property:*

$$T_1 : \beta^\partial_* I_k L^2_{\nu_\partial}(\beta^{\partial *}K^\partial_\delta; \partial\widetilde{X}_1 \cap \beta^{\partial *}K^\partial_\delta) \cap L^2_-(K^\partial_\delta) \longrightarrow \beta_* I_k L^2_\nu(\beta^* K_\delta; \partial\widetilde{X}_1 \cap \beta^* K_\delta)$$

where $\beta_*\nu = dxdy$, $\beta^\partial_*\nu_\partial = \imath^* dxdy$, $\imath : \partial X \hookrightarrow X$.

Proof. By the comment following the proof of Lemma 5.4, the map

$$T_1 : L^2_-(K^\partial_\delta) \longrightarrow \mathcal{D}'(K_\delta)$$

is well defined and, since we know the existence in L^2, we have obtained $u \in L^2_-(K_\delta)$ such that

$$u = T_1 f|_{K_\delta} \quad \text{if} \ \ f|_{K^\partial_\delta} \in L^2_-(K^\partial_\delta).$$

The conormal spaces in the statement of the proposition are complete (as they can be identified with $H^b_{(k)}$ of a manifold with corners – see Appendix B) and $\beta^* C^\infty(X)|_{\beta^* N_\delta}$, $\beta_1^{\partial *} C^\infty(\partial X)|_{\beta_1^{\partial *} K^\partial_\delta}$ are dense in

$$I_k L^2_\nu(\beta^* K_\delta; \partial \widetilde{X}_1 \cap \beta^* K_\delta) \quad \text{and} \quad I_k L^2_{\nu_\partial}(\beta^{\partial *} K^\partial_\delta; \partial \widetilde{X}_1 \cap \beta^{\partial *} K^\partial_\delta) \cap L^2_-(K^\partial_\delta),$$

respectively. This can be seen using the density of $\beta^* C^\infty_{0,0}(X)$, where $C^\infty_{0,0}$ are all functions vanishing to infinite order at $\Gamma = \{(0,y) : y \in \mathbb{R}^{n-3}\}$ which follows from the density of $\mathcal{S}(\mathbb{R}^n)$ in $H_{(k)}(\mathbb{R}^n)$ (where we again use the b-Sobolev spaces on manifolds with corners and the identification with the usual Sobolev spaces through a logarithmic change of variables).

Thus, if $F \in I_k L^2_{\nu_\partial}(\beta^{\partial *} K^\partial_\delta; \partial \widetilde{X}_1 \cap \beta^{\partial *} K^\partial_\delta) \cap L^2_-(K^\partial_\delta)$, $f \in \beta_* F$, then there exist $F_N \in \beta^{\partial *} C^\infty(\partial X)|_{K^\partial_\delta}$ such that

$$F_N \longrightarrow F \ \text{ in } \ I_k L^2_{\nu_\partial}(\beta^{\partial *} K^\partial_\delta; \partial \widetilde{X}_1 \cap \beta^{\partial *} K^\partial_\delta) \cap L^2_-(K^\partial_\delta), \quad N \longrightarrow \infty.$$

We then consider $u_N = T_1 \beta^\partial{}_* F_N$, where using the *a priori* estimate of Lemma 5.8, we conclude that $u \in I_k L^2_\nu(\beta^* K_\delta; \partial \widetilde{X}_1 \cap \beta^* K_\delta)$. □

6. The extension property

The purpose of this section is to construct an extension map from the marked Lagrangian spaces in X defined in Section 4 into the conormal space $(\beta_5)_*(I_k L^2_{\nu_5}(\widetilde{X}_5, \mathcal{S}_5))$. Let us first fix our notation (see Sect.2):

$$\Lambda_R = N^* \widetilde{R} \setminus 0, \quad \Lambda_F = N^* \widetilde{F} \setminus 0, \quad \Lambda_S = N^* \widetilde{S} \setminus 0, \quad \Lambda_H = N^* H \setminus 0. \tag{6.1}$$

The main result is

Theorem 6.1. *There exist linear and continuous maps*

$$\begin{gathered} E_1 \colon I^b_k L^2_c(X, \Lambda_S \sqcup \Lambda_H) + I^b_k L^2_c(X, \Lambda_S; \Lambda_S \cap \Lambda_F) \longrightarrow (\beta_5)_*(I_k L^2_{\nu_5}(\widetilde{X}_5, \mathcal{S}_5)), \\ \textit{and} \\ E_2 \colon I^b_k L^2_c(X, \Lambda_F \sqcup \Lambda_R) \longrightarrow (\beta_5)_*(I_k L^2_{\nu_5}(\widetilde{X}_5, \mathcal{S}_5)) \end{gathered} \tag{6.2}$$

such that

$$E(u) = u \ \textit{in} \ X. \tag{6.3}$$

Before constructing the extension maps E_1 and E_2 we need to present a result on the commutation of blow–ups.

In this section we will modify the notation introduced in Sect.3 and apply the blow-ups symmetrically as in the construction of $\widetilde{X}_5$ (see Fig.5 and Fig.9):

$$\widetilde{X}_5 \to \widetilde{X}_3 \to \widetilde{X}_2 \to \widetilde{X}_1 \to \widetilde{X}.$$

Thus, $\widetilde{X}_2$ is a manifold with corners defined by the $1-2-3$ blow–up of Γ followed by the $2-1-1$ blow–up of $\beta_1^* D \cap \partial \widetilde{X}_1$ with $\beta_1 \circ \beta_{12} : \widetilde{X}_2 \longrightarrow \widetilde{X}_1$ a corresponding blow–down map. Recall also that $\widetilde{X}_3$ is a manifold with corners obtained from $\widetilde{X}_2$ by the $2-1-0$ blow–up of $D^{(2)} = (\beta_1 \circ \beta_{12})^* D$ and $\beta_{23} : \widetilde{X}_3 \longrightarrow \widetilde{X}_2$ is the corresponding blow–down map.

The problem of constructing the extension using $\beta_1 \circ \beta_{12} \circ \beta_{23}$ is that the boundary of X does not lift to a smooth hypersurface under this map. Thus we will need an alternative hierarchy of blow-ups.

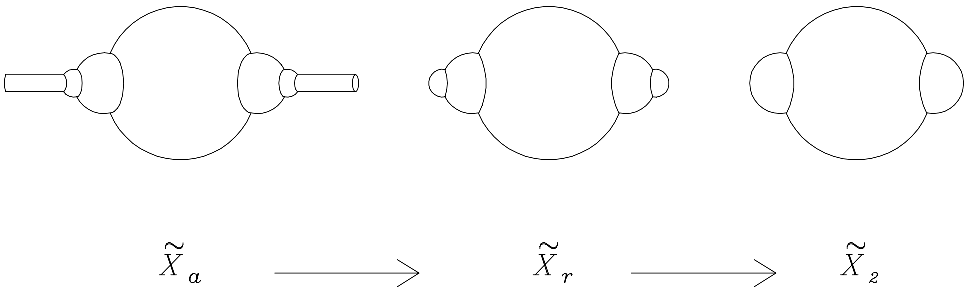

FIGURE 9. The auxiliary blow-ups: $\widetilde{X}_a \stackrel{\beta_{ra}}{\to} \widetilde{X}_r \stackrel{\beta_{2r}}{\to} \widetilde{X}_2$

Let $D_0^{(2)} = D^{(2)} \cap \partial \widetilde{X}_2$ and (x_1, x_2, x_3, y) be local coordinates given in $i)$ of Proposition 3.3. Let (z', y', s) be the corresponding projective coordinates near $D_0^{(2)}$ chosen so that $\beta_{12}(z', y', s) \mapsto (s^2 z', sy', s)$. Then

$$D^{(2)} = \{y' = z' = 0\}, \quad D_0^{(2)} = \{s = y' = z' = 0\}.$$

Let

$$\mathbb{S}^2_{6-3-1_\pm} = \{\omega \in \mathbb{R}^3 : \omega_1^2 + \omega_2^4 + \omega_3^{12} = 1, \quad \pm\omega_3 \geq 0\}$$

and define the space obtained by blowing up the two components of $D_0^{(2)}$:

$$\widetilde{X}_r = \widetilde{X}_2 \setminus D_0^{(2)} \sqcup (S^2_{6-3-1_+} \times \mathbb{R}^{n-3}) \sqcup (S^2_{6-3-1_-} \times \mathbb{R}^{n-3})$$

with the C^∞ structure obtained as before. The blow–down map near $D_0^{(2)}$ now takes form:

$$\beta_{2r} \colon \widetilde{X}_r \longrightarrow \widetilde{X}_2$$
$$\beta_{2r}(\omega, \rho) = (\rho^6 \omega_1, \rho^2 \omega_2, \rho\omega_3) = (z', y', s).$$

If $D^{(3)} = \beta_{2r}^* D^{(2)}$, we define the auxiliary manifold with corners $\widetilde{X}_a$ as the one obtained from $\widetilde{X}_r$ by blowing–up the submanifold $D^{(3)}$ with homogeneity $2-1-0$ and let β_{ra} denote the corresponding blow–down map. Let

$$\beta_{2a} \colon \widetilde{X}_a \longrightarrow \widetilde{X}_2,$$
$$\beta_{2a} = \beta_{2r} \circ \beta_{ra}$$

be the corresponding blow–down map.

Lemma 6.1. *If* $Q = \{x_2^2 - x_1 x_3 = 0\}$ *and* $H = \{x_1 = 0\}$, *then the smooth hypersurfaces* $Q^{(2)} = \beta_2^* Q$ *and* $H^{(2)} = \beta_2^* H$ *lift under* β_{2a} *to disjoint smooth hypersurfaces intersecting* $\partial \widetilde{X}_a$ *transversally.*

Proof. In the projective coordinates (s, y', z') defined in a neighbourhood of $D^{(2)}$

$$Q^{(2)} = \{z' = y'^2\}, \quad H^{(2)} = \{z' = 0\}. \tag{6.4}$$

As before we shall define projective coordinates for β_{2r}. Thus let

$$\begin{aligned}
&\rho = s, \quad y'' = \frac{y'}{\rho^3}, \quad z'' = \frac{z'}{\rho^6}. \\
&\rho = |y'|^{\frac{1}{3}}, \quad s' = \frac{s}{\rho}, \quad z'' = \frac{z'}{\rho^6}. \\
&\rho = |z'|^{\frac{1}{6}}, \quad y'' = \frac{y'}{\rho^3}, \quad s' = \frac{s}{\rho}.
\end{aligned} \tag{6.5}$$

Each valid in the region $|\omega_i| > 0, i = 1, 2, 3$. In each of these coordinate systems we obtain

$$\begin{gathered}
\beta_{2r}^* Q^{(2)} = \{z'' = y''^2\}, \quad \beta_{2r}^* H^{(2)} = \{z'' = 0\}. \\
\beta_{2r}^* Q^{(2)} = \{z'' = 1\}, \quad \beta_{2r}^* H^{(2)} = \{z'' = 0\}. \\
\beta_{2r}^* Q^{(2)} = \{y'' = 1\}.
\end{gathered}$$

Hence in the first coordinate system $\beta_{2r}^* Q^{(2)}$ and $\beta_{2r}^* H^{(2)}$ are simply tangent and thus lift to smooth hypersurfaces in $\widetilde{X}_a$ by the $2-1-0$ blow–up. In the second and third coordinate systems $\beta_{2r}^* Q^{(2)}$ and $\beta_{2r}^* H^{(2)}$ are clearly smooth. This concludes the proof of the lemma. $\square$

Proposition 6.1. *Let $\mathcal{V}_a = \mathcal{V}(\beta_{2a}^* Q^{(2)} \sqcup \beta_{2a}^* H^{(2)} \sqcup \partial \widetilde{X}_a)$ be the Lie algebra of smooth vector fields in $\widetilde{X}_a$ tangent to $\beta_{2a}^* Q^{(2)}$, $\beta_{2a}^* H^{(2)}$ and $\partial \widetilde{X}_a$ and let $\mathcal{V}_3 = \mathcal{V}(\beta_{23}^* Q^{(2)} \sqcup \beta_{23}^* H^{(2)} \sqcup \partial \widetilde{X}_3)$ be the corresponding Lie algebra in $\widetilde{X}_3$. Let ν_a be the lift of the Lebesgue measure in $\mathbb{R}^n$ to $\widetilde{X}_a$. Then*

$$(\beta_{23})_* I_k L^2_{\nu_3}(\widetilde{X}_3, \mathcal{V}_3) = (\beta_{2a})_* I_k L^2_{\nu_a}(\widetilde{X}_a, \mathcal{V}_a). \tag{6.6}$$

Proof. Observe that in coordinates (s, y', z') such that (6.4) holds, the map $F(s, y', z') = (s, y', z' - y'^2)$ is a diffeomorphism that preserves $D^{(2)}$ and maps $Q^{(2)}$ into $H^{(2)}$. The homogeneity of the variables s, y' and z' shows that F lifts respectively under β_{23} and β_{2a} to smooth diffeomorphisms mapping the lift of $Q^{(2)}$ into the corresponding lift of $H^{(2)}$. Since the pairs $\beta_{23}^* Q^{(2)}$ and $\beta_{23}^* H^{(2)}$ are disjoint, and so are $\beta_a^* Q^{(2)}$ and $\beta_a^* H^{(2)}$, we only need to prove (6.6) for the Lie algebras $\mathcal{V}_3$ and $\mathcal{V}_a$ replaced by $\mathcal{V}_{3,H} = \mathcal{V}(\beta_{23}^* H^{(2)} \sqcup \partial \widetilde{X}_3)$ and $\mathcal{V}_{a,H} = \mathcal{V}(\beta_a^* H^{(2)} \sqcup \partial \widetilde{X}_a)$.

The next step is to show that both sides of (6.6) have the same characterization in terms of singular vector fields.

Lemma 6.2. *Let $u \in L^2_{\nu_2}(\widetilde{X}_2)$. Then $u \in (\beta_{23})_* I_k L^2_{\nu_3}(\widetilde{X}_3, \mathcal{V}_{3,H})$ if and only if in the projective coordinates (s, y', z') in which (6.4) holds*

$$(s\partial_s, z'\partial_{z'}, |z'|^{\frac{1}{2}}\partial_{y'}, y'\partial_{y'})^\alpha u \in L^2_{\nu_2}(X_2), \quad |\alpha| \le k. \tag{6.7}$$

Proof. Indeed, if u satisfies (6.7), then in projective coordinates

$$t = |y'|, \quad z'' = \frac{z'}{t^2} \text{ and } t = |z'|^{\frac{1}{2}}, \quad y'' = \frac{y'}{t} \tag{6.8}$$

the vector fields in (6.7) lift to

$$s\partial_s, z''\partial_{z''}, t\partial_t, |z''|^{\frac{1}{2}}(s\partial_s - 2z''\partial_{z''}), \quad \text{and} \quad s\partial_s, t\partial_t, \partial_{y''} \tag{6.9}$$

respectively, and these span $\mathcal{V}_{3,H}$. Therefore $\beta_{23}^* u \in I_k L^2_{\nu_3}(\widetilde{X}_3, \mathcal{V}_{3,H})$.

Conversely if $\beta_{23}^* u \in I_k L^2_{\nu_3}(\widetilde{X}_3, \mathcal{V}_{3,H})$, then $\beta_{23}^* u$ is stable under the application of the vector fields in (6.9) and therefore u satisfies (6.7). $\square$

Similarly we obtain

Lemma 6.3. *Let $u \in L^2_{\nu_2}(\widetilde{X}_2)$. Then $u \in (\beta_{2a})_* I_k L^2_{\nu_a}(\widetilde{X}_a, \mathcal{V}_{a,H})$ if and only if u satisfies (6.7).*

The two lemmas complete the proof of Proposition 6.1 $\square$

We also need a corollary of Seeley's extension theorem, with the proof being immediate from the arguments of [46]. Let $\mathbb{R}^n_+ = \{(x', x_n) : x_n > 0\}$ and for $s \in \mathbb{R}$, let $\bar{H}_{(s)}(\mathbb{R}^n_+)$ be the space of restrictions to $\mathbb{R}^n_+$ of elements in $H_{(s)}(\mathbb{R}^n)$

Proposition 6.2. *There exists a linear and continuous map*

$$S\colon L^2_c(\mathbb{R}^n_+) \longrightarrow L^2_c(\mathbb{R}^n)$$

such that if $u \in L^2_c(\mathbb{R}^n_+)$ satisfies

$$Q_1(x', D_{x'}) \dots Q_l(x', D_{x'})(D^r_{x_n})^j u \in \bar{H}_{(-(r-1)j)}(\mathbb{R}^n_+) \tag{6.10}$$

for $Q_i(x', D_{x'}) \in \mathrm{Diff}^{\bullet}(\mathbb{R}^{n-1}), 1 \leq i \leq m$, $j + l \leq k$ and $r \in \mathbb{N}$, $r \geq 1$. Then

$$Q_1(x', D_{x'}) \dots Q_l(x', D_{x'})(D^r_{x_n})^j S(u) \in H_{(-(r-1)j)}(\mathbb{R}^n). \tag{6.11}$$

We can now start the construction of the extension map and it is convenient to introduce the following notation:

$$X_i = \beta_i^* X, \quad \partial X_i = \beta_i^* \partial X.$$

The X_i's are manifolds with corners and in the definition of the extendible $(H^b_{(s)})$ and partial $(H^b_{(s,m)})$ b-Sobolev spaces (see Appendix B) are taken with respect to the boundary face given by ∂X_i. The first half of the theorem is given by

Proposition 6.3. *There exists a linear and continuous map*

$$E_1\colon I^b_k L^2_c(X, \Lambda_S \sqcup \Lambda_H) + I^b_k L^2_c(X, \Lambda_S; \Lambda_S \cap \Lambda_F) \longrightarrow (\beta_5)_*(I_k L^2_{\nu_5}(\widetilde{X}_5, \mathcal{S}_5)) \tag{6.12}$$

such that

$$E_1(u) = u \text{ in } X \tag{6.13}$$

Proof. Let us define the following Lagrangian variety

$$\mathcal{L} = \{\Lambda_S \sqcup \Lambda_H, \Lambda_S \cap \Lambda_F\},$$

where the marking of the union is disjoint from the second component. It easily follows that if u satisfies the hypothesis of Proposition 6.3, then $u \in I^b_k L^2(X, \mathcal{L})$. We shall use this to construct $\tilde{u}$ satisfying (6.13). Let us assume that $\mathrm{supp}(u)$ is contained in a small neighbourhood of Γ.

Proposition 3.3 guarantees the existence of smooth coordinates (x, y) in a neighbourhood of Γ such that

$$\widetilde{S} = \{x_2^2 = x_1 x_3\}, \quad B = \widetilde{F} \cap \widetilde{S} = \{x_2 = x_3 = 0\}, \quad H = \{x_1 = 0\} \tag{6.14}$$

and the defining function ρ of ∂X , i.e $X=\{\rho>0\}$, $\partial X=\{\rho=0\}$ satisfies

$$\rho(x,y)=x_2+c_1x_1^2+c_2x_2^2+c_3x_3^2+\sum_{i<j}a_{ij}x_ix_j+O(|x|^3),\quad c_1,c_3>0, \tag{6.15}$$

see Remark 3.1.

Direct computations show that $\Psi_b^{1,\bullet}(X,\mathcal{L})$ is the $\Psi_b^{0,\bullet}(X)$ span of

$$\begin{gathered} V_1=3x_3\partial_{x_3}+2x_2\partial_{x_2}+x_1\partial_{x_1},\quad V_2=2x_1\partial_{x_1}+x_2\partial_{x_2},\\ V_3=(x_2^2-x_1x_3)\partial_{x_1},\quad V_4=(x_2^2-x_1x_3)\partial_{x_2},\\ V_5=(x_2^2-x_1x_3)\partial_{x_3},\quad AL, L=\partial_{x_2}^2-4\partial_{x_1}\partial_{x_3}. \end{gathered} \tag{6.16}$$

where $A\in\Psi_b^{-1}(X)$ is elliptic. By Proposition 4.5, if $V=(V_1,\dots,V_5)$, we find that

$$V^\alpha L^a u=\sum_{|\beta|\leq a}(\partial_{x_1},\partial_{x_2},\partial_{x_3})^\beta u_\beta,\quad u_\beta\in L^2(X),\quad |\alpha|+a\leq k. \tag{6.17}$$

The first step is to analyze the lift of (6.17) by β_1. Let $X_1=\{r^{-2}\beta_1^*\rho>0\}$, $V_i'=\beta_1^*V_i, 1\leq i\leq 5$, $L'=r^4\beta_1^*L$, $W_j=r^j\beta_1^*\partial_{x_j}$, $1\leq j\leq 3$. We deduce from (6.17) that if $u_1=\beta^*u$, $V'=(V_1',\dots,V_5')$ then

$$\begin{gathered} V'^\alpha L'^a u_1=r^{4a}\sum_{|\beta|\leq a}r^{-3\beta_1-2\beta_2-\beta_3}W^\beta u_\beta^1,\\ |\alpha|+a\leq k,\quad u_\beta^1\in r^{-3}L_b^2(X_1). \end{gathered} \tag{6.18}$$

Since u is supported near Γ we may assume that $r<1$ on $\operatorname{supp}(u_1)$. As $4a-3\beta_1-2\beta_2-\beta_3\geq 0$, we obtain from (6.18) that

$$V'^\alpha L'^a u_1\in r^{-3}\bar{H}^b_{(-a)}(X_1), \tag{6.19}$$

where L_b^2 and $H_{(s)}^b$ are the b-Sobolev spaces defined in Appendix B. Then we analyze (6.19) in projective coordinates

$$\begin{gathered} r=|x_3|^{\frac13},\quad Y=\frac{x_2}{r^2},\quad Z=\frac{x_1}{r}\\ r=|x_2|^{\frac12},\quad X=\frac{x_3}{r^3},\quad Z=\frac{x_1}{r}\\ r=|x_1|,\quad X=\frac{x_3}{r^3},\quad Y=\frac{x_2}{r^2} \end{gathered} \tag{6.20}$$

In the third set of coordinates we obtain from (6.19)

$$(r\partial_r,2X\partial_X+Y\partial_Y)^\alpha[(\partial_Y+2Y\partial_X)\partial_Y]^a u_1\in r^{-3}\bar{H}^b_{(-a)}(X_1),\quad |\alpha|+a\leq k \tag{6.21}$$

and

$$\rho_1=r^{-2}\beta_1^*\rho=Y+c_1+O(r). \tag{6.22}$$

Thus for small r, $Y\neq 0$ near ∂X_1. The operators in (6.21) span the space of totally characteristic operators in $\Psi_b^1(\widetilde{X}_1,{}^b\mathcal{L})$ where ${}^b\mathcal{L}$ is the marked Lagrangian variety formed by ${}^bN^*\widetilde{S}^{(1)}\sqcup{}^bN^*H^{(1)}$ marked by ${}^bN^*\widetilde{S}^{(1)}\cap{}^bN^*B^{(1)}$.

Near ∂X_1, $\rho = Y$, $x' = X/Y^2$, give a smooth coordinate system in which (6.21) can be written as

$$(r\partial_r, \partial_\rho)^\alpha [x'(1-x')\partial^2_{x'}]^a u_1 \in r^{-3}\bar{H}^b_{(-a)}(X_1)$$
$$\text{and} \tag{6.23}$$
$$\partial X_1 = \{\rho + c_1 + O(r) = 0\}$$

If $\rho' = 0$ defines the boundary then $\rho' = \rho + c_1 + O(r)$ and we can use it as a new coordinate. Then we deduce from (6.23) that

$$(r\partial_r, \partial_{\rho'})^\alpha (x'(1-x')\partial^2_{x'})^a u_1 \in r^{-3}\bar{H}^b_{(-a)}(X_1), \quad \partial X_1 = \{\rho' = 0\}. \tag{6.24}$$

Since $\partial_{\rho'}$ is transversal to ∂X_1 we deduce from Proposition 6.2, or rather its easy modification to the case of a manifold with corners, that there exists $\tilde{u}$ such that

$$\tilde{u} = u \text{ in } X_1, \quad (r\partial_r, \partial_{\rho'})^\alpha (x'(1-x')\partial^2_{x'})^a \tilde{u} \in r^{-3}\bar{H}^b_{(-a)}(\widetilde{X}_1). \tag{6.25}$$

Now we proceed by a microlocal partition of unity of ${}^bT^*\widetilde{X}_1$. In the region where either $r\partial_r$ or $\partial_{\rho'}$ is elliptic it follows from (6.25) that $\tilde{u}$ is conormal to $\partial\widetilde{X}_1$. In the region where $\partial_{x'}$ is elliptic we obtain from (6.25) that

$$(r\partial_r, \partial_{\rho'})^\alpha (x'(1-x')\partial_{x'})^a \tilde{u} \in r^{-3}L^2_b(\widetilde{X}_1). \tag{6.26}$$

Going back to the original coordinates this provides the condition required in the extension.

In the region where the coordinates (r, X, Z) can be used

$$\partial X_1 = \{1 + c_1 Z^2 + O(r) = 0\}.$$

Thus for small r, $Z \neq 0$ near ∂X_1 and therefore the set of projective coordinates (r, Y, X) can also be used there and the extension is constructed as above.

In the first set of projective coordinates in (6.20) we obtain,

$$\begin{gathered}(r\partial_r, 2Z\partial_Z + Y\partial_Y, r^2(Z-Y^2)\partial_Y, r^3(Z-Y^2)\partial_Z)^\alpha \\ [(\partial_Y + 2Y\partial_Z)\partial_Y]^a u_1 \in r^{-3}\bar{H}^b_{(-a)}(X_1).\end{gathered} \tag{6.27}$$

and the lift of the boundary of ∂X is given by

$$\partial X_1 = \{Y + c_1 Z^2 + a_{12} rYZ + a_{13} r^2 Z + b_1 r Z^3 + b_2 r^2 Y Z^2 + f(Y, Z, r) r^3 = 0\},$$
$$f \in C^\infty, \quad f(0,0,0) = 0.$$

Since we are only concerned with the region $r \sim 0$, we find that if $|Y| > 0$ near ∂X_1, then $|Z| > 0$ near ∂X_1. Therefore in this case the extension map is constructed in the third set of coordinates. Thus we may restrict our analysis to a small neighbourhood of $Y = Z = 0$.

To start we need to analyze the lift of (6.27) under the $2-1-1$ blow–down map. For that we use projective coordinates

$$\begin{gathered} s = r, \quad y' = \frac{Y}{s}, \quad z' = \frac{Z}{s^2}. \\ s = |Y|, \quad r' = \frac{r}{s}, \quad z' = \frac{Z}{s^2}. \\ s = |Z|^{\frac{1}{2}}, \quad r' = \frac{r}{s}, \quad y' = \frac{Y}{s}. \end{gathered} \tag{6.28}$$

In the third set of coordinates of (6.28)

$$\partial X_2 = \{y' + O(s^3) = 0\}. \tag{6.29}$$

We deduce from (6.27) that $u_2 = \beta_{12}^* u_1$ satisfies

$$(r\partial_{r'}, s\partial_s)^\alpha [(1 \mp y'^2)\partial_{y'}^2]^a u_2 \in r'^{-3} s^{-\frac{9}{2}} \bar{H}^b_{(-a)}(X_2), \quad |\alpha| + a \le k. \tag{6.30}$$

Since r' and s are small, $y' \sim 0$ near ∂X_2 and thus $|1 \mp y'^2| > 0$ there. Therefore near ∂X_2

$$(r'\partial_{r'}, s\partial_s)^\alpha \partial_{y'}^{2a} u_2 \in r'^{-3} s^{-\frac{9}{2}} \bar{H}^b_{(-a)}(X_2), \quad |\alpha| + a \le k. \tag{6.31}$$

Let $y'' = y' + O(s^3)$ be a defining function of ∂X_2. We deduce from (6.31) that

$$\begin{gathered}(r'\partial_{r'} + f_1\partial_{y''}, s\partial_s + f_2\partial_{y''})^\alpha \partial_{y''}^{2a} u_2 \in r'^{-3} s^{-\frac{9}{2}} \bar{H}^b_{(-a)}(X_2), \quad |\alpha| + a \le k, \\ \partial X_2 = \{y'' = 0\}\end{gathered} \tag{6.32}$$

where f_1 and f_2 are smooth functions. Thus, completing the squares, we see from (6.32) that

$$((r'\partial_{r'})^2, (s\partial_s)^2, \partial_{y''}^2)^\alpha u_2 \in r'^{-3} s^{-\frac{9}{2}} \bar{H}^b_{(-|\alpha|)}(X_2), \quad |\alpha| \le k. \tag{6.33}$$

It follows from Proposition B.1, applied to the noncharacteristic operator $\partial_{y''}^{2a}$, that

$$\left((r'\partial_{r'})^2, (s\partial_s)^2\right)^\alpha \partial_{y''}^a u_2 \in r'^{-3} s^{-\frac{9}{2}} \bar{H}^b_{(-|\alpha|)}(X_2), \quad |\alpha| + a \le k. \tag{6.34}$$

Proposition 6.2 gives $\tilde{u}_2 \in r'^{-3} s^{-\frac{9}{2}} L^2_b(\widetilde{X}_2)$ such that

$$\left((r'\partial_{r'})^2, (s\partial_s)^2)\right)^\alpha \partial_{y''}^a u_2 \in r'^{-3} s^{-\frac{9}{2}} H^b_{(-|\alpha|)}(\widetilde{X}_2), \quad |\alpha| + a \le k. \tag{6.35}$$

Since at any point $q \in {}^bT^*\widetilde{X}_2$ one of the three operators $s\partial_s$, $r'\partial_{r'}$ or $\partial_{y''}$ is elliptic one deduces from (6.35) that

$$\widetilde{u}_2 \in r'^{-3} s^{-\frac{9}{2}} H^b_{(k)}(\widetilde{X}_2)$$

Simple computations show that, for r' and s' small, ∂X_2 does not intersect the region where the second coordinate system in (6.28) holds.

Next we consider the region where the first coordinate system in (6.28) is used. We find that in these coordinates

$$\partial X_2 = \{y' + c_1 s^3 + s^3 f(r', z', s) = 0\}, \quad f \in C^\infty, \quad f(0,0,0) = 0 \tag{6.36}$$

and we obtain from (6.19) that

$$(s\partial_s, 2z'\partial_{z'} + y'\partial_{y'})^\alpha [(\partial_{y'} + 2y'\partial_{z'})\partial_{y'}]^m u_2 \in s^{-\frac{9}{2}} \bar{H}^b_{(-m)}(X_2), \quad |\alpha| + m \le k. \tag{6.37}$$

To analyze $\widetilde{X}_3$ we need to blow–up $D^{(2)}$ with homogeneity $1-2$. However as we already mentioned ∂X_2 lifts under β_{23} to a singular hypersurface. This is where we use the result on the commutation of blow–ups proved in Proposition 6.1. We shall prove that there exists an extension of $\beta_{2a}^* u_2$ into the conormal space $I_k L^2_{\nu_a}(X_a, \mathcal{V}_a)$. It follows from Proposition 6.2 that this gives an extension of u_2 to the conormal space.

In the first set of coordinates (6.5) we obtain that $u_3=\beta_{2r}^* u_2$ satisfies

$$(\rho\partial_\rho, 2z''\partial_{z''}+y''\partial_{y''})^\alpha[(\partial_{y''}+2y''\partial_{z''})\partial_{y''}]^m u_3 \in \rho^{-9}\bar{H}^b_{(-m)}(X_r), \quad |\alpha|+m\leq k.$$

and

$$\partial X_r=\{y''+c_1+O(\rho)=0\}. \tag{6.38}$$

Thus for small ρ, $y''\neq 0$ near ∂X_r. Then we blow–up $D^{(r)}=\beta_{2r}^* D^{(2)}=\{y''=z''=0\}$ with homogeneity $2-1-0$. In projective coordinates

$$t=|y''|, \quad \tilde{z}=\frac{z''}{t^2} \text{ and } \tilde{y}=\frac{y''}{t}, \quad t=|z''|^{\frac{1}{2}}. \tag{6.39}$$

The condition (6.37) gives, for $|\alpha|+m\leq k$

$$\begin{aligned}(\rho\partial_\rho, t\partial_t)^\alpha[\tilde{z}(\tilde{z}-1)\partial_{\tilde{z}}^2]^m\beta_{ra}^* u_3 \in t^{-\frac{3}{2}}\rho^{-9}\bar{H}_{(-m)}(X_a).\\ (\rho\partial_\rho, t\partial_t)^\alpha[\tilde{y}(\tilde{y}-1)\partial_{\tilde{y}}^2]^m\beta_{ra}^* u_3 \in t^{-\frac{3}{2}}\rho^{-9}\bar{H}_{(-m)}(X_a).\end{aligned} \tag{6.40}$$

Hence, away from ∂X_a, we conclude by a simple microlocal partition of unity argument that $\beta_{ra}^* u_3 \in I_k L_{\nu_a}(X_a,\mathcal{V}_a)$.

Since $y''\neq 0$ near ∂X_r, the first set of coordinates in (6.39) can be used and

$$\partial X_a=\{t+c_1+O(\rho)=0\}.$$

We deduce from (6.40) that

$$(\rho\partial_\rho, \partial_t)^\alpha[\tilde{z}(\tilde{z}-1)\partial_{\tilde{z}}^2]^m\beta_{ra}^* u_3 \in \rho^{-9}\bar{H}_{(-m)}(X_a). \tag{6.41}$$

Since ∂_t is transversal to ∂X_3, one deduces from Proposition 6.2 that there exists an extension $\tilde{u}_3$ of $\beta_{ra}^* u_3$ to $\widetilde{X}_a$ such that

$$(\rho\partial_\rho, t\partial_t)^\alpha[\tilde{z}(\tilde{z}-1)\partial_{\tilde{z}}^2]^m\tilde{u}_3 \in \rho^{-\frac{13}{2}}\bar{H}_{(-m)}(\widetilde{X}_a). \tag{6.42}$$

Next we proceed by a microlocal partition of unity. In the region where either $\rho\partial_\rho$ or $t\partial_t$ is elliptic $\tilde{u}_3$ is conormal to the boundary of $\widetilde{X}_a$. In the region where $\partial_{\tilde{z}}$ is elliptic we find that

$$(\rho\partial_\rho, t\partial_t)^\alpha[\tilde{z}(\tilde{z}-1)\partial_{\tilde{z}}]^a\tilde{u}_3 \in \rho^{-9}L^2_b(\widetilde{X}_a). \tag{6.43}$$

This shows that $\tilde{u}_3$ is conormal to the lifting of the cone and the plane.

Simple computations show that for ρ and s' small, the boundary ∂X_r does not intersect the region where the second coordinate system in (6.5) holds. In the third set of coordinates

$$(\rho\partial_\rho, s\partial_s)^\alpha[(1\mp y''^2)\partial_{y''}^2]^m u_3 \in s'^{-\frac{9}{2}}\rho^{-9}\bar{H}^b_{(-m)}(X_r), \quad m+|\alpha|\leq k$$

and

$$\partial X_r=\{y''+O(\rho)=0\} \tag{6.44}$$

Away from ∂X_r, a microlocal partition of unity argument gives that $u_3\in I_k L_{\nu_r}(X_r,\mathcal{V}_r)$. On the other hand $1\mp y''\neq 0$ near ∂X_r. Thus it follows from Proposition 6.2 that there exists an extension $\tilde{u}_3$ of u_3 to $\widetilde{X}_r$ such that

$$(\rho\partial_\rho, s\partial_s)^\alpha\partial_{y''}^{2m}\tilde{u}_3 \in s'^{-\frac{9}{2}}\rho^{-9}\bar{H}^b_{(-m)}(X_r), \quad m+|\alpha|\leq k. \tag{6.45}$$

Now a simple microlocal partition of unity argument analogous to the one used above shows that $\tilde{u}_3\in\rho^{-11}s'^{-\frac{5}{2}}H^b_{(k)}(X_r)$. This concludes the proof of Proposition 6.3.

Let us observe that the linearity and continuity of E_1 follow from its construction and the linearity and continuity of the Seeley map S in Proposition 6.2. $\square$

Next we consider the Lagrangians corresponding to the direct and reflected fronts, that is the second part of Theorem 6.1.

Proposition 6.4. *There exists a linear and continuous map*

$$E_2\colon I_k^b L^2(X,\Lambda_F\sqcup\Lambda_R)\longrightarrow(\beta_5)_*(I_k L^2_{\nu_5}(\widetilde{X}_5,\mathcal{S}_5)) \tag{6.46}$$

such that

$$E_2(u)=u \ \mathit{in}\ X, \tag{6.47}$$

$$\tag{6.48}$$

Proof. We can assume that $\operatorname{supp}(u)$ is contained in a small neighbourhood of Γ. Let (x_1,x_2,x_3,y) be local coordinates in a neighbourhood of Γ given by $ii)$ of Proposition 3.3. Hence,

$$\widetilde{R}=\{x_2^3=x_3^2\},\quad \widetilde{F}=\{2x_3+x_1^3-3x_1x_2=0\}, \tag{6.49}$$

and the boundary of X is given by

$$\partial X=\{x_2=\frac{1}{4}x_1^2+g\},\quad g\in M_3^{3-2-1}. \tag{6.50}$$

In these coordinates the space $\Psi_b^{1,\bullet}(X,\Lambda_F\sqcup\Lambda_R)$ is spanned over $\Psi_b^{0,\bullet}(X)$ by

$$\begin{gathered}AP, AL,\ P=4\partial_{x_2}^2-9x_2\partial_{x_3}^2-6\partial_{x_1}\partial_{x_3},\quad L=(2\partial_{x_2}+3x_1\partial_{x_3})\partial_{x_1},\\ V_1=3x_3\partial_{x_3}+2x_2\partial_{x_2}+x_1\partial_{x_1},\quad \partial_{y_j}.\end{gathered} \tag{6.51}$$

where $A\in\Psi_b^{-1}(X)$ is elliptic. To see that one needs to observe that in coordinates $(x_1,x_2,x_3,y,\xi_1,\xi_2,\xi_3,\eta)$ in $T^*\mathbb{R}^n\setminus 0$

$$\begin{gathered}\Lambda_R=\{4\xi_2^2-9x_2\xi_3^2=0,\ \ 3x_3\xi_3+2x_2\xi_2=0,\ \ \xi_1=0,\ \ \eta=0,\ \ \xi_3\neq 0\},\\ \Lambda_F=\{2\xi_2+3x_1\xi_3=0,\ \ 2\xi_1-3(x_1^2-x_2)\xi_3=0, 3x_3\xi_3+2x_2\xi_2+x_1\xi_1=0,\ \ \eta=0\}.\end{gathered}$$

We then notice that

$$\Lambda_F\cup\Lambda_R\subset M=\{4\xi_2^2-9x_2\xi_3^2-6\xi_1\xi_3=0,\ \ 3x_3\xi_3+2x_2\xi_2+x_1\xi_1=0,\eta=0,\xi_3\neq 0\}$$

Then M is a smooth submanifold of $T^*\mathbb{R}^n\setminus 0$ and in M

$$\Lambda_F=\{\xi_1=0\},\quad \Lambda_R=\{2\xi_2+3x_1\xi_2=0\}.$$

Therefore if $f\in C^\infty(T^*\mathbb{R}^n\setminus 0)$ vanishes on $\Lambda_F\cup\Lambda_R$

$$\begin{gathered}f=a_1(4\xi_2^2-9x_2\xi_3^2-6\xi_1\xi_3)+a_2(3x_3\xi_3+2x_2\xi_2+x_1\xi_1)+a_3(2\xi_2+3x_1\xi_2)\xi_1+\\ \sum a_j\eta_j,\quad a_i\in C^\infty(T^*\mathbb{R}^n\setminus 0).\end{gathered}$$

This shows that if $B\in\Psi_b^{m,k}(X,\Lambda_F\sqcup\Lambda_R)$ then B is in the span of P, L and V_1. Here and in what follows we should neglect the trivial generators ∂_{y_j}. It will also be useful for us to observe that

$$V_2=3x_2^2\partial_{x_3}+2x_3\partial_{x_2}+(2x_2-x_1^2)\partial_{x_1}\in\Psi_b^{1,1}(X,\Lambda_F\sqcup\Lambda_R).$$

Therefore if $u\in I_k^b(X,\Lambda_F\sqcup\Lambda_R)$ then Proposition 4.5 shows that

$$(P,L)^a(V_1,V_2)^\alpha u\in\bar{H}_{(-|a|)}(X),\quad |a|+|\alpha|\le k, \tag{6.52}$$

where V_2 is used as a generator only for convenience.

To construct the extension map we need to examine the lift of (6.52) under the blow–down map β_1. Thus let

$$\begin{gathered} P_1 = r^4 \beta_1^* P, \quad L_1 = r^3 \beta_1^* L, \\ V_i' = \beta_1^* V_i,\ i = 1, 2, \quad W_j = r^j \beta_1^* \partial_{x_j},\ j = 1, 2, 3. \end{gathered} \tag{6.53}$$

We deduce from (6.52) that, for $u_1 = \beta_1^* u$ and $W = (W_1, W_2, W_3)$

$$\begin{gathered} (P_1, L_1)^a (V_1', V_2')^\alpha u_1 = r^{4a_1 + 3a_2} \sum_{|\beta| \le |a|} r^{-3\beta_1 - 2\beta_2 - \beta_3} W^\beta u_\beta, \\ u_\beta \in r^{-3} L_b^2(X_1), \quad |a| + |\alpha| \le k. \end{gathered} \tag{6.54}$$

Since u is supported in a small neighbourhood of Γ, we may assume that $r < 1$ on $\mathrm{supp}(u_1)$. Therefore,

$$(P_1, L_1)^a (V_1', V_2')^\alpha u_1 \in r^{-3} \bar{H}^b_{(-|a|)}(X_1), \quad |\alpha| + |a| \le k. \tag{6.55}$$

Since $x_2 \ge 0$ on $\widetilde{R}$, we first consider the region where the projective coordinates $r = x_2^{\frac{1}{2}}$, $X = x_3 / r^3$, $Z = x_1 / r^3$ can be used. In terms of these we find that:

$$\begin{gathered} P_1 = (r\partial_r - 3X\partial_X - Z\partial_Z)^2 - 9\partial_X^2 - 6\partial_X \partial_Z, \quad L_1 = (r\partial_r + 3(Z - X)\partial_X - Z\partial_Z)\partial_Z, \\ V_1' = r\partial_r, \quad V_2' = r\left(3(1 - X^2)\partial_X + (2 - Z^2 - XZ)\partial_Z\right), \end{gathered} \tag{6.56}$$

where we neglected the $r\partial_r$ terms in V_2'. The boundary of X_1 and the lifted hypersurfaces are defined by

$$\begin{gathered} \partial X_1 = \{Z^2 = 4 + r\phi_1\}, \quad \phi_1 \in C^\infty(\widetilde{X}_1), \\ \beta_1^* \widetilde{F} = \{2X + Z^3 - 3Z = 0\}, \quad \beta_1^* \widetilde{R} = \{X^2 - 1 = 0\}. \end{gathered} \tag{6.57}$$

One can write

$$\beta_1^* \widetilde{F} = \{2(X \mp 1) + (Z \mp 1)^2 (Z \mp 2) = 0\}$$

which shows that $\beta_1^* \widetilde{F}$ and $\beta_1^* \widetilde{R}$ are simply tangent along $B^{(1)} = \{X = Z = \pm 1\}$ and intersect transversally at $K^{(1)} = \{X = \pm 1,\ Z = \mp 2\}$, see Fig 10.

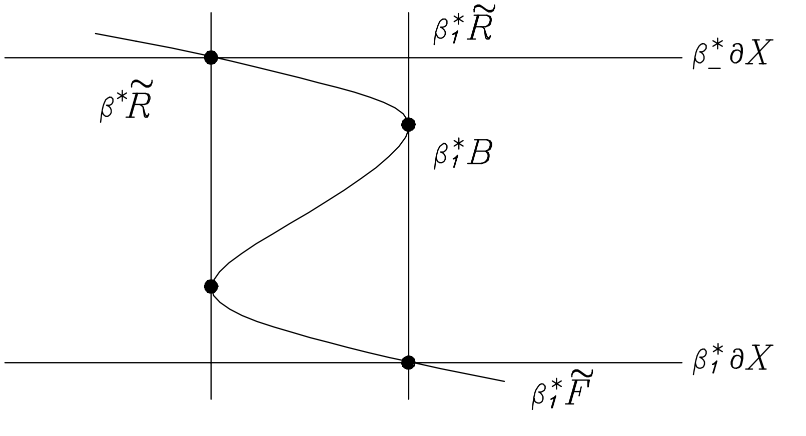

FIGURE 10. Intersections of $\beta^* \widetilde{F}$ and $\beta^* \widetilde{R}$

Since r is small in $\mathrm{supp}(u_1)$ we may write

$$\partial X_1 = \{Z = -2 + r\phi_2\} \cup \{Z = 2 + r\phi_3\}, \quad \phi_2, \phi_3 \in C^\infty(\widetilde{X}_1). \tag{6.58}$$

Let

$$P_1' = (3X\partial_X + Z\partial_Z)^2 - 9\partial_X^2 - 6\partial_X\partial_Z, \quad L_1' = (3(Z-X)\partial_X - Z\partial_Z)\,\partial_Z. \tag{6.59}$$

Direct computations show that, modulo lower order terms,

$$\begin{gathered} P_1' + ZL_1' = Q_1' = 3U_2\partial_X, \\ U_2 = \frac{1}{r}V_2' = 3(1-X^2)\partial_X + (2 - Z^2 - XZ)\partial_Z. \end{gathered} \tag{6.60}$$

Hence we obtain from (6.56),(6.59) and (6.60) that

$$L_1^{a_1}(\partial_X U_2)^{a_2}(V_1', V_2')^\alpha u_1 \in r^{-3}\bar{H}^b_{(-|a|)}(X_1), \quad |a| + |\alpha| \le k. \tag{6.61}$$

We shall prove that in fact

$$L_1^{a_1}U_2^{a_2}(V_1', V_2')^\alpha u_1 \in r^{-3}\bar{H}^b_{(-a_1)}(X_1), \quad |a| + |\alpha| \le k. \tag{6.62}$$

First we observe that for $a_1 + \alpha_1 \le k$

$$L_1^{a_1}U_2^{\alpha_2}(V_1')^{\alpha_1} u_1 \in r^{-3}\bar{H}^b_{(-a_1-\alpha_2)}(X_1), \quad a_1 + \alpha_1 \le k. \tag{6.63}$$

From (6.61) and the fact that $V_2' = rU_2$, we deduce for $l + |\alpha| \le k$

$$\begin{gathered} \partial_X^{\alpha_2} L_1^l (V_1', U_2)^\alpha u_1, \quad (r\partial_r)^{\alpha_2} L_1^l (V_1', U_2)^\alpha u_1, \\ (r\partial_Z)^{\alpha_2} L_1^l (V_1', U_2)^\alpha u_1 \in r^{-3}\bar{H}^b_{(-l-\alpha_2)}(X_1). \end{gathered} \tag{6.64}$$

In the neighborhood where $Z \sim \pm 2$ we consider a change of variables

$$\Psi(r, X, Z) = (r, X, Z - r\phi_i), \quad i = 2, 3.$$

Ψ is a diffeomorphism preserving $\{r = 0\}$ and $X_1^\# = \Psi(X_1) = \{Z \le 2\}$. Let $U_2^\# = \Psi_* U_2$, $V_1^\# = \Psi_* V_1'$, $L_1^\# = \Psi_* L_1$ and $u_1^\# = \Psi_* u_1$. We deduce from (6.64) that for $l + |\alpha| \le k$

$$\partial_X^{\alpha_2} {L_1^\#}^l (U_2^\#, V_1^\#)^\alpha u_1^\#, \quad (r\partial_r)^{\alpha_2} {L_1^\#}^l (U_2^\#, V_1^\#)^\alpha u_1^\# \in r^{-3}\bar{H}^b_{(-l-\alpha_2)}(X_1^\#). \tag{6.65}$$

We obtain from (6.63) that

$${L_1^\#}^{a_1} (V_1^\#, U_2^\#)^\alpha u_2 \in r^{-3}\bar{H}^b_{(-a_1-\alpha_2)}(X_1^\#), \quad a_1 + \alpha_1 \le k. \tag{6.66}$$

It follows, in the notation of Appendix B, that (6.65) is equivalent to

$${L_1^\#}^l (V_1^\#, U_2^\#)^\alpha u_2 \in r^{-3}\bar{H}^b_{(-l-\alpha_2, \alpha_2)}(X_1^\#), \quad l + \alpha_1 + \alpha_2 \le k. \tag{6.67}$$

Let us observe that $L_1^\#$ is a second order differential operator for which $\partial X^\# = \{Z = 2\}$ is non–characteristic. It follows from (6.67) and Proposition B.1 that by taking $a_1 = l + \alpha_2$

$${L_1^\#}^l (V_1^\#, U_2^\#)^\alpha u_1^\# \in r^{-3}\bar{H}^b_{(-l)}(X_1^\#), \quad l + |\alpha| \le k. \tag{6.68}$$

Thus the pull–back of (6.68) by Ψ gives (6.62).

The vector fields $r\partial_r$, U_2 and ∂_Z span all smooth vector fields in $\widetilde{X}_1$ tangent to $\beta_1^*\widetilde{R}$, while $r\partial_r$, U_2 and $3(Z-X)\partial_X - Z\partial_Z$ span the ones tangent to $\beta_1^*\widetilde{F}$. Therefore we conclude that the operators in (6.62) span $\Psi_b^1(\widetilde{X}_1, {}^bN^*\beta_1^*\widetilde{F} \sqcup {}^bN^*\beta_1^*\widetilde{R})$.

Another simple computation shows that

$$\begin{aligned}(Z-X)U_2\partial_Z - (1-X^2)L_1 &= -(2X+Z^3-3Z)\partial_Z^2,\\ (2-XZ-Z^2)L_1 + ZU_2\partial_Z &= -3(2X+Z^3-3Z)\partial_Z\partial_X\end{aligned} \tag{6.69}$$

Since ∂_Z is transversal to ∂X_1, the same method as the one used in the proof of (6.62) shows that if $U_3 = (2X+Z^3-3Z)\partial_Z$, then

$$(r\partial_r, U_2, U_3)^\alpha u_1 \in r^{-3}L^2(X_1), \quad |\alpha| \le k. \tag{6.70}$$

The vector fields $r\partial_r, U_2$ and U_3 span all the vector fields tangent to $\beta_1^*\widetilde{R}$ and $\beta_1^*\widetilde{F}$. We conclude that away from $\beta_1^*\widetilde{F}$ and $\beta_1^*\widetilde{R}$, $u_1 \in r^{-3}\bar{H}^b_{(k)}(X_1)$ and the extension is trivial. Therefore we may restrict our analysis to a neighbourhood of the intersections of the hypersurfaces and ∂X_1.

Let us first consider the case where $\beta_1^*\widetilde{R}$ intersects ∂X_1 away from $\beta_1^*\widetilde{F}$. Since this intersection is transversal one can introduce local coordinates
$x' = X \mp 1, z' = Z \mp r\phi_i$, $i=2,3$ in which

$$\beta_1^*\widetilde{R} = \{x'=0\}, \quad \partial X_1 = \{z'=0\}.$$

Since away from $\beta_1^*\widetilde{F}$ the vector fields in (6.70) span all the vector fields tangent to $\beta_1^*\widetilde{R}$, we find that

$$(r\partial_r, x'\partial_{x'}, \partial_{z'})^\alpha \tilde{u}_1 \in r^{-3}L^2(X_1), \quad |\alpha| \le k. \tag{6.71}$$

Now we deduce from Proposition 6.2 that there exists $\tilde{u}_1 \in r^{-3}L_b^2(\widetilde{X}_1)$ such that

$$\tilde{u}_1 = u_1 \text{ in } X_1 \text{ and } (r\partial_r, x'\partial_{x'}, \partial_{z'})^\alpha \tilde{u}_1 \in r^{-3}L_b^2(\widetilde{X}_1), \quad |\alpha| \le k. \tag{6.72}$$

Finally we analyze the case where $\beta_1^*\widetilde{F}$ and $\beta_1^*\widetilde{R}$ intersect the boundary simultaneously. Let us concentrate on the part of the boundary given by $Z = 2 + r\phi_3$ as the other case is analogous. Consider the change of variables

$$x' = 2\frac{X+1}{(Z+1)^2}, \quad z' = Z-2,$$

which is smooth since $Z+1>0$ near the intersection in question. In these coordinates

$$\begin{gathered}\beta_1^*\widetilde{F} = \{x' = -z'\}, \quad \beta_1^*\widetilde{R} = \{x'=0\}, \text{ and}\\ \partial X_1 = \{z' = r\phi_4\}, \quad \phi_4 \in C^\infty(\widetilde{X}_1).\end{gathered} \tag{6.73}$$

From the fact that $r\partial_r, U_2$ and U_3 span all the vector fields tangent to $\beta_1^*\widetilde{R}$ and $\beta_1^*\widetilde{F}$ we deduce that

$$(r\partial_r, (x'+z')\partial_{z'}, x'(\partial_{x'} - \partial_{z'}))^\alpha u_1 \in r^{-3}L_b^2(\widetilde{X}_1), \quad |\alpha| \le k. \tag{6.74}$$

Then we blow–up the submanifold $\{x' = z' = r' = 0\}$ with homogeneity $1-1-1$. Homogeneity of the vector fields in (6.74) and the fact that ∂X_1 is non–characteristic for $\partial_{z'}$ give that the vector fields in (6.74) lift under the $1-1-1$ blow–up to smooth vector fields tangent to the lifts of the hypersurfaces $\beta_1^*\widetilde{R}$ and $\beta_1^*\widetilde{F}$ and transversal to the lift of ∂X_1. Now we can use Proposition 6.2 to extend the lift of u_1 across the lift of ∂X_1 to be conormal to the lifts of $\beta_1^*\widetilde{R}$ and $\beta_1^*\widetilde{F}$. Then we can show that this in fact gives an extension of u_1 across ∂X_1 into the conormal space to the hypersurfaces $\beta_1^*\widetilde{R}$ and $\beta_1^*\widetilde{F}$.

We still need to construct the extension map in the region where projective coordinates $r = |x_3|^{\frac{1}{3}}\, Y = x_2/r^2$, $Z = x_1/r$ are used. In these coordinates

$$\begin{aligned} \beta_1^*\widetilde{R} &= \{Y^3 - 1 = 0\}, \quad \beta_1^*\widetilde{F} = \{\pm 2 + Z^3 - 3YZ = 0\}, \\ \partial X_1 &= \{Y = \frac{1}{4}Z^2 + r\phi_1\}, \quad \phi_1 \in C^\infty(\widetilde{X}_1). \end{aligned} \tag{6.75}$$

Observe that $Y > 0$ near $\beta_1^*\widetilde{R}$ and near the intersection of $\beta_1^*\widetilde{F}$ and ∂X_1. Therefore coordinates (r, X, Z) can be used there and the extension can be constructed as above. Away from the two hypersurfaces $u_1 \in r^{-3}\bar{H}^b_{(k)}(X_1)$ and the extension is trivial.

Similarly in the region where coordinates $r = |x_1|, Y = x_2/r^2, X = x_3/r^3$ are valid, the intersections of the boundary and the hypersurfaces are contained in the region where $|Y| > 0$ and the extension is constructed as above. Away from the hypersurfaces $u_1 \in r^{-3}\bar{H}^b_{(k)}(X_1)$ and the extension is trivial.

Let us remark again that the linearity and continuity of E_2 follow from those of the map S of Proposition 6.2 This concludes the proof of Proposition 6.4. □

7. Estimates for the Dirichlet Problem

7.1. To solve the mixed problem with the Dirichlet boundary condition we proceed by solving

$$Pu = 0 \quad \text{in} \quad X, \quad u\restriction_{\partial X} = f, \quad u\restriction_{X_-} = 0$$

where f comes from restricting the solution of the free equation obtained in Theorems 3.2 and 5.1 Thus we first need to characterize f.

Let (x, y) be the coordinates in Proposition 3.3 (i) which were used in the definition of $\widetilde{X}_4$. Hence

$$\partial X = \{\rho = 0\} \cap \widetilde{X}, \quad \Gamma = D \cap \partial X, \quad d\rho\restriction_\Gamma = dx_2,$$

and (x_1, x_3, y) gives a coordinate system on ∂X. In X_r defined in the beginning of Sect.6, the 1-2-3 blow-up followed by the 2-1-1 and 6-3-1 blow-ups induce a 1-3 blow-up followed by 2-1 and 6-1 blow-ups on the boundary $\beta_r^*\partial X$. Thus, let us write $Y = \partial X$ and define

$$Y_1 = (Y \setminus \Gamma) \sqcup (\mathbb{S}^1_{1-3} \times \mathbb{R}^{n-3}) \simeq \mathbb{R}_+ \times \mathbb{S}^1_{1-3} \times \mathbb{R}^{n-3}, \quad \mathbb{S}^1_{1-3} = \{\omega \in \mathbb{R}^2 : \omega_1^{12} + \omega_2^4 = 1\}$$

with the C^∞-structure given by the second identification. The blow-down map β_1^∂ : $Y_1 \to Y$ is given by

$$\beta_1^\partial : (r, \omega, y) \longmapsto (r\omega_1, r^3\omega_2, y).$$

We then define Y_2 similarly to $\widetilde{X}_2$ by blowing-up $\beta_1^{\partial *}\{x_1 = 0,\ x_3 \geq 0\} \cap \partial Y_1 = \Gamma_1$:

$$Y_2 = (Y_1 \setminus \Gamma_1) \sqcup (\mathbb{S}^1_{2-1_+} \times \mathbb{R}^{n-3}), \quad \mathbb{S}^1_{2-1_+} = \{\omega \in \mathbb{R}^2 : \omega_1^2 + \omega_2^4 = 1, \omega_2 \geq 0\},$$

with

$$Y_2 \xrightarrow{\beta_{12}^\partial} Y_1 \xrightarrow{\beta_1^\partial} Y, \qquad \beta^\partial = \beta_1^\partial \circ \beta_{12}^\partial$$

where $\beta_{12}^\partial(\rho, \omega, y) = (\rho^2\omega_1, \rho\omega_2, y)$ and where the coordinates in Y_1 near Γ_1 are chosen so that $\beta_1^\partial(X_1, r, y) = (rX_1, r^3, y)$.

Finally, we have $Y_3 \xrightarrow{\beta^\partial_{23}} Y_2$,

$$Y_3 = (Y_2 \setminus (\beta_2^{\partial^*}\{x_1 = 0, x_3 \geq 0\} \cap \partial Y_2)) \sqcup (\mathbb{S}_{6-1_+} \times \mathbb{R}^{n-3}),$$

defined as a C^∞ manifold with corners in a similar way.

We recall [28, 30] that a diffeomorphism between manifolds with corners is a homeomorphism which, together with its inverse, induces C^∞ maps on all the boundary faces. The induced blow-up of the boundary is made precise in the following

Lemma 7.1. *There exists a diffeomorphism $f : \beta_r^* \partial X \to Y_3$ such that $\beta_r \circ \imath_1 = \beta^\partial \circ f$, where $\imath_1 : \beta_r^* \partial X \hookrightarrow \widetilde{X}_r$.*

Proof. The boundary blow-up was defined using the coordinates (x_1, x_3, y) used also in the definition of $\widetilde{X}_r$. Thus it suffices to check that, after normalizations according to homogeneity, $\beta_r^* x_1, \beta_r^* x_3, \beta_r^* y$ give coordinates on $\beta_r^* \partial X$. Since $d\rho \restriction_\Gamma = dx_2$ this is immediately verified in each projective coordinate system for $\widetilde{X}_r$. $\square$

Hence we can identify Y_3 and $\beta_r^* \partial X$ so that the diffeomorphism f will be omitted below.

This suggests the definition

$$\mathcal{S}_\partial = \beta^{\partial *}(\widetilde{S}_+ \cap \partial X) \sqcup \beta^{\partial *}(\widetilde{F} \cap \partial X) \sqcup \beta^{\partial *}((\widetilde{R} \setminus \widetilde{F}) \cap \partial X) \sqcup \beta^{\partial *}(H \cap \partial X) \sqcup \partial \widetilde{X}_4 \tag{7.1}$$

where we note that $\mathcal{S}_\partial = \imath_1^* \beta_{2r}^* (\beta_{24})_* \mathcal{S}$, with $\mathcal{S}$ defined by (3.7) and appearing in Definition 3.1 of $J_k L^2(\widetilde{X}, H)$.

Recalling that the conormal spaces with non-integral orders of regularity are defined by complex interpolation we can state the restriction result:

Proposition 7.1. *If $u \in J_k^1 L_c(\widetilde{X}, H)$, $1 \leq j \leq n$, then*

$$u \restriction_{\partial X} \in \beta_*^\partial I_{k+1/2} L^2_{\nu_\partial}(Y_3, \mathcal{V}(Y_3, \mathcal{S}_\partial^+))$$

where

$$\begin{aligned} \mathcal{S}_\partial^+ \quad = \quad & \partial Y_3 \sqcup \beta^{\partial *}(\widetilde{S}_+ \cap \partial X) \sqcup \beta^{\partial *}(\widetilde{F} \cap \partial X) \sqcup \\ & \beta^{\partial *}(((\widetilde{R} \setminus \widetilde{F}) \setminus K) \cap \partial X) \sqcup \beta^{\partial *}((H \setminus K) \cap \partial X), \end{aligned} \tag{7.2}$$

$\beta_*^\partial \nu_\partial = dx_1 dx_3 dy$, and K in Definition 5.1, with ϵ chosen as in

The geometry of $\mathcal{S}_\partial^+$ is shown in Fig.11. The proof will follow easily from the following

Lemma 7.2. *If $\beta^* u, \beta^* D_j u \in I_k L^2_\nu(\widetilde{X}_4, \mathcal{V}(\widetilde{X}_4, \mathcal{S}))$ then, with S_∂ is given by (7.1),*

$$\beta_r^* u \restriction_{\beta_r^* \partial X} \in I_{k+1/2} L^2_{\nu_\partial}(Y_3, \mathcal{V}(Y_3, \mathcal{S}_\partial)).$$

Proof. Let us recall the definition (3.9)

$$\mathcal{S} = \beta^* \widetilde{F} \sqcup \beta^* \widetilde{R} \sqcup \beta^* \widetilde{R} \sqcup \beta^* \widetilde{S}_+ \sqcup \beta^* H \sqcup \beta^*(\widetilde{F} \cap \widetilde{R} \setminus B) \sqcup \beta^*(\widetilde{S}_+ \cap \widetilde{R} \setminus B),$$

and introduce the notation

$$\mathcal{S}_r = \beta_{2r}^* \circ (\beta_{24})_* \mathcal{S}, \quad \mathcal{S}_\partial = \imath_1^* \mathcal{S}_r.$$

Since the only surfaces intersecting $\beta^* D$ are $\beta^* \widetilde{S}$ and $\beta^* H$, Proposition 6.4 shows that

$$\beta^* v \in I_k L^2_\nu(\widetilde{X}_4, \mathcal{V}(\widetilde{X}_4, \mathcal{S})) \Longrightarrow \beta_a^* v \in I_k L^2_{\nu_a}(\widetilde{X}_a, \mathcal{V}(\widetilde{X}_a, \beta_{ra}^* \mathcal{S}_r)).$$

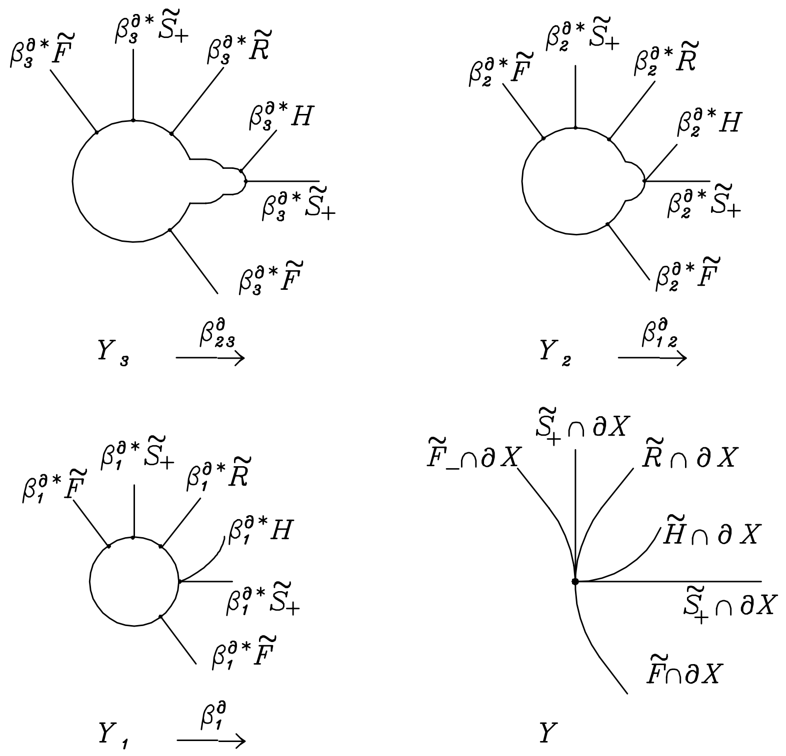

FIGURE 11. The blow-ups on the boundary and $\mathcal{S}_\partial^+$ in Y_3

Since in a neighbourhood of $\beta_r^*\partial X = \beta_a^*\partial X$, $\widetilde{X}_a$ and $\widetilde{X}_r$ are equal, the assumptions of the lemma imply that

$$\chi\beta_r^* u \in I_k H^b_{(1),\nu_r}(\widetilde{X}_r, \mathcal{S}_r),$$

for some $\chi \in C^\infty(\widetilde{X}_r)$, $\chi \equiv 1$ near $\beta_r^*\partial X$.

The surfaces $\beta_r^*\widetilde{S}_+, \beta_r^* H$ and $\beta_r^*\widetilde{F}, \beta_r^*\widetilde{R}$ away from $\beta_r^*(\widetilde{F} \cap \widetilde{R} \setminus B)$, intersect $\beta_r^*\partial X$ transversally and are separated from each other. Thus, away from $\beta_r^*(\widetilde{F} \cap \widetilde{R} \setminus B)$ (see Fig.4), $V \in \mathcal{V}(Y_3, \mathcal{S}_\partial)$ extends to a vector field $\widetilde{V} \in \mathcal{V}(\mathcal{S}_r)$, $\widetilde{V} \restriction_{\beta_r^*\partial X} = V$, so that

$$V^l(u \restriction_{\beta_r^*\partial X}) = (\widetilde{V}^l u) \restriction_{\beta_r^*\partial X} \in H^b_{(\frac{1}{2}),\nu_\partial}(Y_3),\ l \le k$$

as $Y_3 \simeq \beta_r^*\partial X \subset \widetilde{X}_r$ and $\widetilde{V}^l u \in H^b_{(1),\nu_r}(\widetilde{X}_r)$. If U is a sufficiently small neighbourhood of $\beta_r^*(\widetilde{F} \cap \widetilde{R} \setminus B)$ in $\widetilde{X}_r$ then

$$\mathcal{S}_r \restriction_U = (\beta_r^*\widetilde{R} \sqcup \beta_r^*\widetilde{F} \sqcup \beta_r^*(\widetilde{F} \cap \widetilde{R} \setminus B)) \restriction_U .$$

By Proposition 3.3 (see Fig.4) $\beta_r^*\widetilde{F}$, $\beta_r^*\widetilde{R}$ and $\beta_r^*\partial X$ intersect transversally at $\beta_r^*\widetilde{F} \cap \beta_r^*\partial X$. We conclude that again $V \in \mathcal{V}(U \cap \beta_r^*\partial X, \mathcal{S}_\partial \restriction_U)$ extends to $\widetilde{V} \in \mathcal{V}(\widetilde{X}_r, \mathcal{S}_r)$, so that the previous argument is applicable. Thus, it follows by induction that

$$u \restriction_{\beta_r^*\partial X} \in I_k H^b_{(\frac{1}{2}),\nu_\partial}(Y_3, \mathcal{S}_\partial).$$

We observe that

$$\begin{array}{ccc} I_{k+1}L^2_{\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial)) & \longrightarrow & I_kL^2_{\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial)) \\ \Big\uparrow & & \Big\uparrow Id \\ I_kH^b_{(1),\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial)) & \longrightarrow & I_kL^2_{\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial)) \end{array} \tag{7.3}$$

and

$$u\in I_kH^b_{(s),\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial)) \Longleftrightarrow (I+A)^s u\in I_kL^2_{\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial)),$$

if $A\in\Psi^1_b(Y_3)$ is elliptic and $A\geq 0$. Hence we have

$$\begin{aligned} I_kH^b_{(\frac12)}(Y_3,\mathcal{V}(\mathcal{S}_\partial)) &= [I_kH^b_{(1),\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial)), I_kL^2_{\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial))]_{\frac12}\subset \\ & [I_{k+1}L^2_{\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial)), I_kL^2_{\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial))]_{\frac12} = I_{k+\frac12}L^2_{\nu_\partial}(Y_3,\mathcal{V}(\mathcal{S}_\partial)). \end{aligned}$$

□

To complete the proof of Proposition 7.1 we observe that (see also Fig.11)

$$\mathcal{S}^+_\partial\restriction_{Y_3\setminus\beta^{\partial*}(K\cap\partial X)}=\mathcal{S}_\partial\restriction_{Y_3\setminus\beta^{\partial*}(K\cap\partial X)},\quad \mathcal{S}^+_\partial\restriction_{\beta^{\partial*}(K\cap\partial X)}=\beta^{\partial*}(\widetilde{F}_-\cap\partial X)\restriction_{\beta^{\partial*}(K\cap\partial X)},$$

Proceeding as in the proof of Lemma 7.2 shows that if $u_1\restriction_{K_1}\in I_kH^b_{(1),\nu_r}(\widetilde{X}_r,(\beta_r^*\widetilde{F}\sqcup\partial\widetilde{X}_r\restriction_{K_1}))$, $K_1=\beta_r^*K$, then

$$(u\restriction_{\beta_r^*\partial X})\restriction_{\beta^{\partial*}(K\cap\partial X)}\in I_{k+\frac12}L^2_{\nu_\partial}(\beta^{\partial*}(K\cap\partial X),\beta^{\partial*}(\widetilde{F}\cap\partial X)\sqcup\partial Y_3).$$

Combined with Lemma 7.2 these observations give $u\restriction_{\partial X}\in\beta^\partial_* I_{k+\frac12}L^2_{\nu_\partial}(Y_3,\mathcal{V}(Y_3,\mathcal{S}^+_\partial))$.

Proposition 7.1 motivates the following definition:

Definition 7.1. The conormal space on the boundary, $J_sL^2_c(\partial X,H)$, is defined as

$$J_sL^2_c(\partial X,H)=\beta^\partial_* I_sL^2_{\nu_\partial}(Y_3,\mathcal{V}(Y_3,\mathcal{S}^+_\partial))$$

where the variety $\mathcal{S}^+_\partial$ is given by (7.2).

A slightly stronger formulation of the restriction result is now given in

Theorem 7.1. *If $u\in J^1_sL^2(\widetilde{X},H)$ then $u\restriction_{\partial X}\in J_{s+\frac12}L^2(\partial X,H)$.*

Proof. The conormal space in the definition of $J_tL^2(\partial X,H)$ is an interpolations space – see Appendix B. Since for non-integral orders $J^1_sL^2(\widetilde{X},H)$ is defined by interpolation between $[s]$ and $[s]+1$ the theorem follows from Proposition 7.1. □

The main result of this section is

Theorem 7.2. *If $u\in L^2_{loc}(X)$ is the solution of*

$$Pu=0\quad in\quad X,\quad u\restriction_{\partial X}=f\in J_sL^2_c(\partial X,H),\quad u\restriction_{X_-}=0,\quad f\restriction_{(\partial X)_-}=0 \tag{7.4}$$

then there exists $\tilde u\in J_sL^2_{loc}(\widetilde{X},H)$ such that $u=\tilde u\restriction_X$.

Theorem 7.1 is proved by finding a microlocally characterized space containing $J_kL^2_c(\partial X,H)$ (Proposition 7.2) and by using microlocal models for the components of that space. The propagation estimates obtained in the Friedlander model (Proposition 7.3) give u in terms of marked Lagrangian spaces on a manifold with boundary (Proposition 7.6). The extension property for those spaces obtained in Sect.6 completes the proof.

Let us start by recalling from Section 2 the defintion of the following smooth Lagrangian submanifolds of $T^*\partial X \setminus 0$:

$$\Lambda_{00} = N^*\Gamma, \quad \Lambda_{11} = N^*(\widetilde{F} \cap \partial X), \quad \Lambda_{31} = N^*((\widetilde{R} \setminus \widetilde{F}) \cap \partial X), \quad \Lambda_{13} = N^*(H \cap \partial X), \tag{7.5}$$

and

$$\Lambda_{21} \cup \Lambda_{23} = N^*(\widetilde{S} \cap \partial X), \quad \Lambda_{11} \cap \Lambda_{23} = \emptyset, \quad \Lambda_{13} \cap \Lambda_{21} = \emptyset.$$

From these we obtain Lagrangians with boundaries:

$$\Lambda_{ij}^{\pm} = \Lambda_{ij} \cap \{\pm x_j \geq 0\}, \quad j = 1, 3, \ i = 1, 2, 3.$$

in terms of coordinates (x, y) of Proposition 3.1.

The sub– and super–marked Lagrangian spaces introduced in Sect.4 now enter in

Proposition 7.2. *If $J_k L^2_c(\partial X, H)$ is given by Definition 7.1, then*

$$\begin{aligned} J_k L^2_c(\partial X, H) \ \subset \ & \sum_{i=1,3} I_k L^2(\partial X; \Lambda_{00}, J^1_{\Lambda_{00} \cap \Lambda_{2i}} \Lambda_{2i}, 2) + \sum_{i=1,3} I_k L^2(\partial X; \Lambda^+_{2i}, 2) + \\ & I_k L^2(\partial X; \Lambda^+_{3i}, 2) + I_k L^2(\partial X; \Lambda^+_{13}, 2) + \sum_{i=1}^{2} I_k L^2(\partial X; \Lambda_{11}^{\mathrm{sgn}((-1)^i)}, 2). \end{aligned} \tag{7.6}$$

Proof. We need to consider another blow-up $\beta^\partial_{34} : Y_4 \longrightarrow Y_3$, $\beta^\partial_4 = \beta^\partial \circ \beta^\partial_{34}$, obtained by successive 2-1 and 6-1 blow-ups of $\beta^{\partial *}(\{x_1 = 0, x_3 \leq 0\}) \cap \partial Y_3$ identical to the blow-ups used in the construction of Y_3.

We easily see that

$$\beta^\partial_{4*} I_k L^2_{\nu_{\partial 4}}(Y_4; \mathcal{V}(\beta^{\partial *}_{34} \mathcal{S}^+_\partial \sqcup \partial Y_4)) \supset J_k L^2_c(\partial X, H), \quad \beta^\partial_{4*}\nu_{\partial 4} = dx_1 dx_3 dy,$$

and we shall prove that the left hand side is contained in the right hand side of (7.6). We first prove that

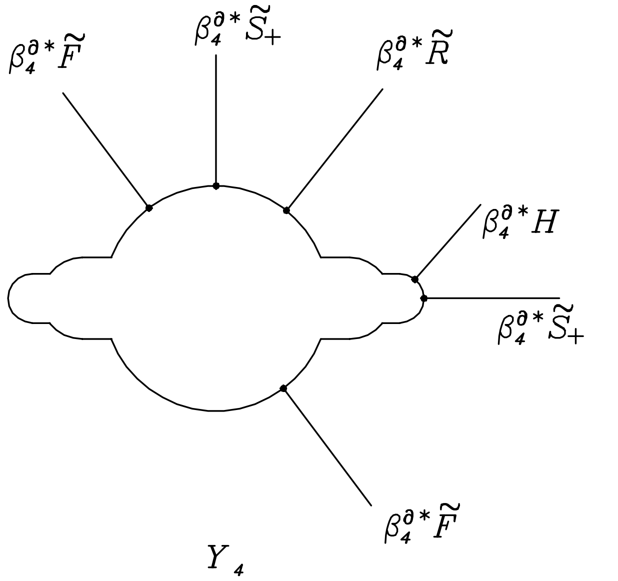

FIGURE 12. The geometry in Y_4

$$I_k L^2_{\nu_{\partial 4}}(Y_4; \mathcal{V}(\beta_4^{\partial *}(\widetilde{F} \cap \partial X) \sqcup \partial Y_4)) \xrightarrow{\beta_{4*}}$$
$$\sum_{i=1,3} I_k L^2(\partial X; \Lambda_{00}, J^1_{\Lambda_{00}\cap\Lambda_{2i}}\Lambda_{2i}, 2) + I_k L^2(\partial X; \Lambda_{11}, 2). \tag{7.7}$$

In particular this gives

$$I_k L^2_{\nu_{\partial 4}}(Y_4; \partial Y_4) \xrightarrow{\beta_{4*}} \sum_{i=1,3} I_k L^2(\partial X; \Lambda_{00}, J^1_{\Lambda_{00}\cap\Lambda_{2i}}\Lambda_{2i}, 2) + I_k L^2(\partial X; \Lambda_{11}, 2). \tag{7.8}$$

We note that in view of Proposition 4.3 we can replace the second term on the right hand side by the last term in (7.6). To prove (7.7) we first observe that we can change coordinates in ∂X such that $\widetilde{F} \cap \partial X = \{x_3 = 0\}$, $\Gamma = \{x_1 = x_3 = 0\}$ and $J^1_{\Lambda_{00}\cap\Lambda_{23}}\Lambda_{23} = J^1_{\Lambda_{00}\cap\Lambda_{23}}(N^*\{x_1 = 0\})$ and that this change of coordinates lifts to a smooth diffeomorphism on Y_4. To construct such a change of variables we let (x_1, x_3) be smooth coordinates near Γ in which $H \cap \partial X = \{x_1 = 0\}$. Since $\widetilde{F} \cap \partial X$ is second order tangent to $H \cap \partial X$ at Γ, it follows that $\widetilde{F} \cap \partial X = \{x_3 + f(x,y)x_1^3 = 0\}$, with $f(x,y) \neq 0$. For simplicity we assume that $f(0,0) = 1$. Now a direct calculation shows that there exists a smooth function $A(s, x_1, x_3)$ for $s \in [0,1]$ and x_1, x_3 small, such that

$$A(s, x_1, x_3)x_1\partial_{x_1}\left(x_3 + x_1^3 + s(f(x,y)-1)x_1^3\right) = -(f(x,y)-1)x_1^3$$

Thus, as in section 3, we obtain, by integrating the vector field $V_s = A(s, x_1, x_3)x_3\partial_{x_3}$, a one parameter family smooth diffeomorphisms ϕ_s fixing $\{x_1 = 0\}$ and satisfying

$$\phi_s^*\left(x_3 + x_1^3 + s(f(x,y)-1)x_1^3\right) = x_3 + x_1^3.$$

In particular the map ϕ_1 fixes $\{x_1 = 0\}$ and $\phi_1^*\left(\widetilde{F} \cap \partial X\right) = \{x_3 + x_1^3 = 0\}$. We are going to show below that the vector field $x_1\partial_{x_1}$ lifts to a smooth vector field in Y_4 which is tangent to ∂Y_4, therefore it follows that ϕ_1 coordinates lifts to a smooth diffeomorphism on Y_4 preserving ∂Y_4.

Proposition 4.3 shows that the right hand side of (7.7) is equal to

$$I_k L^2(\partial X, \Lambda_{00} \sqcup \Lambda_{11}) + I_k L^2(\partial X, \Lambda_{00}, J^1_{\Lambda_{00}\cap\Lambda_{23}}\Lambda_{23}, 2).$$

Thus Definition 4.4, a microlocal partition of unity and the proof of Lemma 4.2 show that

$$u \in I_k L^2(\partial X, \Lambda_{00} \sqcup \Lambda_{11}) + I_k L^2(\partial X, \Lambda_{00}, J^1_{\Lambda_{00}\cap\Lambda_{23}}\Lambda_{23}, 2) \Longleftrightarrow$$
$$(x_3 D_{x_3})^{k_1}(x_1 D_{x_1})^{k_2}(x_3^3 D_{x_1})^{k_3} u \in L^2(\partial X),\ k_1 + k_2 + k_3 \leq k. \tag{7.9}$$

To prove that (7.7) holds one needs to show that for $u \in I_k L^2_{\nu_{\partial 4}}(Y_4; \mathcal{V}(\beta_4^{\partial *}(\widetilde{F} \cap \partial X) \sqcup \partial Y_4))$ and $v = \beta_{4*}u$ one has

$$(x_3 D_{x_3})^{k_1}(x_1 D_{x_1})^{k_2}(x_3^3 D_{x_1})^{k_3} v \in L^2(\partial X),\ k_1 + k_2 + k_3 \leq k. \tag{7.10}$$

Since the operators in (7.10) are smooth vector fields, (7.7) is a consequence of

$$\beta_4^{\partial *}(x_3\partial_{x_3}, x_1\partial_{x_1}, x_3^3\partial_{x_1}) \subset \mathcal{V}(\beta_\triangle^{\partial\,*}\left(\widetilde{\mathcal{F}} \cap \partial \mathcal{X}\right) \sqcup \partial \mathcal{Y}_\triangle). \tag{7.11}$$

In other words one needs to show that the vector fields in (7.9) lift under β_4^∂ to smooth vector filelds in Y_4 tangent to $\beta_4^{\partial *}(\widetilde{F} \cap \partial X)$. To prove (7.11) we compute the lifts of these vector fields in twelve projective coordinate systems. Consider the first set of projective coordinates

$$(\rho, X_1) \mapsto (\rho X_1, \pm\rho^3) = (x_1, x_3),$$
$$(\rho, X_3) \mapsto (\pm\rho, \rho^3 X_3) = (x_1, x_3).$$

In these coodinates the lift of the vector fields in (7.11) are spanned, over $C^\infty(Y_1)$, by

$$\begin{gathered} \rho\partial_\rho, \quad X_1\partial_{X_1}, \rho^8\partial_{X_1}, \\ \rho\partial_\rho, \quad X_3\partial_{X_3}. \end{gathered} \tag{7.12}$$

The vector fields $\rho\partial_\rho$ and $X_3\partial_{X_3}$ are tangent to $\beta_1^{\partial *}(\widetilde{F} \cap \partial X) \sqcup \partial Y_1$ which can be identified with $\beta_4^{\partial *}(\widetilde{F} \cap \partial X) \sqcup \partial Y_4$ away from $\{X_1 = \rho = 0\}$.

Next we blow-up $\{X_1 = \rho = 0\}$ with homogeneity $2 - 1$. Consider the second set of projective coordinates

$$(R, x_1') \mapsto (R, R^2 x_1') = (\rho, X_1)$$
$$(R, \rho') \mapsto (\rho' R^2, \pm R^2) = (\rho, X_1).$$

The lift of the vector fields in (7.12) are spanned by

$$\begin{gathered} R\partial_R, \; ; \; x_1'\partial_{x_1'}, \quad R^6\partial_{x_1'}, \\ R\partial_R, \rho'\partial_{\rho'} \end{gathered} \tag{7.13}$$

Away from $\{x_1' = R = 0\}$ Y_2 can be identified with Y_4 and one easily sees that the second set of vector fields clearly span $\mathcal{V}(\mathcal{Y}_\triangle)$. Finally one needs to blow-up the submanifold $\{x_1' = R = 0\}$ with homogeneity $6 - 1$. Consider coordinates

$$(r, T) \mapsto (\pm T^6, rT) = (x_1', R)$$
$$(r_1, T_1) \mapsto (T_1^6 r_1, T_1) = (x_1', R).$$

It is easy to see that the lift of the vector fields in (7.13) is in the span of

$$\begin{gathered} T\partial_T, r\partial_r \\ \partial_{r_1}, T_1\partial_{T_1} \end{gathered}$$

Therefore they are in $\mathcal{V}(\partial\mathcal{Y}_\triangle)$. This concludes the proof of (7.7).

We now want

$$\begin{gathered} I_k L^2_{\nu_{\partial 4}}(Y_4; \mathcal{V}(\beta_4^{\partial *}((\widetilde{R} \setminus \widetilde{F}) \cap \partial X \cap \{x_1 \geq 0\}) \sqcup \partial Y_4)) \xrightarrow{\beta_{4*}} \\ \sum_{i=1,3} I_k L^2(\partial X; \Lambda_{00}, J^1_{\Lambda_{00} \cap \Lambda_{2i}} \Lambda_{2i}, 2) + I_k L^2(\partial X, \Lambda_{00} \sqcup \Lambda_{11}) + \\ I_k L^2(\partial X; \Lambda_3^+, 2) + I_k L^2(\partial X, \Lambda_{11}, 2), \end{gathered} \tag{7.14}$$

To prove (7.14) we proceed by a partition of unity in Y_4. Let $\phi \in C_c^\infty(Y_4)$ be such that $\phi = 1$ near $\beta_4^{\partial *}\left((\widetilde{R} \setminus \widetilde{F}) \cap \partial X\right) \cap \{\beta_4^{\partial *} x_1 \geq 0\}$ and ϕ is supported away from the boundary faces introduced by the $2-1$ and $6-1$ blow-ups of $\beta_3^{\partial *}(\{x_1 = 0, x_3 \leq 0\}) \cap \partial Y_3$. Let $u \in I_k L^2_{\nu_{\partial 4}}(Y_4; \mathcal{V}(\beta_4^{\partial *}((\widetilde{R} \setminus \widetilde{F}) \cap \partial X \cap \{x_1 \geq 0\}) \sqcup \partial Y_4))$. Then by (7.8) $\beta^{\partial}_{4*}(1 - \phi)u \in \sum_{i=1,3} I_k L^2(\partial X; \Lambda_{00}, J^1_{\Lambda_{00} \cap \Lambda_{2i}} \Lambda_{2i}, 2) + I_k L^2(\partial X; \Lambda_{11}, 2)$.

From the support properties of ϕ we deduce that

$$\beta^{\partial}_{4\,*}(\phi u) = \beta^{\partial}_{3\,*}(\phi u).$$

Now it follows from Lemma 4.2 that

$$\beta^{\partial}_{4\,*}(\phi u) \in I_k L^2_{\mathrm{loc}}(\partial X, \Lambda_3^+, 2) + \beta_3^* H^b_{(k)}(\partial Y_3).$$

Thus (7.14) follows from (7.8).

The push forward of the other components of $\beta_{34}^* \mathcal{S}_\partial^+$ are handled in a similar way and the details will be left to the reader. □

Remark 7.1. The space $J_k L^2(\partial X, H)$ is strictly contained in the microlocal space defined in Proposition 7.2. The extra terms are in $I_k L^2(\partial X; \Lambda_{23}^+, 2)$ and $I_k L^2(\partial X; \Lambda_{13}^+, 2)$. They are explained by the additional blow-ups in Y_4 needed for the first term in the right hand side of (7.6) and the singularities on $\partial Y_4 \setminus \beta_{34}^{\partial *} \partial Y_3$ produced by $I_k L^2(\partial X; \Lambda_{23}^+, 2)$ and $I_k L^2(\partial X; \Lambda_{13}^+, 2)$. Otherwise the push-forward is sharp.

To solve (7.4) we use the forward Melrose-Taylor diffractive parametrix [35] which we shall briefly recall:

$$T = \widetilde{T} \circ L, \quad L \in I^0(\mathbb{R}^{n-1} \times \partial X, \mathcal{J}^{-1\prime}), \quad \widetilde{T} : \mathcal{E}'(\mathbb{R}^{n-1}) \longrightarrow \mathcal{N}(X), \tag{7.15}$$

where $P\widetilde{T}v \in C^\infty(X)$, $\widetilde{T}v \restriction_{\partial X} = Jv$, $\widetilde{T}v \restriction_{X_-} = 0$, $J \in I^0(\partial X \times \mathbb{R}^{n-1}, \mathcal{J}')$ is an elliptic (in an appropriate cone) Fourier Integral Operator and L its microlocal inverse. The space $\mathcal{N}(X)$ used here gives, in an invariant way, the functions 'smooth in the direction normal to the boundary' – see [14, 25, 28].

Let $\Gamma \subset T^*\mathbb{R}^{n-1} \setminus 0$ be a small conic neighbourhood of $(0; (0, 1, 0, \cdots, 0))$. Then for $f \in \mathcal{E}'(\mathbb{R}^{n-1})$, $WF(f) \subset \Gamma$,

$$\widetilde{T} f(z) = \int (g(z,\xi) A_+(\zeta) + h(z,\xi) A'_+(\zeta)) A_+(\zeta_0)^{-1} e^{i\theta(z,\xi)} \hat{f}(\xi) d\xi, \tag{7.16}$$

with the phase functions $\zeta, \zeta_0 = \zeta \restriction_{\partial X \times \mathbb{R}^{n-1}}$ and θ homogeneous of degree 2/3 and 1 respectively. The amplitudes $g \in S^0(X; \mathbb{R}^{n-1})$ and $h \in S^{-1/3}(X, \mathbb{R}^{n-1})$ are supported in a conic neighbourhood of $(z_0, (0, \cdots, 0, 1))$, $z_0 \in \partial X$ and satisfy appropriate transport equations. Most importantly $g(0, (0, 1, 0, \cdots, 0)) = 1$ and $h \restriction_{\partial X \times \mathbb{R}^{n-1}} = 0$.

The construction of ζ and θ exploits the equivalence of glancing hypersurfaces which we will now discuss (see [25, 35], for detailed presentation and proofs). Let $\widetilde{\Gamma} \subset T^*\mathbb{R}^n \setminus 0$ be an open cone, $\widetilde{\Gamma} \restriction_{x_2 = 0} \subset \Gamma$, where we denote the coordinates in $\mathbb{R}^n$ by (x, y), $x \in \mathbb{R}^3$, $y \in \mathbb{R}^{n-3}$ with $\mathbb{R}^{n-1} = \partial \mathbb{R}^n_+ = \{x_2 = 0\} \cap \mathbb{R}^n$ (compare Proposition 3.1) and let $m \in T^*_{\partial X} \widetilde{X} \setminus 0$ be a glancing point for $p = \sigma_2(P)$. Then there exists a canonical transformation

$$\tilde{\chi} : \tilde{\Gamma} \longrightarrow T^*\widetilde{X} \setminus 0, \quad \tilde{\chi}(0; (0, 1, 0, \cdots, 0)),$$

such that

$$\begin{aligned} &\tilde{\chi}(\{\xi_2^2 - x_2 \xi_3^2 - \xi_3 \xi_1 = 0\} \cap \widetilde{\Gamma}) \subset p^{-1}(0), \\ &\tilde{\chi}(\{x_2 = 0\} \cap \widetilde{\Gamma}) \subset T^*_{\partial X} \widetilde{X}. \end{aligned} \tag{7.17}$$

The equivalence $\tilde{\chi}$ induces a canonical transformation on the boundary

$$\chi_\partial : \Gamma \longrightarrow T^*\partial X \setminus 0,$$

and the graph of χ_∂, $\mathcal{J}$, gives then the canonical relation of the elliptic Fourier Integral Operator J. The main geometric property of χ_∂ is the intertwining of the billiard ball maps defined in Sect.2:

$$\chi_\partial^{-1} \circ \delta^\pm \circ \chi_\partial = \delta_0^\pm, \tag{7.18}$$

where $\delta_0^\pm$ corresponds to the model glancing hypersurfaces in the left hand side of (7.17).

There is a substantial amount of freedom in choosing $\tilde{\chi}$. Thus, in addition to (7.17) we can also have

$$\chi_\partial(\Lambda_0 \cap \Gamma) \subset \Lambda_{00} \quad \text{or} \quad \chi_\partial(\Lambda_1 \cap \Gamma) \subset \Lambda_{1i}, \quad i = 1 \ \text{ or } \ 3, \tag{7.19}$$

where $\Lambda_0 = N^*\{x_1 = x_3 = 0\} \subset T^*\mathbb{R}^{n-1} \setminus 0$ and $\Lambda_1 = N^*\{x_3 + x_1^3/3 = 0\} \subset T^*\mathbb{R}^{n-1} \setminus 0$. In fact, one can apply Theorem 4.2.6 of [36] in the same way as in Sect.3 of [37] and Sect.3 of [51]. We should note that in (7.19) one needs to choose a different χ_∂ in the case of each Lagrangian $\Lambda_0, \Lambda_{1i}, i = 1, 3$.

We recall from Sect.2 that

$$\Lambda_{31}^\pm = \delta^\pm \Lambda_{11}^\pm, \quad \Lambda_{21}^\pm \cup \Lambda_{23}^\pm = \delta^\pm \Lambda_0$$

and

$$\Lambda_{33}^\pm = \delta^\pm \Lambda_{13}^\pm$$

where Λ_{33} is a Lagrangian simply tangent to Λ_{23} at $\Lambda_{23} \cap \Lambda_{13}$. It corresponds to the reflection of the false front H.

Thus, thanks to (7.18) in each case of (7.19) we also obtain

$$\begin{aligned} \chi_\partial(\Lambda_2 \cap \Gamma) &\subset \Lambda_{2i}, \quad i = 1 \ \text{ or } \ 3, \\ \chi_\partial(\Lambda_3 \cap \Gamma) &\subset \Lambda_{3i}, \quad i = 1 \ \text{ or } \ 3, \end{aligned}$$

where $\Lambda_m = N^*\{x_3 + x_1^3/(3m^2) = 0\}, m \neq 0$.

If the phases in (7.16) are considered formally then the expected wave front set relation for $\widetilde{T}$ becomes $\mathcal{F}^+ \circ \mathcal{C}_0^-$, where $\mathcal{C}_0$ is the model square root of the billiard ball map (see [36] and (7.26) below) and

$$\mathcal{F}^\pm = \left\{ (z, \phi_z^\pm; \phi_\xi^\pm, \xi) : \phi^\pm(z,\xi) = \theta \mp \frac{2}{3}(-\zeta(z,\xi))^{3/2} \right\} \subset T^*\widetilde{X} \setminus 0 \times T^*\mathbb{R}^{n-1} \setminus 0$$

with $\mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^-$, a smooth canonical relation generated by $\tilde{\phi}(z,\xi,\tau) = -\xi_1^{-2}\tau^3/3 - \tau\xi_1^{-2/3}\zeta(z,\xi) + \theta(z,\xi)$. We also have $\mathcal{F}_0 = \mathcal{F}_0^+ \cup \mathcal{F}_0^-$ for the model case.

The construction of ζ and θ in [35] shows that

$$\mathcal{F} = \{(r,s) \in \tilde{\chi}(\widetilde{\Gamma}) \times \Gamma \subset T^*\widetilde{X} \setminus 0 \times T^*\mathbb{R}^{n-1} \setminus 0 : (\widetilde{\chi}^{-1}(r), s) \in \mathcal{F}_0\}, \tag{7.20}$$

and we observe that consequently for each choice of χ_∂ in (7.19)

$$\begin{aligned} &\chi_\partial(\Lambda_1 \cap \Gamma) \subset \Lambda_{11} \Longrightarrow N^*\widetilde{F} \cap \widetilde{\chi}(\widetilde{\Gamma}) \subset \mathcal{F}(\Lambda_0 \cap \Gamma),\ N^*\widetilde{R} \cap \widetilde{\chi}(\widetilde{\Gamma}) \subset \mathcal{F}(\Lambda_2 \cap \Gamma) \\ &\chi_\partial(\Lambda_0 \cap \Gamma) \subset \Lambda_{00} \Longrightarrow N^*\widetilde{S} \cap \widetilde{\chi}(\widetilde{\Gamma}) \subset \mathcal{F}(\Lambda_1 \cap \Gamma) \\ &\chi_\partial(\Lambda_1 \cap \Gamma) \subset \Lambda_{13} \Longrightarrow N^*H \cap \widetilde{\chi}(\widetilde{\Gamma}) \subset \mathcal{F}(\Lambda_0 \cap \Gamma),\ N^*H_+ \cap \widetilde{\chi}(\widetilde{\Gamma}) \subset \mathcal{F}(\Lambda_2 \cap \Gamma), \end{aligned}$$

where H_+ is the reflection[1] of H by ∂X and where in the second case we consider different choices of $\widetilde{\chi}$ and $\widetilde{\Gamma}$ giving localization near B and D respectively. We note

[1] see Sect.2; we will refer to this artificial surface only once, in the last part of the proof of Theorem 7.2.

that by choosing different conic neighbourhoods Γ and $\widetilde{\Gamma}$ we can reverse the inclusions above.

7.2. In this subsection we shall consider the model problem in

$$\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_2 > 0\}$$

with $P_0 = D^2_{x_2} - x_2 D^2_{x_3} - D_{x_1} D_{x_3}$,

$$P_0 u = 0 \ \text{ in } \mathbb{R}^3_+, \quad u\restriction_{\partial \mathbb{R}^3_+} = u_0 \in \mathcal{E}'(\mathbb{R}^2), \quad u\restriction_{\phi(x)\ll 0} = 0, \tag{7.21}$$

where $\phi(x)$ is the time function for P_0, near zero chosen to be $x_1 + x_3$. The 1-2-3 homogeneity of the problem has already been stressed in Sect.3 and here we shall use it microlocally near $m_0 = (0; (0,0,1))$.

The solution operator for (7.21) is given explicitly as

$$u = T_0 u_0, \quad T_0 u_0(x) = \frac{1}{(2\pi)^2} \int \frac{A_+(\zeta)}{A_+(\zeta_0)} \hat{u}_0(\xi_1, \xi_3) e^{i(\xi_1 x_1 + \xi_3 x_3)} d\xi_1 d\xi_3, \tag{7.22}$$

where

$$\zeta = -\xi_3^{-1/3}(\xi_1 + x_2 \xi_3), \quad \zeta_0 = \zeta\restriction_{\partial \mathbb{R}^3_+ \times \mathbb{R}^3} = -\xi_3^{-1/3} \xi_1.$$

Let $\Gamma_0 \subset T^* \partial \mathbb{R}^3_+ \setminus 0$ be a small connected open conic neighbourhood of $(0, (0,1))$ and let $\Gamma \subset T^* \mathbb{R}^3 \setminus 0$ be a connected open conic neighbourhood of $(0; (0,0,1))$ such that

$$WF_b(\chi T u_0) \subset \jmath_* \Gamma \quad \text{if} \quad WF(u_0) \subset \Gamma_0, \ \ \chi \in C_0^\infty(\mathbb{R}^3), \quad \operatorname{supp} \chi \ \text{ near } 0,$$

and where $\jmath$ is the natural inclusion $\jmath : T^* \mathbb{R}^3_+ \setminus 0 \to {}^b T^* \mathbb{R}^3_+ \setminus 0$. The existence of Γ follows from propagation of singularities for the diffractive boundary value problem and in this case can be easily seen directly.

We also define the microlocal parametrix near Γ_0:

$$T_0^\sharp u_0(x) = \frac{1}{(2\pi)^2} \int \beta(\xi_3) \psi(\xi_3^{-1} \xi_2) \frac{A_+(\zeta)}{A_+(\zeta_0)} \widehat{u}_0(\xi_1, \xi_3) e^{i(x_1 \xi_1 + x_3 \xi_3)} d\xi_1 d\xi_3, \tag{7.23}$$

$$\beta \in C^\infty([1,\infty)), \ \beta(t) = 1 \quad \text{if } t > 2, \quad \psi \in C_0^\infty(\mathbb{R}), \ \psi = 1 \ \text{ near } 0$$

and we are interested in its mapping properties.

To exploit the homogeneity in a systematic way, let us now introduce

$$Z = [1, \infty) \times \mathbb{R}, \qquad Z_+ = [1,\infty) \times \mathbb{R} \times \mathbb{R}_+$$

$$(\lambda, x) \in Z, \qquad (\lambda, x, y) \in Z_+$$

and the isometries

$$W : L^2(\mathbb{R}^2) \longrightarrow \lambda^{\frac{1}{6}} L^2(Z), \quad W_+ : L^2(\mathbb{R}^3_+) \longrightarrow \lambda^{\frac{1}{2}} L^2(Z_+)$$

extending the maps defined for Schwartz function as

$$\begin{aligned} u_0 \ &\longmapsto \ W u_0(\lambda, x) = \sqrt{2\pi} \int u_0(\lambda^{-\frac{1}{3}} x, x_3) e^{-i\lambda x_3} dx_3, \\ u \ &\longmapsto \ W_+ u(\lambda, x, y) = 2\pi \int u(\lambda^{-\frac{1}{3}} x, \lambda^{-\frac{2}{3}} y, x_3) e^{-i\lambda x_3} dx_3. \end{aligned}$$

This corresponds to a non-homogeneous blow-up on the Fourier transfer side with

$$(\lambda^{\frac{1}{3}}, \xi, \eta), \quad \xi \ \text{ dual to } x, \ \eta \ \text{ dual to } y$$

giving the projective coordinate near the lift of $(0,0,1)$. We observe that because of the cut-off β in (7.23), there exists a unique operator

$$S : \mathcal{S}(Z) \longrightarrow C^\infty(Z_+)$$

such that $SW = W_+ T^\sharp$. The formula for $T^\sharp$ provides an explicit expression for S:

$$Sv(\lambda, x, y) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \psi(\lambda^{-\frac{2}{3}}\xi)\beta(\lambda)\frac{A_+(-\xi - y)}{A_+(-\xi)}\widehat{v}(\xi)e^{ix\xi}d\xi \tag{7.24}$$

and since our considerations are local in the original coordinates (x_1, x_2, x_3) we want to look at

$$S^\sharp v(\lambda, x, y) = \psi(\lambda^{-\frac{2}{3}}y)\psi(\lambda^{-\frac{1}{3}}x)Sv(\lambda, x, y). \tag{7.25}$$

The billiard ball maps for the model problem (7.21) have the following well known form

$$\delta_0^\pm(x_1, x_3; \xi_1, \xi_3) = \left(x_1 \pm 2\left(\frac{\xi_1}{\xi_3}\right)^{\frac{1}{2}}, x_3 \mp \frac{2}{3}\left(\frac{\xi_1}{\xi_3}\right)^{\frac{3}{2}}; \xi_1, \xi_3 \right).$$

In the new projective coordinates we obtain

$$\gamma^\pm(x,\xi) = (x \pm 2\xi^{\frac{1}{2}}, \xi), \quad \xi \geq 0, \quad x = \xi_3^{\frac{1}{3}} x_1, \quad \xi = \xi_3^{-\frac{1}{3}}\xi_1,$$

that is $\gamma^\pm \circ \mathcal{W} = \mathcal{W} \circ \delta_0^\pm$ near Γ_0, where $\mathcal{W}(x_1, x_3; \xi_1, \xi_3) = (\xi_3^{\frac{1}{3}} x_1, \xi_3^{-\frac{1}{3}}\xi_1)$. Similarly, the chosen square root of the billiard ball map, $\mathcal{C}_0 = \mathcal{C}_0^+ \cup \mathcal{C}_0^-$,

$$\begin{aligned}
\mathcal{C}_0^\pm \quad &= \quad \{((x_1, x_3; \xi_1, \xi_3), (y_1, y_3; \eta_1, \eta_3)) : x_1 - y_1 = \pm\left(\frac{\xi_1}{\xi_3}\right)^{\frac{1}{2}}, \qquad (7.26)\\
&(x_3 - y_3) = \mp\frac{1}{3}\left(\frac{\xi_1}{\xi_3}\right)^{\frac{3}{2}}, \quad \xi_i = \eta_i \geq 0\}
\end{aligned}$$

can be rewritten as

$$\mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-, \quad \mathcal{G}^\pm = \{((x,\xi),(x',\xi')) : \xi = \xi' \geq 0, x - x' = \pm\xi^{\frac{1}{2}}\},$$

in the sense that $\mathcal{G}^\pm \circ \mathcal{W} = \mathcal{W} \circ \mathcal{C}_0^\pm$. We also define (near Γ):

$$\mathcal{W}_+(x_1, x_2, x_3; \xi_1, \xi_2, \xi_3) = (\xi_3^{\frac{1}{3}} x_1, \xi_3^{\frac{2}{3}} x_2; \xi_3^{-\frac{1}{3}}\xi_1, \xi_3^{-\frac{2}{3}}\xi_2)$$

and $\mathcal{H}$ such that

$$\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-, \quad \mathcal{H}^\pm \circ \mathcal{W} = \mathcal{W}_+ \circ \mathcal{F}_0^\pm$$

with $\mathcal{F}_0^\pm \subset T^*\mathbb{R}^3 \backslash 0 \times T^*\mathbb{R}^2$ defined earlier and generated by $x_1\xi_1 + x_3\xi_3 \mp \frac{2}{3}(-\zeta(x,\xi))^{\frac{3}{2}}$. The relations $\mathcal{H}^\pm$ also take a very simple form

$$\mathcal{H}^\pm = \{((x,y;\xi,\eta);(x',\xi')) : \xi = \xi', \ x = x' \pm (\xi + y)^{\frac{1}{2}}, \ \eta = \mp(\xi + y)^{\frac{1}{2}}\}. \tag{7.27}$$

Let $\Lambda_m^\pm = N^*\{x_3 + \frac{1}{3m^2}x_1^3 = 0\} \cap \{\pm x_1 \geq 0\} \subset T^*\mathbb{R}^2\backslash 0, \ m > 00$, be the model boundary Lagrangians first introduced in (7.19):

$$\Lambda_m^\pm = (\mathcal{C}_0^\pm)^m \Lambda_0, \quad \Lambda_0 = N^*\{x_1 = x_3 = 0\}. \tag{7.28}$$

We then consider $\Xi_m^\pm \subset T^*\mathbb{R}$,

$$\Xi_m^\pm = \mathcal{W}(\Lambda_m^\pm) = \{(x,\xi) : \xi = m^{-2}x^2, \ \pm x \geq 0\}, \ m > 0, \ \Xi_0 = \{(0,\xi) : \xi \in \mathbb{R}\}. \tag{7.29}$$

If $\widetilde{\Lambda}_m = \mathcal{F}_0(\Lambda_m),\ \Lambda_m = \Lambda_m^+ \cup \Lambda_m^-$, then (7.27) immediately yields

$$\widetilde{\Xi}_0 = \mathcal{W}_+(\widetilde{\Lambda}_0) = \mathcal{H}(\Xi_0) = \{(x,y;\xi,\eta) : \eta^2 - y - \xi = 0,\ x + \eta = 0\}, \tag{7.30}$$
$$\widetilde{\Xi}_m = \mathcal{W}_+(\widetilde{\Lambda}_m) = \mathcal{H}(\Xi_m) = \{(x,y;\xi,\eta) : \eta^2 - y - \xi = 0,\ m^{-2}(x+\eta)^2 = \xi\},$$

when $m \neq 0$. Using this and the observation that, by homogeneity,

$$(3x_3\xi_3 + 2x_2\xi_2 + x_1\xi_1)\restriction_{\widetilde{\Lambda}_m} = 0$$

we obtain explicit expressions for $\widetilde{\Lambda}_m$ in terms of generators or as conormal bundles:

$$\begin{aligned}
\widetilde{\Lambda}_0 &= N^*(\{x_3 + \frac{1}{3}x_1^3 - x_1x_2 = 0\}) \\
\widetilde{\Lambda}_1 &= \mathrm{cl}[N^*(\{x_3x_1 + \frac{1}{12}(x_2 - x_1^2)^2 - \frac{1}{3}x_1(x_2 + x_1^2) = 0\} \setminus \{x_1 = x_2 = x_3 = 0\})] \\
\widetilde{\Lambda}_2 &= \mathrm{cl}[N^*(\{27x_3 + 9x_1x_2 + 5x_1^3)^2 - 16(3x_2 - x_1^2)^3 = 0\} \setminus \\
&\qquad\qquad \{3x_2 - x_1^2 = 27x_3 + 9x_1x_2 + 5x_1^3 = 0\})].
\end{aligned}$$

The Lagrangians (7.28), $m \leq 3$, are grouped in two pairs related by the billiard ball map:

$$\{\Lambda_0, \Lambda_2\}, \quad \{\Lambda_1, \Lambda_3\}, \quad \Lambda_2^\pm = \delta_0^\pm\Lambda_0, \quad \Lambda_3^\pm = \delta_0^\pm\Lambda_1,$$

and we study these separately as they cannot be simultaneous models – see (7.19) and the discussion following it.

The main result of this subsection is

Proposition 7.3. *The Dirichlet problem parametrix for* (7.21), $T_0^\sharp$, *given by* (7.23) *has the following mapping properties for k even:*

$$T_0^\sharp : I_kL_c^2(\mathbb{R}^2; \Lambda_0, J^1_{\Lambda_0\cap\Lambda_2}\Lambda_2, 2) + I_kL_c^2(\mathbb{R}^2; \Lambda_2^+, 2) \longrightarrow$$
$$I_k^bL^2_{\mathrm{loc}}(\mathbb{R}^3_+; \mathcal{F}_0(\Lambda_1), \mathcal{F}_0(\Lambda_1) \cap \mathcal{F}_0(\Lambda_0)),$$
$$T_0^\sharp : I_kL_c^2(\mathbb{R}^2, \Lambda_1^-, 2) + I_kL_c^2(\mathbb{R}^2, \Lambda_3^+, 2) \longrightarrow I_k^bL^2_{\mathrm{loc}}(\mathbb{R}^3_+; \mathcal{F}_0(\Lambda_2), \mathcal{F}_0(\Lambda_0) \cap \mathcal{F}_0(\Lambda_2)),$$

and

$$T_0^\sharp : I_kL_c^2(\mathbb{R}^2, \Lambda_1^+, 2) \longrightarrow I_k^bL^2_{\mathrm{loc}}(\mathbb{R}^3_+; \mathcal{F}_0(\Lambda_0) \sqcup \mathcal{F}_0(\Lambda_\sharp)),$$

where $\Lambda_\sharp \subset T^*\mathbb{R}^2 \setminus 0$ is any *C^∞ homogeneous Lagrangian tangent to Λ_1 at $\Lambda_1 \cap \Lambda_0$.*

We want to reduce the proof to an estimate for S defined by (7.24) and for that we need a characterization of the spaces above. Thus we define

$$I_kL^2(Z, \Xi_m^\pm) = \{v \in L^2(Z) : (\lambda D_\lambda)^{k_0}(D_x - m^{-2}x^2)^{k_1} \cdot$$
$$(x(D_x - m^{-2}x^2))^{k_2}(\chi(\mp x)x)^{k_3}v \in L^2, k_0 + k_2 + \frac{2}{3}k_1 + \frac{1}{3}k_3 \leq k\}$$

for k even, and by complex interpolation between the even indexed neighbours for k odd. Similarly (but for all $k \in \mathbb{N}_0$),

$$I_kL^2(Z, \Xi_0) = \{v \in L^2(Z) : (\lambda D_\lambda)^{k_0}(xD_x)^{k_1}x^{k_2}v \in L^2, k_0 + k_1 + \frac{1}{3}k_2 \leq k\}.$$

Definitions 4.3 and 4.4 easily give

Lemma 7.3. *If $WF(u) \subset \Gamma_0,\ u \in \mathcal{E}'(\mathbb{R}^2)$ then*

$$\begin{aligned} u &\in I_kL^2(\mathbb{R}^2; \Lambda_m^{\pm}, 2) \Longleftrightarrow Wu \in I_kL^2_{\nu_{1/6}}(Z, \Xi_m^{\pm}) \\ u &\in I_kL^2(\mathbb{R}^2; \Lambda_0, J^1_{\Lambda_0\cap\Lambda_1}\Lambda_2, 2) \Longleftrightarrow Wu \in I_kL^2_{\nu_{1/6}}(Z, \Xi_0), \end{aligned}$$

where $\nu_\alpha = \lambda^{-2\alpha} d\lambda dx$.

We proceed similarly for $\widetilde{\Lambda}_m$ by defining

$$I_kL^2(Z_+, \widetilde{\Xi}_0) = \{u \in L^2(Z_+) : (\lambda D_\lambda)^{k_0}(D_y^2 - y - D_x)^{k_1}(D_y + x)^{k_2} \cdot ((x + D_y)D_x)^{k_3} u \in L^2(Z_+),\ k_0 + \frac{2}{3}k_1 + \frac{1}{3}k_2 + k_3 \le k\},$$

$$I_kL^2(Z_+, \widetilde{\Xi}_m) = \{u \in L^2(Z_+) : (\lambda D_\lambda)^{k_0}(D_y^2 - y - D_x)^{k_1}(D_x - m^{-2}(x + D_y)^2)^{k_2} \cdot [(x + D_y)(D_x - m^{-2}(x + D_y)^2)]^{k_3} \in L^2(Z_+),\ k_0 + \frac{2}{3}k_1 + \frac{2}{3}k_2 + k_3 \le k\},$$

for k even and by complex interpolation for k odd. The analogue of Lemma 7.3 does not hold in full generality but the following lemma is precisely what we need. For the notational convenience it is stated, as the rest of this subsection, for $\mathbb{R}^3_+$ but the generalization to $\mathbb{R}^n_+$ is easy.

Lemma 7.4. *If for some $\chi \in C_0^\infty(\mathbb{R}^3)$, $\chi(0) = 1$, $WF_b(\chi u) \subset \jmath_*\Gamma$ and $P_0 u \in C^\infty(\bar{\mathbb{R}}^3_+)$, then for k even*

$$\begin{aligned} W_+(\varphi u) \in I_kL^2_{\nu^+_{\frac12}}(Z, \widetilde{\Xi}_m) \qquad &\text{for any } \varphi \in C_0^\infty(\bar{\mathbb{R}}^3_+) \Longrightarrow \\ &u \in I_k^bL^2_{\mathrm{loc}}(\mathbb{R}^3_+, \mathcal{F}_0(\Lambda_m), \mathcal{F}_0(\Lambda_m) \cap \mathcal{F}_0(\Lambda_0)), \quad m \neq 0, \\ W_+(\varphi u) \in I_kL^2_{\nu^+_{\frac12}}(Z, \widetilde{\Xi}_0) \qquad &\text{for any } \varphi \in C_0^\infty(\bar{\mathbb{R}}^3_+) \Longrightarrow \\ &u \in I_k^bL^2_{\mathrm{loc}}(\mathbb{R}^3_+, \mathcal{F}_0(\Lambda_0) \sqcup \mathcal{F}_0(\Lambda_\sharp)), \end{aligned}$$

$\nu^+_\alpha = \lambda^{-2\alpha} d\lambda dx dy$, and where $\Lambda_\sharp \subset T^*\mathbb{R}^2\backslash 0$ is any *C^∞ homogeneous Lagrangian tangent to Λ_1 at $\Lambda_0 \cap \Lambda_1$.*

Proof. We start with the case $m \neq 0$. Near $\jmath_*\Gamma$, ${}^b\mathcal{M}_1^\sharp(\mathcal{F}_0(\Lambda_m), \mathcal{F}_0(\Lambda_m) \cap \mathcal{F}(\Lambda_0))$ is generated by (see (7.29))

$$\begin{aligned} &p_0(x,\xi) = \xi_2^2 - x_2\xi_3^2 - \xi_3\xi_1, \quad a_1(x,\xi) = \sum_{j=1}^{3} j x_j \xi_j, \\ &a_2(x,\xi)^2, \quad a_3(x,\xi) = (x_1\xi_3 + \xi_2)a_2(x,\xi) \end{aligned} \tag{7.31}$$

where $a_2(x,\xi) = \xi_1\xi_3 - m^{-2}(x_1\xi_3 + \xi_2)^2$. We shall denote by P_0 and A_i the differential operators corresponding to p_0, a_i. Since $P_0 u \equiv 0$, it suffices to have

$$A_1^{\alpha_1} A_2^{\alpha_2} A_3^{\alpha_3} u \in \bar{H}^{\mathrm{loc}}_{(-\frac{3}{2}\alpha_2 - 2\alpha_3)}(\mathbb{R}^3_+), \quad \alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3 \le k = 2l. \tag{7.32}$$

In fact, $[P_0, A_j] = -4i\delta_{1j}P_0$ so that $P_0 A_1^{\alpha_1} A_2^{\alpha_2} A_3^{\alpha_3} u \equiv 0$. Thus, (7.32) implies that $A_1^{\alpha_1} A_2^{\alpha_2} A_3^{\alpha_3} u \in \bar{H}^{\mathrm{loc}}_{(\alpha_1 + 2\alpha_2 + 3\alpha_3, -\alpha_1 - \frac{7}{2}\alpha_2 - 5\alpha_3)}(\mathbb{R}^3_+)$ which in turn shows that $BA_1^{\alpha_1} A_2^{\alpha_2} A_3^{\alpha_3} u$

$\in L^2_{\mathrm{loc}}(\mathbb{R}^3_+)$ for $B \in \Psi_b^{-\frac{3}{2}\alpha_2-2\alpha_3,\alpha_1+2\alpha_1+3\alpha_3}(\mathbb{R}^3_+)$ and gives the defining condition for $I^b_k L^2_{\mathrm{loc}}(\mathbb{R}^3_+, \mathcal{F}_0(\Lambda_m), \mathcal{F}_0(\Lambda_m)\cap\mathcal{F}_0(\Lambda_0))$:

$$(B_1A_1)^{\alpha_1}(B_2A_2)^{\alpha_2}(B_3A_3)^{\alpha_3}u \in L^2_{\mathrm{loc}}(\mathbb{R}^3_+), \quad \alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3 \le k,$$

where $B_1 \in \Psi_b^0(\mathbb{R}^3_+),\ B_2 \in \Psi_b^{-\frac{3}{2}}(\mathbb{R}^3_+),\ B_3 \in \Psi_b^{-2}(\mathbb{R}^3_+)$. On the other hand the definition of $I_k L^2_{\nu^+_{\frac{1}{2}}}(Z_+, \tilde{\tilde{\Xi}}_m)$ and the assumptions on u imply that

$$A_1^{\beta_1}A_2^{\beta_2}A_3^{\beta_3}u \in \bar{H}^{\mathrm{loc}}_{(0,-\frac{4}{3}\beta_2-2\beta_3)}(\mathbb{R}^3_+), \quad \beta_1 + \frac{2}{3}\beta_2 + \beta_3 \le k = 2l. \tag{7.33}$$

Let us observe that α_2 in (7.32) can be assumed to be even and thus we have two cases

$$\alpha_2 = 4m + p, \quad p = 0 \quad \text{or} \quad p = 2.$$

If $p = 0$ then

$$A_1^{\alpha_1}A_2^{\alpha_2}A_3^{\alpha_3}u = A_1^{\alpha_1}A_2^{3m}A_2^{m}A_3^{\alpha_3}u \in \bar{H}^{\mathrm{loc}}_{(-2m,-4m-2\alpha_3)}(\mathbb{R}^3_+) \subset \bar{H}^{\mathrm{loc}}_{(-\frac{3}{2}\alpha_2-2\alpha_3)}(\mathbb{R}^3_+)$$

by (7.33) with $\beta_i = \alpha_i, i = 1,3$ and $\beta_2 = 3m$. Thus it remains to analyse the case $p = 2$ which is more involved and which by the above ($p = 0$) argument reduces to $\alpha_2 = 2$, $k = 2$ in (7.32). Let us note the following identities:

$$A_2 = D_{x_1}D_{x_3} - m^{-2}A_0^2,\ A_3 = A_0A_2,\ a_0(x,\xi) = x_1\xi_3 + \xi_2,\ a_0 = \sigma_1(A_0). \tag{7.34}$$

The desired property (7.32) reduces to

$$A_iA_2^2u \in \bar{H}^{\mathrm{loc}}_{(-3-2\delta_{3i})}(\mathbb{R}^3_+), \quad i = 1,3, \tag{7.35}$$

which for $i = 3$ follows by writing $A_3A_2^2u = A_0A_2^3u$ with $A_2^3u \in \bar{H}^{\mathrm{loc}}_{(0,-4)}$ by (7.33). Thus in view of the assumption $WF_b(u) \subset \jmath_*\Gamma$, it remains to establish

$$\|\langle D_{x_3}\rangle^{-3}\chi_0A_1A_2^2u\|^2_{L^2(\mathbb{R}^3_+)} \le C \sum_{\beta_1+\frac{2}{3}\beta_2+\beta_3\le 2} \|\langle D_{x_3}\rangle^{-\frac{4}{3}\beta_2-2\beta_3}\chi_1A_1^{\beta_1}A_2^{\beta_2}A_3^{\beta_3}u\|^2_{L^2(\mathbb{R}^3_+)}, \tag{7.36}$$

where $\chi_i(x,D) \in \Psi^0(\partial\mathbb{R}^3_+)$, $\operatorname{supp}\chi_i$, $i = 0,1$ is in a neighbourhood of Γ_0 (which is a conic neighbourhood of $(0;(0,0,1))$ and $\chi_1 \equiv 1$ on $\operatorname{supp}\chi_0$. We will denote the right hand side in (7.36) by $M(u)$. We first obtain an *a priori* inequality in which we assume $u \in C^\infty(\mathbb{R}^3_+)$ and start by using (7.34) to rewrite the left hand side of (7.36) as

$$\langle D_{x_3}\rangle^{-3}\langle\chi_0A_1A_2^2u, \langle D_{x_3}\rangle^{-3}\chi_0A_1A_2D_{x_1}D_{x_3}u - m^{-2}\langle D_{x_3}\rangle^{-3}\chi_0A_1A_2A_0^2u\rangle$$

which modulo commutator terms bounded by $M(Bu)$, $B \in \Psi_b^{0,K}(\mathbb{R}^3_+)$ for some K (see the proof of Proposition 4.5), is equal to

$$\begin{aligned}
&\langle\langle D_{x_3}\rangle^{-3}\chi_0D_{x_1}D_{x_3}A_1A_2^2u, \langle D_{x_3}\rangle^{-3}\chi_0A_1A_2u\rangle - \\
&m^{-2}\langle\langle D_{x_3}\rangle^{-3}\chi_0A_1A_2^2u, \langle D_{x_3}\rangle^{-3}\chi_0A_1A_3A_0u\rangle \equiv \\
&\langle\langle D_{x_3}\rangle^{-3}\chi_0A_2^3u, \langle D_{x_3}\rangle^{-3}\chi_0A_1^2A_2u\rangle + \\
&m^{-2}\langle\langle D_{x_3}\rangle^{-3}\chi_0A_0A_3A_1A_2u, \langle D_{x_3}\rangle^{-3}\chi_0A_1A_2u\rangle - \\
&m^{-2}\langle\langle D_{x_3}\rangle^{-3}\chi_0A_1A_2^2u, \langle D_{x_3}\rangle^{-3}\chi_0A_1A_3A_0u\rangle.
\end{aligned}$$

The integrations by parts without boundary contributions were allowed as we used only the tangential derivatives and $x_2 D_{x_2}$ in A_1 which vanishes at the boundary. Thus the following estimate holds

$$\begin{aligned}\|\langle D_{x_3}\rangle^{-3}\chi_0 A_1 A_2^2 u\| \;\le\; & \sum_{j=1}^{J} M(B_j u) + \|\langle D_{x_3}\rangle^{-4}\chi_0 A_2^3 u\|\|\langle D_{x_3}\rangle^{-2}\chi_0 A_1^2 A_2 u\| + \\ & \|\langle D_{x_3}\rangle^{-4}\chi_0 A_0 A_3 A_1 A_2 u\|\|\langle D_{x_3}\rangle^{-2}\chi_0 A_1 A_2 u\| + \\ & \|\langle D_{x_3}\rangle^{-3}\chi_0 A_1 A_2^2 u\|\|\langle D_{x_3}\rangle^{-3}\chi_0 A_1 A_3 A_0 u\|,\end{aligned}$$

where the norms are in $L^2(\mathbb{R}^3_+)$. An approximation argument gives

$$\|\langle D_{x_3}\rangle^{-3}\chi_0 A_1 A_2^2 u\| \le 2\sum_{j=1}^{\tilde{J}} M(B_j u),$$

for any u satisfying (7.33) with $k=2$. As in the proof of sufficiency of (7.32) $P_0 u \equiv 0$ implies now that $M(Bu) \le CM(u)$ for $B \in \Psi_b^{0,K}(\mathbb{R}^3_+)$. In fact, commuting B through gives

$$M(Bu) \le C \sum_{\beta_1+\frac{2}{3}\beta_2+\beta_3\le 2} \sum_{j=1}^{J_\beta} \|B_j^\beta A_1^{\beta_1} A_2^{\beta_2} A_3^{\beta_3} u\|^2,$$

where $B_j^\beta \in \Psi_b^{-\frac{4}{3}\beta_2-3\beta_3, K+\beta_1+2\beta_2+3\beta_3}(\mathbb{R}^3_+)$ and hence we need to show that for $E \in \Psi_b^{m,k}(\mathbb{R}^3_+)$ and v such that $P_0 v \in C^\infty$ near the boundary,

$$\|Ev\| \le C\|v\|_{(0,m)} \quad \text{for} \quad m \le 0.$$

This in turn follows from $\|Ev\| \le C\|v\|_{(k,m-k)}$, $m-k \le 0$, and Theorem B.2.9 of [14]. Thus we obtain (7.35), $i=1$, which concludes the proof of (7.32) and of the lemma for $m>0$.

For the case $m=0$ we need to describe the generators of ${}^b\mathcal{M}_1^\sharp(\mathcal{F}_0(\Lambda_0)\sqcup\mathcal{F}_0(\Lambda_\sharp))$ near $\jmath_*\Gamma$. The Lagrangian $\mathcal{F}(\Lambda_0) = \widetilde{\Lambda}_0$ is given near Γ, a conic neighbourhood of $(0;(0,0,1))$ by the zeros of p_0, a_0, a_1 defined in (7.31) and (7.34). Since $\Lambda_\sharp$ is tangent to Λ_1 at $\Lambda_0\cap\Lambda_1$, it follows that $\Lambda_\sharp = N^*\{x_3 + x_1^3 f(x_1) = 0\}, f \in C^\infty$. Thus, a computation based on this and the definition of $\mathcal{F}_0$ shows that, near Γ, $\mathcal{F}_0(\Lambda_\sharp)$ is given by the zeros of $p_0, a_1^\sharp, a_2^\sharp$ where

$$a_1^\sharp(x,\xi) = a_1(x,\xi) + \xi_3^{-2} a_0(x,\xi)^3 h_1(x,\xi), \quad a_2^\sharp(x,\xi) = \xi_1\xi_3 - a_0(x,\xi)^2 h_2(x,\xi), \tag{7.37}$$

with

$$\begin{aligned} h_1(x,\xi) &= \xi_3^{-1} a_0(x,\xi) f'(x_1 + \xi_3^{-1}\xi_2)/3 \\ h_2(x,\xi) &= f(x_1 + \xi_3^{-1}\xi_2) + \xi_3^{-1} a_0(x,\xi) f'(x_1+\xi_3^{-1}\xi_2). \end{aligned}$$

To obtain the generators of ${}^b\mathcal{M}_1^\sharp(\mathcal{F}_0(\Lambda_\sharp))$ near $\jmath_*\Gamma$ we write

$$h_i(x,\xi) = h_i^e(x,\xi_1,\xi_3(\xi_2/\xi_3)^2,\xi_3) + (\xi_2/\xi_3) h_i^o(x,\xi_1,\xi_3(\xi_2/\xi_3)^2,\xi_3),$$

so that using p_0, h_i in $a_i^\sharp$ can be replaced by

$$h_i^\sharp(x,\xi) = h_i^e(x;\xi_1,\xi_3 x_1+\xi_1,\xi_3) + \xi_2\xi_3^{-1} h_i^o(x;\xi_1,\xi_3 x_1+\xi_1,\xi_3)$$

which near $\jmath_*\Gamma$ are in $S^{0,1}_{\mathrm{hg}}({}^bT^*\bar{\mathbb{R}}^3_+)$, so that the corresponding $a_1^\sharp \in S^{1,4}_{\mathrm{hg}}({}^bT^*\bar{\mathbb{R}}^3_+)$ and $a_2^\sharp \in S^{2,3}_{\mathrm{hg}}({}^bT^*\bar{\mathbb{R}}^3_+)$.

We observe that we can find operators $H_i \in \Psi_b^{0,1}(\mathbb{R}^3_+)$ such that ${}^b\sigma_{0,1}(H_i) = h_i^\sharp$ in $\jmath\Gamma$ and

$$[P_0, H_i] = D_{x_2}^2 B^1_{-1} + B^2_{-1} D^2_{x_2} + B^2_0 D_{x_2} + B^1_1, \quad B^j_i \in \Psi^i_b(\mathbb{R}^3_+),\ WF_b(B^j_i) \cap \jmath\Gamma = \emptyset, \tag{7.38}$$

where $WF_b(B)$ denotes the essential support of B as a b-pseudo-differential operator. The wave front set conditions on u and B imply that $Bu \in \dot{\mathcal{A}}(\mathbb{R}^3_+)$. Since $u \in \mathcal{N}(\mathbb{R}^3_+)$ it then follows that $Bu \in C^\infty(\bar{\mathbb{R}}^3_+)$ as $B : \mathcal{N}(\mathbb{R}^3_+) \subset \mathcal{A}'(\mathbb{R}^3_+) \to \mathcal{A}'(\mathbb{R}^3_+)$ and $\mathcal{A}'(\mathbb{R}^3_+) \cap \dot{\mathcal{A}}(\mathbb{R}^3_+) = C^\infty(\bar{\mathbb{R}}^3_+)$ (see the references given above and also Subsection 7.3 below). Hence $[P_0, H_i]u \in C^\infty(\bar{\mathbb{R}}^3_+)$ and we can commute P_0 through as in the proof of (7.32).

From (7.37) we obtain the generators of ${}^b\mathcal{M}_1^\sharp(\mathcal{F}_0(\Lambda_0) \sqcup \mathcal{F}_0(\Lambda_\sharp))$ in a neighbourhood of $\jmath_*\Gamma$:

$$\xi_3^{-1} p_0, \quad a_1^\sharp, \quad a_2^\sharp a_0.$$

The assumption $W_+ u \in I_k L^2_{\nu_{\frac{1}{2}}}(Z_+, \widetilde{\Xi}_0)$ implies

$$A_0^{\beta_0} A_1^{\beta_1} (D_{x_1} A_0)^{\beta_2} u \in \bar{H}^{\mathrm{loc}}_{(0, -\frac{2}{3}\beta_0 - \beta_2)}, \quad \frac{1}{3}\beta_0 + \beta_1 + \beta_2 \le k. \tag{7.39}$$

Thus, as in the case of (7.32), it suffices to prove that (7.39) implies

$$(A_1^\sharp)^{\alpha_1} (A_2^\sharp A_0)^{\alpha_2} u \in \bar{H}^{\mathrm{loc}}_{(-m, -2\alpha_2 + m)}, \quad {}^b\sigma_{1,4}(A_1^\sharp) = a_1^\sharp,\ {}^b\sigma_{2,3}(A_2^\sharp) = a_2^\sharp, \tag{7.40}$$

for some m. However, (7.37) shows that

$$A_1^\sharp \equiv \langle D_{x_3} \rangle^{-2} (D^2_{x_3} A_1 + A_0^3 H_1), \quad A_2^\sharp A_0 \equiv D_{x_3} D_{x_1} A_0 - A_0^3 H_2, \quad H_i \in \Psi_b^{0,1}(\bar{\mathbb{R}}^3_+).$$

and thus (7.40) follows from (7.39). □

The proof of Proposition 7.3 reduces to the proof of

Proposition 7.4. *If $S^\sharp$ is defined by (7.25) then for k even*

$$S^\sharp : I_k L^2_{\nu_\alpha}(Z; \Xi^\pm_m) \longrightarrow I_k L^2_{\nu^+_{\alpha+\frac{1}{3}}}(Z_+; \widetilde{\Xi}_{m\pm 1}) \tag{7.41}$$

$$S^\sharp : I_k L^2_{\nu_\alpha}(Z; \Xi_0) \longrightarrow I_k L^2_{\nu^+_{\alpha+\frac{1}{3}}}(Z_+; \widetilde{\Xi}_1) \tag{7.42}$$

where $\nu_\alpha = \lambda^{-2\alpha} d\lambda dx$, $\nu^+_\beta = \lambda^{-2\beta} d\lambda dx dy$.

We start with the second mapping property where we can give a direct proof.

Proof of (7.42). Since λ does not appear in S other than in the cut-off function ψ, the stability under λD_λ is clear and for simplicity of notation we shall omit that variable. Thus we consider

$$\widetilde{S}\widehat{v}(\xi, y) = \int Sv(x, y) e^{-ix\xi} dx, \quad \widetilde{S}\widehat{v}(\xi, y) = \psi(\lambda^{-\frac{2}{3}}\xi) \frac{A_+(-\xi - y)}{A_+(-\xi)} \widehat{v}(\xi)$$

and recalling the definition of $I_k L^2_{\nu_+}(Z_+, \widetilde{\Xi}_1)$ we want to apply to it the operators

$$\xi - (D_y - D_\xi)^2, \qquad (\xi - (D_y - D_\xi)^2)(D_y - D_\xi)$$

with weights $\frac{2}{3}$ and 1, respectively, while the stability under $D_y^2 - y - \xi$ is clear from the Airy equation. We claim that

$$((\xi - (D_y - D_\xi)^2)(D_y - D_\xi))^{k_2}(\xi - (D_y - D_\xi)^2)^{k_1}\widetilde{S}\widehat{v}(\xi, y) \equiv \widetilde{S}\widehat{v}_{k_1,k_2}(\xi, y),$$

where $v_{k_1,k_2} \in L^2_\nu(Z)$ if $v \in I_k L^2_\nu(Z, \Xi_0)$ and $\frac{2}{3}k_1 + k_2 \leq k$ and where we omitted the irrelevant terms with differentiation falling on ψ. In fact, we can proceed by induction, noting that the order in the iteration does not matter: $(\xi - (D_y - D_\xi)^2)\widetilde{S}\widehat{v}_{k_1-1,0}(\xi, y) \equiv \widetilde{S}\widehat{v}_{k_1 0}(\xi, y)$, where by a simple computation

$$\widehat{v}_{k_1 0}(\xi) = 2(D\Phi_+)(-\xi)\widehat{v}_{k_1-1,0}(\xi) - 2\Phi_+(-\xi)D_\xi\widehat{v}_{k_1-1,0}(\xi) - D_\xi^2\widehat{v}_{k_1-1,0}(\xi). \tag{7.43}$$

Here

$$\Phi_+(t) = \frac{1}{i}\frac{A_+'(t)}{A_+(t)}, \qquad -D_t\Phi_+(t) = \Phi_+(t)^2 + t, \qquad \Phi_+ \in S^{\frac{1}{2}}(\mathbb{R}).$$

Thus, to obtain the boundedness of the second term in (7.43), we need the stability of $\widehat{v}_{k_1-1,0}(\xi)$, $\frac{2}{3}k_1 \leq k$ under $\langle\xi\rangle^{\frac{1}{2}}D_\xi$ with weight $\frac{2}{3}$. This follows easily from the stability under ξD_ξ with weight 1, D_ξ with weight $1/3$ and an interpolation argument (see the proof of Lemma 7.5).

We now turn to $(D_y - D_\xi)(\xi - (D_y - D_\xi)^2)\widetilde{S}\widehat{v}_{k_1,k_2-1}(\xi, y) \equiv \widetilde{S}\widehat{v}_{k_1 k_2}(\xi, y)$ where now

$$\begin{aligned}\widehat{v}_{k_1 k_2}(\xi) &= -2D^2\Phi_+(-\xi)\widehat{v}_{k_1,k_2-1}(\xi) + 4D\Phi_+(-\xi)D_\xi\widehat{v}_{k_1,k_2-1}(\xi) \\ &\quad -2\Phi_+(-\xi)D_\xi^2\widehat{v}_{k_1,k_2-1}(\xi) - D_\xi^3\widehat{v}_{k_1,k_2-1}(\xi),\end{aligned}$$

and we use the stability of $\widehat{v}$ under $\langle\xi\rangle^{\frac{1}{2}}D_\xi^2$ with weight 1. The proof is concluded by observing that $\psi(\lambda^{-\frac{2}{3}}y)\widetilde{S} = \mathcal{O}(1) : \lambda^{\frac{1}{6}}L^2(\mathbb{R}_\xi) \longrightarrow \lambda^{\frac{1}{2}}L^2(\mathbb{R}_\xi \times \mathbb{R}_{+y})$. $\square$

The presence of Lagrangians with boundaries in (7.41) presents difficulties which seem to prevent a computational proof similar to the one above, and the following characterization of the partial Fourier transform of $I_k L^2(Z, \Xi_m^\pm)$ will allow us to overcome them. We need yet another two spaces of functions:

$$S_k^\alpha L^2(Z) = \{a \in \lambda^\alpha L^2(Z) : (\lambda D_\lambda)^{k_0}(\xi D_\xi)^{k_1}D_\xi^{k_2}a(\lambda, \xi) \in \lambda^\alpha L^2(Z), \quad k_0 + k_1 + \frac{1}{3}k_2 \leq k\}$$

and

$$\mathcal{S}_k^\alpha L^2(Z) = \{a \in \lambda^\alpha L^2(Z) : (\lambda D_\lambda)^{k_0}\xi^{k_1}D_\xi^{k_2}a \in \lambda^\alpha L^2(Z), \quad k_0 + \frac{2}{3}k_1 + \frac{1}{3}k_1 \leq k\}$$

with the latter space defined only for even k and then for odd k by complex interpolation between $k-1$ and $k+1$.

Using this we now have the crucial

Lemma 7.5. *Let $\widehat{v}(\lambda, \xi)$ be the Fourier transform of $v(\lambda, x)$ in the second variable. Then for k even*

$$v \in I_k L^2_{\nu_\alpha}(Z; \Xi_m^\pm) \Longleftrightarrow \widehat{v}(\lambda, \xi) = g(\lambda, \xi) + e^{\mp\frac{2}{3}im\xi^{\frac{3}{2}}}f(\lambda, \xi) \tag{7.44}$$

where $f \in S_k^\alpha L^2(Z)$, $\operatorname{supp} f \subset \{\xi \geq 1\}$ *and*

$$g \in \begin{cases} S_k^\alpha L^2(Z) & m = 0 \\ \mathcal{S}_k^\alpha L^2(Z) & m > 0, \end{cases}$$

$\Xi_0^\pm \equiv \Xi_0, \quad \nu_\alpha = \lambda^{-2\alpha}d\lambda dx.$

Proof. We observe that the case $m = 0$ follows immediately from the definition and we shall first prove that the left hand side in (7.44) implies the right hand side, that is, we assume that $v \in I_k L^2_{\nu_\alpha}(Z; \Xi^{\pm}_m)$ for $k = 2l$ and $m > 0$. To simplify the notation we will allow $m \in \mathbb{Z}$ and define $\Xi_m \stackrel{\text{def}}{=} \Xi^{\text{sgn}(m)}_{|m|}$, and also put $\alpha = \frac{1}{6}$. Thus we take $v \in I_{2l} L^2_{\nu_{\frac{1}{6}}}(Z, \Xi_,)$.

Let $\chi \in C^\infty(\mathbb{R})$, supp $\chi \subset [1, \infty)$ be such that $\chi \equiv 1$ for $x > 2$ and let us define

$$v_+(x) = \chi(\text{sgn}(m)x)v(x), \quad v_- = v - v_+$$

The term v_- satisfies the estimates $(\lambda D_\lambda)^{k_0} x^{k_1} D_x^{k_2} v_- \in \lambda^{1/6} L^2(Z)$ for $k_0 + \frac{1}{3}k_1 + \frac{2}{3}k_2 \leq 2l$, which immediately implies that $\hat{v}_- \in \mathcal{S}^{1/6}_{2l} L^2(Z)$.

We can easily construct a canonical transformation $\chi_m : T^*\mathbb{R}^2 \setminus 0 \longrightarrow T^*\mathbb{R}^2 \setminus 0$ such that $\chi_m : N^*\{x_1 = 0\} \longrightarrow \Lambda_m$:

$$\chi_m : (x_1, x_2; \xi_1, \xi_2) \longmapsto (x_1 - \frac{1}{3m^2}x_2^3, x_2; \xi_1, \xi_2 + \frac{1}{m^2}x_2^2\xi_1) = (y_1, y_2; \eta_1, \eta_2) \tag{7.45}$$

which is generated by $\phi_m = \frac{1}{3m^2}y_2^3\xi_1 + y_2\xi_2 + y_1\xi_1$. The definition of $I_k L^2_{\nu_\alpha}(Z; \Xi^{\pm}_m)$ then shows that (see Lemma 7.3)

$$v(\lambda, x) = e^{\frac{1}{3}im^{-2}x^3} W u_1(\lambda, x), \quad u_1 \in I_k L^2(\mathbb{R}^2; N^*\{x_1 = 0\} \cap \{\pm x_2 \geq 0\}, 2).$$

Consequently, if $w(\lambda, x) = e^{-\frac{1}{3m^2}ix^3} v_+(\lambda, x)$ then since $\hat{v}_- \in \mathcal{S}^{1/6}_{2l} L^2(Z)$,

$$(\lambda D_\lambda)^{k_0} (x D_x)^{k_1} D_x^{k_2} w \in \lambda^{\frac{1}{6}} L^2(Z), \quad k_0 + k_1 + \frac{2}{3}k_2 \leq 2l, \text{ supp } w \subset \text{sgn}(m)[1, \infty). \tag{7.46}$$

Strictly speaking we should now replace $w \in \lambda^{1/6} L^2(Z)$ by a sequence $w_j \in \lambda^{1/6} L^2([1, \infty); \mathcal{S}(\mathbb{R}))$, $w_j \to w$ with the estimate (7.46) satisfied uniformly, but for simplicity we shall write w everywhere. With this understanding we have

$$\widehat{v}_+(\lambda, \xi) = \int e^{\frac{1}{3}im^{-2}x^3 - ix\xi} w(\lambda, x) dx \tag{7.47}$$

and we first consider $\xi > 1$ where we define $f(\lambda, \xi) = \chi(\xi)\exp(\frac{2}{3}im\xi^{\frac{3}{2}})\widehat{v}_+(\lambda, \xi)$. We will check the stability under D_ξ and ξD_ξ (λD_λ is clear) by induction for more general f's of the form

$$f(\lambda, \xi) = \chi(\xi) \int \frac{\xi^{\frac{1}{2}q} x^k}{\xi^{\frac{1}{2}p}(m\xi^{\frac{1}{2}} + x)^r} e^{i(\frac{2}{3}m\xi^{\frac{3}{2}} + \frac{1}{3m^2}x^3 - x\xi)} w(\lambda, x) dx, \tag{7.48}$$

where supp sgn$(m)\,[1, \infty)$, $w \in \lambda^{\frac{1}{6}} L^2([1, \infty), H_{(-s)}(\mathbb{R}))$, and $q + k + 2s \leq r$. We can rewrite this as

$$\mathcal{F}^{-1}(e^{-i\frac{2}{3}m\xi^{\frac{3}{2}}} f)(\lambda, x) = a(x, D_x)^* \left(e^{i\frac{1}{3m^2}(\cdot)^3} w(\lambda, \cdot)\right)$$

where $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier transform and

$$a(x, \xi) = \chi(2\text{sgn}(m)x)\chi(\xi)\frac{x^k \xi^{\frac{1}{2}q}}{(m\xi^{\frac{1}{2}} + x)^r \xi^{\frac{1}{2}p}} \in \langle m|\xi|^{\frac{1}{2}} + |x|\rangle^{-r+k} S^{\frac{1}{2}(q-p)}(\mathbb{R}; \mathbb{R}). \tag{7.49}$$

To see that $f \in \lambda^{\frac{1}{6}} L^2(Z)$ if $w \in \lambda^{\frac{1}{6}} L^2([1,\infty), H_{(-s)}(\mathbb{R}))$, it suffices to check that

$$a \in \langle m|\xi|^{\frac{1}{2}} + |x|\rangle^{-2s} S^0_{1,0}(\mathbb{R},\mathbb{R}) \Longrightarrow e^{-i\frac{1}{3m^2}x^3} a(x,D) : L^2(\mathbb{R}) \longrightarrow H_{(s)}(\mathbb{R}),\ s \geq 0.$$

Since this is clear for $s = 0$ it is enough to establish the mapping property for $s \in \mathbb{N}$ as it will then follow by interpolation. Thus let $s = n$ and we need to verify that $D_x^n(\exp(-i\frac{1}{3m^2}x^3)u) \in L^2(\mathbb{R})$. This however is easy as $|x|^l a \in S_{1,0}^{-n+l/2}$ so that $x^l a(x,D) : L^2(\mathbb{R}) \to H_{(n-l/2)}(\mathbb{R})$.

We will show, using (7.46), that the applications of D_ξ and ξD_ξ do not change the form of f. To do that we first need to strengthen (7.46) slightly:

$$(\lambda D_\lambda)^{k_0}(xD_x)^{k_1} D_x^{k_2} w \in \lambda^{\frac{1}{6}} L^2([1,\infty); H_{(-s)}(\mathbb{R})),\ k_0 + k_1 + \frac{2}{3}k_2 \leq 2l + \frac{2}{3}s, k_1 + k_0 \leq 2l. \tag{7.50}$$

The optimal choice of s is in $\frac{1}{2}\mathbb{N}_0$ and for $s \in \mathcal{N}_0$, (7.50) is clear. For $s \in \mathbb{N}_0 + \frac{1}{2}$ it is derived from the iteration of the following statement

$$\begin{aligned} &V^i u, D_x^j u \in \lambda^{\frac{1}{6}} L^2(Z),\ i \leq 2, j \leq 3 \Longrightarrow \\ &D_x^2 V u \in \lambda^{\frac{1}{6}} L^2([1,\infty); H_{(-\frac{1}{2})}(\mathbb{R})),\ V = xD_x, \lambda D_\lambda. \end{aligned} \tag{7.51}$$

To see this we take the Fourier transform in x and consider $\langle \widehat{V^2 u}, \widehat{D_x^3 u}\rangle$. An integration by parts in ξ or λ depending on V (in the case of λ this means taking an adjoint in of D_λ) and an application of the Cauchy-Schwartz inequality yields $\xi^{\frac{2}{3}}\widehat{Vu} \in \lambda^{\frac{1}{6}} L^2(Z)$ which gives the right hand side of (7.52).

We can now start with

$$\begin{aligned} D_\xi f(\lambda,\xi) &= \int (m\xi^{\frac{1}{2}} - x) e^{i(\frac{2}{3}m\xi^{\frac{3}{2}} + \frac{1}{3m^2}x^3 - x\xi)} a(x,\xi) w(\lambda,x) dx + \\ &\quad \int (D_\xi a)(x,\xi) e^{i(\frac{2}{3}m\xi^{\frac{3}{2}} + \frac{1}{3m^2}x^3 - x\xi)} w(\lambda,x) dx. \end{aligned}$$

The second term is of the desired form, as $D_\xi a$ is of the form (7.49). We rewrite the first one as

$$m^2 \int (m\xi^{\frac{1}{2}} + x)^{-1} a(x,\xi)[(-D_x) e^{i(\frac{2}{3}m\xi^{\frac{3}{2}} + \frac{1}{3m^2}x^3 - x\xi)}] w(\lambda,x) dx$$

where we can integrate by parts introducing new a's of the form (7.49) and $D_x w$. By an interpolation argument (see for instance the proof of (7.52) above) it suffices to discuss $D_\xi^2 f$, D_ξ^2 with weight $\frac{2}{3}$. Repeating the previous computation we see that D_ξ^2 is a sum of terms of the same form as f with w replaced by $D_x w$ or $D_x^2 w$ with r in a (see (7.49)) increased by 1 or 2 respectively. If $w \in \lambda^{\frac{1}{6}} L^2([1,\infty); H_{(-s)})$ then (7.50) shows that $D_x w \in \lambda^{\frac{1}{6}} L^2([1,\infty); H_{(-s)})$ and $D_x^2 w \in \lambda^{\frac{1}{6}} L^2([1,\infty); H_{(-s-1)})$, where we increased k_2 by 1 in agreement with teh weight of D_ξ^2. Hence, indeed $D_\xi^2 f$ is a sum of terms of the form (7.48) and $D_\xi^2 f \in \lambda^{\frac{1}{6}} L^2(Z)$.

We proceed similarly for ξD_ξ. Observing that $\xi D_\xi a \in S^0(\mathbb{R},\mathbb{R})$ and $\xi e^{i(\frac{1}{3m^2}x^3 - x\xi)} = (-D_x + m^{-2}x^2)e^{i(\frac{1}{3m^2}x^3-x\xi)}$, we have

$$\xi D_\xi f(\lambda,\xi) = \int \xi D_\xi a(x,\xi) w(\lambda,x) e^{i(\frac{2}{3}m\xi^{\frac{3}{2}} + \frac{1}{3m^2}x^3 - x\xi)} dx +$$

$$m^2 \int (m\xi^{\frac{1}{2}} + x)^{-1} \left[(-D_x)(-D_x + m^{-2}x^2) e^{i(\frac{2}{3}m\xi^{\frac{3}{2}} + \frac{1}{3m^2}x^3 - x\xi)} a(x,\xi)\right] w(\lambda,x) dx.$$

and again the first term is harmless. Integration by parts in the second term produces terms with new a's satisfying (7.49) and the following terms involving differentiation of w.

$$m^2 \int \left[\left(\frac{2i}{(m\xi^{\frac{1}{2}}+x)^2} D_x w(\lambda,x) + \frac{x}{m^2(m\xi^{\frac{1}{2}}+x)}(xD_x)w(\lambda,x) + \right.\right.$$

$$\left.\left.\frac{1}{(m\xi^{\frac{1}{2}}+x)} D_x^2 w(\lambda,x)\right) a(x,\xi) + \frac{2D_x a(x,\xi)}{m\xi^{\frac{1}{2}}+x} D_x w(\lambda,x)\right] e^{i(\frac{2}{3}m\xi^{\frac{3}{2}} + \frac{1}{3m^2}x^3 - x\xi)} dx.$$

All the terms except for the third one are of the desired form (since we used the generators in (7.46) with weights ≤ 1). To maintain the order of regularity we use (7.50) again so that if $w \in \lambda^{\frac{1}{6}} L^2([1,\infty), H_{(-s)}(\mathbb{R}))$ then $D_x^2 w \in \lambda^{\frac{1}{6}} L^2([1,\infty), H_{(-s-\frac{1}{2})})$. This decreases $2l$ by 1 in (7.50) so the order in the filtration is preserved. Since r is increased by 1 , (7.48) is preserved and $\xi D_\xi f \in \lambda^{\frac{1}{6}} L^2(Z)$.

To see that $(1-\chi(\xi))\widehat{v}_+(\lambda,\xi) \in \mathcal{S}^\alpha_k L^2(Z)$ we could again analyse the integral (7.47) but it also follows from the uniform ellipticity of $D_\xi^2 - m^2\xi$ for $\xi < 0$. In fact, if

$$\text{supp } b \subset (-\infty, 1), \quad (\lambda D_\lambda)^{k_0}(D_\xi^2 - m^2\xi)^{k_1} b(\lambda,\xi) \in \lambda^\alpha L^2(Z), \quad k_0 + \frac{2}{3}k_1 \leq 2l$$

then $b \in \mathcal{S}^\alpha_{2l} L^2(Z)$ since

$$\langle (D_\xi^2 - m^2\xi)b, b\rangle_{L^2(\mathbb{R}_x)} \geq \|D_\xi b\|^2_{L^2(\mathbb{R}_x)} + m^2 \||\xi|^{\frac{1}{2}} b\|^2_{L^2(\mathbb{R}_x)} - C\|w\|^2_{L^2(\mathbb{R}_x)}.$$

Note that using an interpolation argument as in the proof of (7.52) we obtain the correct weights for the operators defining $\mathcal{S}^\alpha_{2l} L^2(z)$.

Thus the desired decomposition follows by taking

$$g(\lambda,\xi) = \widehat{v}_-(\lambda,\xi) + (1-\chi(\xi))\widehat{v}_+(\lambda,\xi), \quad f(\lambda,\xi) = e^{\frac{2}{3}im\xi^{\frac{3}{2}}} \chi(\xi)\widehat{v}_+(\lambda,\xi).$$

We now want to prove the converse, that is, to show that the decomposition on the right hand side of (7.44) implies that $v \in I_k L^2_{\nu_{\frac{1}{6}}} L^2(Z, \Xi_m)$ where as before $\Xi_m \stackrel{\text{def}}{=} \Xi^{\operatorname{sgn}(m)}_{|m|}$, $m \neq 0$. To check the stability under

$$\lambda D_\lambda, \ D_x - m^{-2}x^2, \ x(D_x - m^{-2}x^2)$$

with weights $1, \frac{2}{3}$ and 1 respectively, we move to the Fourier transform side in x and require that

$$(\lambda D_\lambda)^{k_0}(D_\xi^2 - m^2\xi)^{k_1}[(D_\xi^2 - m^2\xi)D_\xi]^{k_2}(\mathcal{F}_2 \circ W)u(\lambda,\xi) \in \lambda^{1/6} L^2(Z), \tag{7.52}$$

with $k_0 + \frac{2}{3}k_1 + k_2 \leq k = 2l$.

The functions in $\mathcal{S}^{1/6}_{2l}L^2(Z)$ satisfy (7.52) and we need to look at $e^{-\frac{2}{3}im\xi^{\frac{3}{2}}}S^{1/6}_{2l}L^2(Z)$. Commutation through the oscillatory factor shows that it suffices to establish the stability of the elements of $S^{1/6}_{2l}L^2(Z)$ under $\langle\xi\rangle^{\frac{1}{2}}D_\xi$ with the weight $\frac{2}{3}$, i.e.,

$$f \in S^{1/6}_{2l}L^2(Z) \Longrightarrow (\langle\xi\rangle^{\frac{1}{2}}D_\xi)^p(\lambda D_\lambda)^{k_0}(\xi D_\xi)^{k_1}D_\xi^{k_2}f \in \lambda^{1/6}L^2(Z),$$

$\frac{2}{3}p + k_0 + k_1 + \frac{1}{3}k_2 \leq 2l$.
From this, one sees that it suffices to have the stability under $\langle\xi\rangle^{\frac{1}{2}}D_\xi^2$ with weight 1 which follows from the stability under ξD_ξ and D_ξ^3 both with weight 1 and the inequality

$$\int_0^\infty t|h'(t)|^2dt \leq \left(\int_0^\infty |h''(t)|^2dt\right)^{\frac{1}{2}}\left(\int_0^\infty t^2|h'(t)|^2dt\right)^{\frac{1}{2}} \leq$$
$$\frac{1}{2}\left(\int_0^\infty |h''(t)|^2dt + \int_0^\infty t^2|h(t)|^2dt\right), \quad h \in C_0^\infty(\mathbb{R}_+).$$

We now need to check the essential support property required in the definition of $I_kL^2_{\nu_{\frac{1}{6}}}(Z, \Xi_m)$. It is easily satisfied for $\mathcal{F}^{-1}\mathcal{S}^{1/6}_kL^2$ and it remains to consider

$$v_+(x) = \int \chi(2\xi)e^{-\frac{2}{3}im\xi^{\frac{3}{2}}+i\xi x}f(\lambda,\xi)d\xi,$$

and to check the stability under $x\cdot\chi(-\mathrm{sgn}(m)x)$ with weight $\frac{1}{3}$. The argument is similar to the analysis of (7.47) and we shall present only the first induction step. Thus we write

$$x\cdot\chi(-\mathrm{sgn}(m)x)u_+(x) = \int \chi(2\xi)\chi(-\mathrm{sgn}(m)x)\frac{x}{x-m\xi^{\frac{1}{2}}}\left[D_\xi e^{i(-\frac{2}{3}im\xi^{\frac{3}{2}}+\xi x)}\right]f(\xi)d\xi$$

where we integrate by parts. If $D_\xi f \in \lambda^{1/6}L^2(Z)$ then so is the left hand side as

$$a(x,\xi) = \chi(2\xi)\chi(-\mathrm{sgn}(m)x)(x-m\xi^{\frac{1}{2}})^{-1}x \in S^0(\mathbb{R},\mathbb{R})$$

and we apply $a(x,D_x)$ to $\mathcal{F}^{-1}(e^{-\frac{2}{3}im(\bullet)^{\frac{3}{2}}}D_\xi f)^\vee(x)$. The obvious inductive argument concludes the proof of the lemma. □

To prove (7.41) we decompose S into the elliptic and hyperbolic components:

$$S = S_e + S_h,\ S_ev(\lambda,x,y) = \frac{1}{(2\pi)^{\frac{2}{3}}}\int \psi(\lambda^{-\frac{2}{3}}\xi)\phi(-\xi-y)\frac{A_+(-\xi-y)}{A_+(-\xi)}\widehat{v}(\xi)e^{i\xi x}d\xi, \tag{7.53}$$

where $\phi \in C^\infty(\mathbb{R},[0,1])$

$$\phi(t) = \begin{cases} 1 & t > 2 \\ 0 & t < 1. \end{cases}$$

We start with the elliptic region estimate:

Lemma 7.6. *If $k \in 2\mathbb{N}_0$ and $k_0 + \frac{2}{3}(k_1+k_4) + \frac{1}{3}(k_2+k_3) \leq k, \quad m \neq 0$, then*

$$(\lambda D_\lambda)^{k_0}D_x^{k_1}x^{k_2}D_y^{k_3}y^{k_4}S_e = \mathcal{O}(1) : I_kL^2_{\nu_\alpha}(Z;\Xi_m^\pm) \longrightarrow \lambda^\alpha L^2([1,\infty)_\lambda \times \mathbb{R}_{+y}; H_{(\frac{1}{4})}(\mathbb{R}_x)) \tag{7.54}$$

Proof. We again consider $\widetilde{S}_e$ defined in the same way as $\widetilde{S}$ in the proof of (7.42) and, using the asymptotic expansions of the Airy functions on $\mathbb{R}_+$, we write (omitting λ)

$$\widetilde{S}_e\widehat{v}(y,\xi) = \psi(\lambda^{-\frac{2}{3}}\xi)e^{-\frac{2}{3}((-\xi)^{\frac{3}{2}}-(-\xi-y)^{\frac{3}{2}})}w_1(-\xi-y)w_2(-\xi)\widehat{v}(\xi),$$

where $w_i \in S^{(-1)^i/4}(\mathbb{R})$, supp $w_i \subset [1,\infty)$, (compare Sect.9 of [49]). From Lemma 7.5 we conclude that

$$g(\xi) = \phi(-2\xi)\widehat{v}(\xi) \in \mathcal{S}^\alpha_k L^2(Z)$$

and we want to examine the application of the operators falling on S_e in (7.54): $\xi,\ D_\xi,\ D_y,\ y$ with the weights $\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}$, respectively. From the definition of $\mathcal{S}^\alpha_k$, the application of $\lambda D_\lambda, \xi$ and D_ξ to g is allowed and thus we only need to examine the effect of the last three operators in (7.54) on the phase. Thus

$$\begin{aligned}
\partial_\xi^{k_2}\partial_y^{k_3}y^{k_4}\widetilde{S}_e\widehat{v}(y,\xi) &\equiv ((-\xi)^{\frac{1}{2}} - (-\xi-y)^{\frac{1}{2}})^{k_2}(-(\xi-y)^{\frac{1}{2}})^{k_3}y^{k_4} \times \\
&\quad e^{-\frac{2}{3}((-\xi)^{\frac{3}{2}}-(-\xi-y)^{\frac{3}{2}})}w_1(-\xi-y)w_2(-\xi)g(\xi) \\
&\equiv \left(\frac{(-\xi-y)^{\frac{1}{2}}}{(-\xi)^{\frac{1}{2}}+(1+y)^{\frac{1}{2}}}\right)^{k_3}(1+y)^{k_4+\frac{1}{2}k_3}\gamma(y,\xi)^{k_2} \times \\
&\quad e^{-\frac{2}{3}((-\xi)^{\frac{3}{2}}-(-\xi-y)^{\frac{3}{2}})}w_1(-\xi-y)w_2(-\xi)g(\xi),
\end{aligned}$$

where $\gamma(y,\xi) = (-\xi)^{\frac{1}{2}} - (-\xi-y)^{\frac{1}{2}}$ and where $\equiv$ means that the expressions differ by terms which can be treated by the analysis for lower k. In fact, for ∂_ξ and y this is immediate, while for ∂_y falling on the phase we write

$$(-\xi-y)^{\frac{1}{2}} = \frac{(-\xi-y)^{\frac{1}{2}}}{(-\xi)^{\frac{1}{2}}+(1+y)^{\frac{1}{2}}}(1+y)^{\frac{1}{2}} + \frac{(-\xi-y)^{\frac{1}{2}}}{(-\xi)^{\frac{1}{2}}+(1+y)^{\frac{1}{2}}}(-\xi)^{\frac{1}{2}}$$

with $(-\xi)^{\frac{1}{2}}$ in the second term absorbed into g (with weight $\frac{1}{3}$) – it is allowed as $g \in \mathcal{S}^\alpha L^2(Z)$. To proceed we need the following

Lemma 7.7. *Let* $f \in C^\infty(\mathbb{R}_+ \times [1,\infty))$ *satisfy*
i) $|f_y^{(k)}(0,t)| \le C_k$
ii) $|f_y^{(k)}(y,t)| \le C_k t^\alpha$, *for some* $\alpha \ge 0$
iii) supp $f \subset \{(y,t) : y \ge 0,\ t \ge 1+y\}$.
Then

$$\int_0^\infty e^{-\frac{4}{3}(t^{\frac{3}{2}}-(t-y)^{\frac{3}{2}})}f(y,t)dy = \mathcal{O}(t^{-\frac{1}{2}}). \tag{7.55}$$

Proof. We can obtain asymptotic expansion for the integral (7.55) using integration by parts:

$$\begin{aligned}
&\int_0^\infty e^{-\frac{4}{3}(t^{\frac{3}{2}}-(t-y)^{\frac{3}{2}})}f(y,t)dy = \\
&+\frac{1}{2}t^{-\frac{1}{2}}f(0,t) + \int_0^\infty e^{-4/3(t^{\frac{3}{2}}-(t-y)^{\frac{3}{2}})}\left[-\frac{1}{4}(t-y)^{-\frac{3}{2}}f(y,t) + \frac{1}{2}(t-y)^{-\frac{1}{2}}f'_y(y,t)\right]dy
\end{aligned}$$

which continued sufficiently many times gives, in view of (i), a term $\mathcal{O}(t^{-\frac{1}{2}})$ and a sum of terms of the form

$$\int_0^\infty e^{-4/3(t^{\frac{3}{2}}-(t-y)^{\frac{3}{2}})}(t-y)^{-\frac{k_1}{2}}f^{(k_2)}(y,t)dy$$

with k_1 large. Using ii) and iii) we can estimate this integral by

$$C_{k_2} t^{\alpha} e^{-4/3t^{\frac{3}{2}}} \int_0^{t-1} e^{4/3(t-y)^{\frac{3}{2}}} (t-y)^{-k_2/2} dy \leq C_{k_2} t^{\alpha} \int_0^{\frac{1}{2}t} (t-y)^{-k_1/2} dy +$$
$$C_{k_2} t^{\alpha} e^{-4/3(1-2^{-\frac{3}{2}})t^{\frac{3}{2}}} = \mathcal{O}(t^{-\frac{1}{2}})$$

if $k_1 \geq 2\alpha + 3$. □

Since $(1+y)^{1/2} \leq (-\xi)^{1/2}$ for $y \geq 0, -\xi \geq -\xi - y \geq 1$, we can apply Lemma 7.7 with $f(y, -\xi)$ given by

$$\left[\left(\frac{(-\xi - y)^{\frac{1}{2}}}{(-\xi)^{\frac{1}{2}} + (1+y)^{\frac{1}{2}}} \right)^{k_3} (1+y)^{k_4 + \frac{1}{2}k_3} \gamma(y, \xi)^{k_2} (-\xi)^{1/4} w_1(-\xi - y) \right]^2,$$

$\alpha = k_4 + \frac{1}{2}(k_2 + k_3)$ to obtain from the above discussion

$$\begin{aligned}
&\int \int_0^{\infty} \langle \xi \rangle^{\frac{1}{2}} \left| (\lambda D_\lambda)^{k_0} D_x^{k_1} x^{k_2} D_y^{k_3} y^{k_4} \widetilde{S}_e \widehat{v}(y, \xi) \right|^2 \lambda^{-2\alpha} dy d\xi d\lambda \\
&\leq C \sum_{k_0' \leq k_0, k_1' \leq \frac{1}{2}k_3 + k_1, k_2' \leq k_2} \int \left| (\lambda D_\lambda)^{k_0'} \xi^{k_1'} D_\xi^{k_2'} g(\xi) \right|^2 d\xi \lambda^{-2\alpha} d\lambda \\
&\leq C \sum_{k_0' + \frac{2}{3}k_1' + \frac{1}{3}k_2' \leq k_0 + \frac{1}{3}k_3 + \frac{2}{3}k_1 + \frac{1}{3}k_2} \int \left| (\lambda D_\lambda)^{k_0'} \xi^{k_1'} D_\xi^{k_2'} g(\xi) \right|^2 d\xi \lambda^{-2\alpha} d\lambda \\
&\leq C \|g\|_{\mathcal{S}_k^{\alpha} L^2(Z)} \leq C \|v\|^2_{I_k L^2_{\nu_\alpha}(Z; \Xi_m^{\pm})},
\end{aligned}$$

where the last inequality is a consequence of Lemma 7.5. □

We factorize the hyperbolic term defined in (7.53) as follows:

$$S_h = G \circ \widetilde{\mathcal{A}}_+^{-1}$$

where

$$Gv(\lambda, x, y) = \frac{1}{2\pi} \int \psi(\lambda^{-\frac{2}{3}}\xi) \Psi_+(\tau) \widehat{v}(\xi) e^{-\frac{1}{3}i\tau^3 + i\tau(\xi+y) + ix\xi} d\tau d\xi, \tag{7.56}$$

$$\Psi_+(\tau) e^{-\frac{1}{3}i\tau^3} = \frac{1}{2\pi} \int ((1-\phi)A_+)(-s) e^{ist} ds, \quad \Psi_+ \in S^0(\mathbb{R}),$$

see [49], Sect.4. The other term $\widetilde{\mathcal{A}}_+^{-1}$ is the non-homogeneous version of the $\mathcal{A}_+^{-1}$ operator:

$$\begin{aligned}
&(\mathcal{A}_+^{-1} u)^\wedge(\xi_1, \xi_2) = \phi(\xi_1) \psi(\xi_1^{-1}\xi_2) (A_+(-\xi_1^{-\frac{1}{3}}\xi_2))^{-1} \widehat{u}(\xi_1, \xi_2), \ u \in \mathcal{E}'(\mathbb{R}^2), \\
&(\widetilde{\mathcal{A}}_+^{-1} v)^\wedge(\xi) = A_+(-\xi)^{-1} \widehat{v}(\xi), \quad v \in \mathcal{E}'(\mathbb{R}).
\end{aligned} \tag{7.57}$$

The more delicate part of the argument involves $\mathcal{A}_+^{-1}$. This operator is associated to a canonical relation with boundary $\mathcal{C}_0^-$ (see (7.26)) which can be regarded as a singular canonical transformation.

The following proposition, which may be of independent interest, quantizes the Lagrangian mapping properties of $\mathcal{C}_0^-$:

$$\mathcal{C}_0^- \Lambda_m^{\pm} = \Lambda_{m\mp1}^{\pm}, \ m \neq 0, \quad \mathcal{C}_0^- \Lambda_0 = \Lambda_1^-.$$

Proposition 7.5. *Let us define the following marked Lagrangian varieties:*

$$\begin{aligned}\mathcal{L}_p &= \{\Lambda_{|p|}^{\mathrm{sgn}(p)}, 2\},\\ \mathcal{L}_0 &= \{\Lambda_0,\ J^1_{\Lambda_0\cap\Lambda_1}\Lambda_1, 2\}\end{aligned}$$

where $\Lambda^{\pm}_m, \Lambda_0$ *are given by* (7.28), *and* $\mathcal{C}_0^- \mathcal{L}_p = \mathcal{L}_{p-1}$. *Then*

$$\mathcal{A}_+^{-1} : I_k H_{(s)}(\mathbb{R}^2; \mathcal{L}_p) \longrightarrow I_k H_{(s-\frac{1}{6})}(\mathbb{R}^2, \mathcal{L}_{p-1}).$$

The proof follows from Lemma 7.3 and

Lemma 7.8. *The multiplier* $\widetilde{\mathcal{A}}_+^{-1}$ *defined by* (7.57) *satisfies*

$$\langle D_x\rangle^{-1/4}\widetilde{\mathcal{A}}_+^{-1} : I_k L^2_{\nu_\alpha}(Z, \Xi^{\pm}_m) \longrightarrow I_k L^2_{\nu_\alpha}(Z, \Xi^{\pm}_{m-1}),\ m > 0,$$

$$\langle D_x\rangle^{-1/4}\widetilde{\mathcal{A}}_+^{-1} : I_k L^2_{\nu_\alpha}(Z, \Xi_0) \longrightarrow I_k L^2_{\nu_\alpha}(Z, \Xi^{-}_1).$$

Proof. This is an easy consequence of Lemma 7.5 and the asymptotic properties of A_+: $A_+^{-1} = G(\xi) + e^{\frac{2}{3}i\xi^{\frac{3}{2}}}F(\xi),\ G \in \mathcal{S}(\mathbb{R})\ F \in S^{1/4}(\mathbb{R}),\ \mathrm{supp}\, F \subset [1, \infty).$ □

Proof of Proposition 7.5. Unless $p = 0$ or $p - 1 = 0$ the statement follows immediately from the interpolation between the even indexed neighbours. In the special cases, the proposition is equivalent to

$$\mathcal{A}_+^{-1}\langle D_x\rangle^{-s} : I_k L^2(\mathbb{R}^2, \mathcal{L}_0) \longrightarrow I_k H_{(s-\frac{1}{6})}(\mathbb{R}^2, \mathcal{L}_{-1}),\ p = 0 \tag{7.58}$$

$$\langle D_x\rangle^{s-\frac{1}{6}}\mathcal{A}_+^{-1} : I_k H_{(s)}(\mathbb{R}^2, \mathcal{L}_1) \longrightarrow I_k L^2(\mathbb{R}^2, \mathcal{L}_0),\ p = 1. \tag{7.59}$$

In fact, Definition 4.4 immediately implies that for $s \in \mathbb{Z}$,

$$u \in I_k H_{(s)}(\mathbb{R}^2, \mathcal{L}_0) \Leftrightarrow D_x^\alpha u \in I_k L^2(\mathbb{R}^2, \mathcal{L}_0),\ |\alpha| \le s \Leftrightarrow \langle D_x\rangle^s u \in I_k L^2(\mathbb{R}^2, \mathcal{L}_0),$$

so that the case of general s follows interpolation. On the other hand Lemma 4.4 shows that $H^b_{(k)}(Y_2) \overset{\beta_*}{\leftrightarrow} I_k L^2(\mathbb{R}^2, \mathcal{L}_0)$, so that by Proposition B.2 the right hand side is an interpolation space in k. Hence (7.58) holds for all k. □

Lemma 7.9. *Let G be defined by* (7.56). *Then for k even*

$$\psi(\lambda^{-\frac{1}{3}}x)G : I_k L^2_{\nu_\alpha}(Z, \Xi^{\pm}_m) \longrightarrow I_k L^2_{\nu^+_{\alpha+\frac{1}{6}}}(Z_+, \widetilde{\widetilde{\Xi}}_m) \tag{7.60}$$

and in particular

$$\psi(\lambda^{-\frac{1}{3}}x)G\langle D_x\rangle^{1/4} : I_k L^2_{\nu_\alpha}(Z, \Xi^{\pm}_m) \longrightarrow I_k L^2_{\nu^+_{\alpha+\frac{1}{3}}}(Z_+, \widetilde{\widetilde{\Xi}}_m) \tag{7.61}$$

Proof. By taking the Fourier transform in y and applying Young's inequality to eliminate ψ we only need to consider $\lambda^{-1/6}\psi(\lambda^{-\frac{1}{3}}x)\Psi_+(\eta)e^{-\frac{1}{3}i\eta^3}v(x+\eta)$ and the operators (see definition of $I_k L^2(Z_+, \widetilde{\widetilde{\Xi}}_m)$)

$$\eta^2 + D_\eta - D_x,\ x + \eta,\ (x+\eta)D_x, \quad m = 0$$

with multiplicities $\frac{2}{3}, \frac{1}{3}, 1$ and

$$\eta^2 + D_\eta - D_x,\ D_x - m^{-2}(x+\eta)^2,\ (x+\eta)(D_x - m^{-2}(x+\eta)^2),\ m \neq 0$$

with multiplicities $\frac{2}{3}, \frac{2}{3}, 1$.

The corresponding assumptions on v gives stability under the operators (see definitions of $I_k L^2(Z, \Xi^{\pm}_m)$: x, xD_x, $m = 0$, with weight $\frac{1}{3}$ and 1), and $D_x - m^{-2}x^2$,

$x(D_x - m^{-2}x^2)$, $m \neq 0$, with weight $\frac{2}{3}$ and 1. It follows that we need to consider the norm of

$$\lambda^{-1/6}\psi(\lambda^{-\frac{1}{3}}x)a(\eta)w(x+\eta), \qquad a \in S^0(\mathbb{R}),\ w \in L^2(\mathbb{R})$$

in $L^2(\mathbb{R}^2_{x,\eta})$ and it is easily seen to be bounded by $\|w\|_{L^2(\mathbb{R})}$. The mapping property (7.61) follows from the frequency cut-off $\psi(\lambda^{-\frac{2}{3}}\xi)$ in G. $\square$

Combining (7.61) and Lemma 7.8 we obtain that

$$\psi(\lambda^{-\frac{1}{3}}x)S_h : I_kL^2_{\nu_\alpha}(Z,\Xi^\pm_m) \longrightarrow I_kL^2_{\nu^+_{\alpha+\frac{1}{3}}}(Z_+,\widetilde{\widetilde{\Xi}}_{m\mp 1}), m > 0$$

$$\psi(\lambda^{-\frac{1}{3}}x)S_h : I_kL^2_{\nu_\alpha}(Z,\Xi_0) \longrightarrow I_kL^2_{\nu^+_{\alpha+\frac{1}{3}}}(Z_+,\widetilde{\widetilde{\Xi}}_1)$$

Thus, using Lemma 7.6 and the definition of $I_kL^2_{\nu^+_{\alpha+\frac{1}{3}}}(Z_+,\widetilde{\widetilde{\Xi}}_m)$ we obtain (7.41) of Proposition 7.4:

$$S^\sharp = \psi(\lambda^{-\frac{1}{3}}x)\psi(\lambda^{-\frac{2}{3}}y)(S_e+S_h) : I_kL^2_{\nu_\alpha}(Z,\Xi^\pm_m) \longrightarrow I_kL^2_{\nu_{\alpha+\frac{1}{3}}}(Z_+,\widetilde{\widetilde{\Xi}}^\pm_m), \quad m \neq 0.$$

Finally, Lemmas 7.3 and 7.4 complete the proof of Proposition 7.3. We should remark that we have not used the full power of Lemma 7.6 which shows precisely the gain of regularity in the elliptic region. A little more will be used in the proof of Proposition 7.7 but even there it would have sufficed to have $\lambda^{\alpha+\frac{1}{6}}L^2([1,\infty)\times\mathbb{R}_+;L^2(\mathbb{R}))$ in the right hand side of (7.54).

7.3. The extension of the equivalence of glancing hypersurfaces to $b-$canonical transformations (see Appendix A and references given there) will now be used to go beyond the model case considered above. Thus we have the following generalization of Proposition 7.3:

Proposition 7.6. *Let the Lagrangians Λ_m be defined by* (7.28) *and the relation $\mathcal{F}$ by* (7.20)*. Then the diffractive parametrix $\widetilde{T}$ given by* (7.2) *has the following mapping properties for k even:*

$$\begin{aligned}
\widetilde{T} \ &: \ I_kL^2_c(\mathbb{R}^{n-1};\Lambda_0,J^1_{\Lambda_0\cap\Lambda_2}\Lambda_2,2) + I_kL^2_c(\mathbb{R}^{n-1};\Lambda_2^+,2) \longrightarrow \\
&\qquad I^b_kL^2_{\mathrm{loc}}(X,\mathcal{F}(\Lambda_1),\mathcal{F}(\Lambda_1)\cap\mathcal{F}(\Lambda_0)),\\
\widetilde{T} \ &: \ I_kL^2_c(\mathbb{R}^{n-1};\Lambda_1^-,2) + I_kL^2_c(\mathbb{R}^{n-1};\Lambda_3^+,2) \longrightarrow I^b_kL^2_{\mathrm{loc}}(X,\mathcal{F}(\Lambda_0)\sqcup\mathcal{F}(\Lambda_2)),\\
\widetilde{T} \ &: \ I_kL^2_c(\mathbb{R}^{n-1};\Lambda_1^+,2) \longrightarrow I^b_kL^2_{\mathrm{loc}}(X,\mathcal{F}(\Lambda_0)\sqcup\mathcal{F}(\Lambda_\sharp)).
\end{aligned}$$

where $\Lambda_\sharp \subset T^*\mathbb{R}^{n-1}\setminus 0$ is any *C^∞ homogeneous Lagrangian tangent to Λ_1 at $\Lambda_1\cap\Lambda_0$.*

Proof. Let $\widetilde{\chi} : \widetilde{\Gamma} \to T^*\widetilde{X}\setminus 0$ be the equivalence of glancing hypersurfaces used in the construction of the parametrix. By Proposition A.1 it can be chosen to extend to a $b-$canonical transformation ${}^b\chi : \Gamma \to {}^bT^*X\setminus 0$. The induced boundary canonical transformation $\chi_\partial : \Gamma_0 \to T^*\partial X\setminus 0$ coincides with $\mathcal{J}$, the canonical transformation of the elliptic Fourier Integral Operator J such that for $v \in \mathcal{E}'(\mathbb{R}^{n-1})$

$$P\widetilde{T}v \equiv 0, \quad \widetilde{T}v\restriction_{X_-} = 0, \quad \widetilde{T}v\restriction_{\partial X} = Jv.$$

By Theorem A.1 (and Remark A.1), ${}^b\chi$ can be chosen so that

$$({}^b\chi)^*p = c(\xi_2^2 - x_3\xi_3^2 - \xi_1\xi_3), \quad c \in S^0_{\mathrm{hg}}(\Gamma_1), \tag{7.62}$$

where we recall that the pull-back by ${}^b\chi$ makes sense as $p \in S^{2,2}_{\mathrm{hg}}({}^bT^*X \setminus 0)$.

Our first goal is to quantize (7.62) using $b-$Fourier Integral Operators. Thus we choose $F \in I^0_b(\mathbb{R}^n_+ \times X; ({}^b\chi^{-1})')$ and $G \in I^0_b(X \times \mathbb{R}^n_+; ({}^b\chi)')$ so that $FG - I = E_1$, $GF - I = E_2$, with $E_1 \in \Psi^0_b(\bar{\mathbb{R}}^n_+)$, $E_2 \in \Psi^0_b(X)$ satisfying $WF_b(E_1) \cap \Gamma = \emptyset, WF_b(E_2) \cap {}^b\chi(\Gamma) = \emptyset$ ($WF_b(E)$ denotes the essential support of the full symbol of E as a $b-$pseudodifferential operator). By the argument used in the proof of Proposition 4.4 (see also [25], III(4.26) and what follows),

$$FPG = C(P_0 + R) + E, \quad R \in \Psi^{1,2}_b(\bar{\mathbb{R}}^n_+), \ E \in \Psi^{-\infty}_b(\bar{\mathbb{R}}^n_+), \tag{7.63}$$

and where $C \in \Psi^0_b(\bar{\mathbb{R}}^n_+)$ is elliptic in Γ and can be chosen (by appropriate modification of F and G) to have the full symbol vanishing outside a neighbourhood of Γ. By absorbing the term $D^2_{x_2}R_{-1}, R_{-1} \in \Psi^{-1}_b(\bar{\mathbb{R}}^n_+)$, in R into P_0 (by writing $D^2_{x_2}R_{-1} = P_0R_{-1} + (x_2D^2_{x_3} + D_{x_1}D_{x_3})R_{-1}$) we can assume that $R \in \Psi^{1,1}_b(\bar{\mathbb{R}}^n_+)$.

The following lemma is based on Proposition A.2 and allows us to eliminate R:

Lemma 7.10. *If $C \in \Psi^0_b(\bar{\mathbb{R}}^n_+)$ is elliptic in Γ and $R \in \Psi^{1,1}_b(\bar{\mathbb{R}}^n_+)$ then there exist $Q_1, Q_2 \in \Psi^0_b(\bar{\mathbb{R}}^n_+)$, elliptic in Γ and $E \in \Psi^{-\infty,1}_b(\bar{\mathbb{R}}^n_+)$ such that*

$$C(P_0 + R) = Q_1P_0Q_2 + E.$$

Proof. It is in fact more convenient to look for T_1 and T_2 in Ψ^0_b, elliptic in Γ and such that

$$T_1(P_0 + R) = P_0T_2 + E, \quad E \in \Psi^{-\infty,1}_b(\bar{\mathbb{R}}^n_+),$$

which can be rewritten as

$$[P_0, T] \equiv TR + P_0B \ (\mathrm{mod}\ \Psi^{-\infty,1}_b), \tag{7.64}$$

where $T = T_1$ and $B = T_1 - T_2$. Thus we are looking for an elliptic operator $T \in \Psi^0_b$ and an operator $B \in \Psi^{-1}_b$ such that (7.64) holds. Identifying $t^0 = {}^b\sigma_0(T) \in S^0({}^bT^*\bar{\mathbb{R}}^n_+)$ and $b^1 = {}^b\sigma_{-1}(B) \in S^{-1}({}^bT^*\bar{\mathbb{R}}^n_+)$ with their pull-backs under $\jmath$ we see that (7.64) implies

$$\frac{1}{i}H_{p_0}t^0 = t^0r + p_0b^1, \quad r = {}^b\sigma_{1,1}(R). \tag{7.65}$$

On the other hand if (7.65) holds and we choose T^0, B^1 so that ${}^b\sigma_0(T^0) = t^0$ and ${}^b\sigma_{-1} = b^1$ then

$$[P_0, T^0] - T^0 - P_0B^1 \in \Psi^{0,1}_b(\bar{\mathbb{R}}^n_+).$$

Proposition A.3 shows that, indeed, we can find such t^0 and b^1, by taking $t^0 = \exp a$ and $b^1 = (\exp a)b$, with a, b given there. We then continue inductively, assuming that

$$E^j = [P_0, T^0 + \cdots + T^j] - (T^0 + \cdots T^j)R - P_0(B^1 + \cdots + B^{j+1}) \in \Psi^{-j,1}_b(\bar{\mathbb{R}}^n_+).$$

If ${}^b\sigma_{-j,1}(E^j) = e^j$ we again use Proposition A.3 to solve

$$\frac{1}{i}H_{p_0}t^{j+1} = -e^j + rt^{j+1} + p_0b^{j+1}$$

since it is equivalent to

$$\frac{1}{i}H_{p_0}q = -(t^0)^{-1}e^j + ps, \quad t^{j+1} = t^0q, \ b^{j+1} = t^0(qb^1 + s), \ q \in S^{-j}(\Gamma), \ s \in S^{-j-1}(\Gamma).$$

The standard argument adapted to the $\Psi^{\bullet,1}_b$ setting concludes the proof. □

By combining Lemma 7.10 with (7.63) we see that there exist $F_1, F_2 \in I^0_b(\bar{\mathbb{R}}^n_+ \times X, ({}^b\chi^{-1})')$ elliptic in Γ and properly supported such that

$$F_1 P = P_0 F_2 + E F_2, \quad E \in \Psi_b^{-\infty,1}(\bar{\mathbb{R}}^n_+) \tag{7.66}$$

where E is also properly supported. We recall from [25], Proposition III.4.18 that the restriction of a $b-$Fourier Integral Operator, F, to the boundary is a Fourier Integral Operator F_∂ with the canonical relation given by the restriction of that for F. Thus, $(F_2)_\partial \in I^0(\mathbb{R}^{n-1} \times X; (\chi_\partial^{-1})')$. We conclude that $(F_2)_\partial \circ J = A \in \Psi^0(\mathbb{R}^{n-1})$, where A is properly supported and

$$P_0 F_2 \widetilde{T} v = -E F_2 \widetilde{T} v + K v, \quad (F_2 \widetilde{T} v)\restriction_{\partial X} = A v, \quad K : \mathcal{E}'(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\bar{\mathbb{R}}^n_+). \tag{7.67}$$

To continue we need to recall two basic facts. If X and Y are manifolds with boundaries, $\mathcal{C}$, a $b-$canonical graph then,

$$\begin{aligned} I^0_b(X \times Y; \mathcal{C}') \ni G &: \mathcal{N}(Y) \longrightarrow \mathcal{N}(X) \\ \Psi_b^{-\infty,k}(X) \ni A &: \mathcal{N}(X) \longrightarrow C^\infty(X). \end{aligned}$$

The space $\mathcal{N}(X)$ is defined in [25] (see also Sect.18.3 of [14]). The first property is clear from III(4.14) in [25], while the second one follows from $\mathcal{N}(X) \cap \dot{\mathcal{A}}(X) \subset \mathcal{B}(X) \cap \dot{\mathcal{A}}(X) = C^\infty(X)$ (see [25], Proposition II.8.8 and [14], Theorem 18.3.24) and the mapping properties $\Psi_b^{\bullet,k} : \mathcal{N} \to \mathcal{N}$, $\Psi_b^{-\infty,k} : \mathcal{E}' \to \dot{\mathcal{A}}$.

Combining this with (7.67) we conclude that

$$F_2 \widetilde{T} = T_0 A + E_0, \quad A \in \Psi^0(\mathbb{R}^{n-1}), \quad E_0 : \mathcal{E}'(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\bar{\mathbb{R}}^n_+).$$

The proof is now concluded through applications of Propositions 4.1,4.2,4.4, Proposition 7.3 and the use of

$$\chi(\mathcal{F}_0(\Lambda) \cap \Gamma) \subset \mathcal{F}(\Lambda),$$

which follows immediately from (7.20). $\square$

Proof of Theorem 7.2. We proceed by showing that the Poisson map $\mathcal{E} : C_0^\infty(\partial X_+) \to C^\infty(X)$:

$$P(\mathcal{E} f \restriction_X) = 0 \text{ in } X, \quad \mathcal{E} f \restriction_{\partial X} = f, \quad \mathcal{E} f \restriction_{X_-} = 0, \tag{7.68}$$

extends for $s \in 2\mathbb{N}_0$ to a continuous map

$$\mathcal{E} : J_s L^2_c(\partial X, H) \longrightarrow (\beta_5)_* I_s L^2_{\nu_5,\mathrm{loc}} L^2(\widetilde{X}_5, \mathcal{S}_5). \tag{7.69}$$

Since both spaces in (7.69) are interpolation spaces, the map is then continuous for all values of s, in particular for $s \in \mathbb{N}_0$. Once we have (7.69) we apply the 'vanishing in the past' Proposition 5.3 to show that the extension of $\mathcal{E} f \restriction_X$ to $(\beta_5)_* I_s L^2_{\nu_5,\mathrm{loc}} L^2(\widetilde{X}_5, \mathcal{S}_5)$ can be modified to lie in $J_s L^2_{comp}(\tilde{X}, H)$. In fact, Definition 7.1 shows that for $s \in \mathbb{N}_0$

$$v \in J_s L^2_{comp}(\tilde{X}, H) \Rightarrow v \restriction_{K_\delta^\partial} \in \beta^\partial_* I_k L^2_{\nu_\partial}(\beta^{\partial *} K^\partial_\delta; \partial \widetilde{X}_1 \cap \beta^{\partial *} K^\partial_\delta).$$

Proposition 5.3 and the spacelike property of K_δ ($K_\delta^\partial = \partial X \cap K_\delta$ – see Proposition 5.1) gives the second condition in the definition of $J_k L^2_{\mathrm{loc}}(\widetilde{X}, H)$ (Definition 3.1; in fact we get a stronger condition: $U \restriction_{K_1} \in I_k L^2_\nu(\widetilde{X}_4, \partial \widetilde{X}_4) \restriction_{K_1}$).

show that the extension of $\mathcal{E} f \restriction_X$ can then be modified to lie in $J_s L^2_{\mathrm{loc}}(\widetilde{X}, H)$ without affecting (7.68).

To establish (7.69) we will reduce it to a microlocal statement. In fact, it suffices to prove (7.69) with the left hand side replaced by the right hand side of (7.6) and the right hand side replaced by the left hand side of (6.2), as we can use Proposition 7.2 and Theorem 6.1, respectively. In that case we consider the full diffractive parametrix T given by (7.15) and will see that for k even

$$\begin{aligned}
T &: I_k L^2_c(\partial X; \Lambda_{00}, J^1_{\Lambda_{00}\cap\Lambda_{2i}}\Lambda_{2i}, 2) + I_k L^2_c(\partial X; \Lambda^+_{2i}, 2) \longrightarrow \\
&\quad I^b_k L^2_{\mathrm{loc}}(X, \Lambda_S, \Lambda_S \cap \Lambda_F), && (7.70)\\
T &: I_k L^2_c(\partial X; \Lambda^-_{1i}, 2) + I_k L^2_c(\partial X; \Lambda^+_{3i}, 2) \longrightarrow I^b_k L^2_{\mathrm{loc}}(X, \Lambda_F \sqcup \Lambda_R), && (7.71)\\
T &: I_k L^2_c(\partial X; \Lambda^+_{11}, 2) \longrightarrow I^b_k L^2_{\mathrm{loc}}(X, \Lambda_F \sqcup \Lambda_R), && (7.72)\\
T &: I_k L^2_c(\partial X; \Lambda^+_{13}, 2) \longrightarrow I^b_k L^2_{\mathrm{loc}}(X, \Lambda_H \sqcup \Lambda_S) && (7.73)
\end{aligned}$$

Recalling (7.21), we see that, after a suitable microlocalization, (7.70),(7.71),(7.72) and (7.73) are a consequence of Proposition 7.6 and the following mapping properties of L in (7.15):

$$\begin{aligned}
L &: I_k L^2_c(\partial X; \Lambda_{00}, J^1_{\Lambda_{00}\cap\Lambda_{2i}}\Lambda_{2i}, 2) + I_k L^2_c(\partial X; \Lambda^+_{2i}, 2) \longrightarrow \\
&\quad I_k L^2_c(\mathbb{R}^{n-1}; \Lambda_0, J^1_{\Lambda_0\cap\Lambda_2}\Lambda_2, 2) + I_k L^2_c(\mathbb{R}^{n-1}; \Lambda^+_2, 2), && (7.74)\\
L &: I_k L^2_c(\partial X; \Lambda^-_{1i}, 2) + I_k L^2_c(\partial X; \Lambda^+_{3i}, 2) \longrightarrow \\
&\quad I_k L^2_c(\mathbb{R}^{n-1}; \Lambda^-_1, 2) + I_k L^2_c(\mathbb{R}^{n-1}; \Lambda^+_3, 2), && (7.75)\\
L &: I_k L^2_c(\partial X; \Lambda^+_{11}, 2) \longrightarrow I_k L^2_c(\mathbb{R}^{n-1}; \Lambda^+_1, 2), && (7.76)\\
L &: I_k L^2_c(\partial X; \Lambda^+_{13}, 2) \longrightarrow I_k L^2_c(\mathbb{R}^{n-1}; \Lambda^+_1, 2). && (7.77)
\end{aligned}$$

The operator L is chosen differently in each case depending on χ and Γ used in the parametrix construction – see the discussion following (7.19). Since L is an elliptic Fourier Integral Operator associated with χ_∂ (see (7.19)) the mapping properties above follow from Propositions 4.1 and 4.2. We then obtain (7.70),(7.71),(7.72) and (7.73) from (7.74),(7.75),(7.76) and (7.77), respectively from Proposition 7.6, once we check, for (7.72), that

$$\Lambda_R \cap \widetilde{\Gamma} \supset \mathcal{F}(\Lambda_\sharp \cap \Gamma),$$

and for (7.73), that

$$\Lambda_S \cap \widetilde{\Gamma} \supset \mathcal{F}(\Lambda_\sharp \cap \Gamma)$$

for some $\Lambda_\sharp$ tangent to Λ_1 at $\Lambda_1 \cap \Lambda_0$ (again, with different choices of χ and $\widetilde{\Gamma}$ in the two cases (7.72) and (7.73)). For (7.72) we can simply take $\Lambda_\sharp = \Lambda_2 = \bigcup_\pm \delta^\pm_0 \Lambda_0$ as then $\mathcal{F}(\Lambda_2 \cap \Gamma) \subset \Lambda_R \cap \widetilde{\Gamma}$. To construct $\Lambda_\sharp$ for (7.73) we observe that if $\mathcal{F}$ and $\mathcal{F}_1$ are defined using canonical transformations $\widetilde{\chi}$ and $\widetilde{\chi}_1$ – see (7.20) – then $\mathcal{F}_1 = \mathcal{F} \circ \chi_\partial^{-1}(\chi_1)_\partial$. If $(\chi_1)_\partial$ is chosen so that $(\chi_1)_\partial(\Lambda_0 \cap \Gamma) \subset \Lambda_{00}$, as it may be by (7.19), then by (7.21)

$$\mathcal{F}_1(\Lambda_1 \cap \Gamma) \subset \Lambda_S \cap \widetilde{\chi}(\widetilde{\Gamma})$$

and we can take $\Lambda_\sharp = \chi_\partial^{-1}(\chi_1)_\partial(\Lambda_1)$. Since Λ_1 is tangent to Λ_2, it remains to check that $\Lambda_\sharp$ is tangent to Λ_2, and that follows from the tangency of $\mathcal{F}(\Lambda_\sharp)$ to $\mathcal{F}(\Lambda_2)$, as can be verified using $\mathcal{F}_0$ or even more easily $\mathcal{H}$ given by (7.27). By the last part of (7.21), $\mathcal{F}(\Lambda_2 \cap \Gamma) \subset \Lambda_{H_+}$ and that tangency follows from the third order tangency of the cone and the reflected front H_+ (see Sect.2). This completes the proof of Theorem 7.2. $\square$

7.4. Because of the restriction to the boundary (see Theorem 7.1) we lose 1/2 of the order of regularity when applying Theorem 7.2. To avoid this loss in the final application to conormal regularity of diffracted waves we need

Proposition 7.7. *If $(F \cap \bar{X}_-) \cap \partial X = \emptyset$ and $u \in L^2(X)$ satisfies*

$$Pu = 0 \quad in \quad X, \; u \restriction_{\partial X} = 0, \; u \restriction_{X_-} = u_0, \quad u_0 \in I_k L^2(X_-; \Lambda_F), \tag{7.78}$$

then

$$u \in {}^b I_k L^2_{\mathrm{loc}}(X; \Lambda_F \sqcup \Lambda_R).$$

Proof. Following the proof of Theorem 7.2 and with the notation of Proposition 7.3, we only need to prove the statement in the model case: $P = P_0$, $\Lambda_F = \mathcal{F}_0(\Lambda_0)$, $\Lambda_R = \mathcal{F}_0(\Lambda_2)$, $X = \mathbb{R}^n_+$, $X_- = \{\phi(x) < -\delta, x_2 > 0\}$, where for small $|x|$ we can take $\phi(x) = x_1 + x_3$. The assumption that $\bar{X}_- \cap F$ is away from the boundary is replaced by its microlocal version:

$$WF(u_0) \subset \mathcal{F}_0(\Gamma_0) \cap \pi^{-1}(X_-) \tag{7.79}$$

where $\Gamma_0 \subset T^*\mathbb{R}^{n-2}$ is small conic neighbourhood of $(0; (1, 0, \cdots, 0))$.

The lemma below provides the needed characterization of the restriction of the free solution which is already essentially contained in Corollary 5.10 of [35] (see also Sect.25.3 of [14] and Sect.3 of [49]).

Lemma 7.11. *If u_0 satisfies* (7.79) *and if*

$$P_0 w = 0 \quad in \quad \mathbb{R}^3, \; w \restriction_{(\mathbb{R}^3_+)_-} = u_0, \; u_0 \in I_k L^2_{\mathrm{loc}}((\mathbb{R}^3)_-, \mathcal{F}_0(\Lambda_0)),$$

then $w \in I_k L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathcal{F}_0(\Lambda_0))$ and

$$w \restriction_{\partial \mathbb{R}^3_+} = \mathcal{A}i w_0 + E u_0, \; w_0 \in I_k H_{(-1/6)}(\mathbb{R}^2, \Lambda_0),$$

where $\mathcal{A}i$ is the multiplier defined by Ai as in (7.57) *and $E : L^2_{\mathrm{loc}}(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$.*

Proof. The first property of w is immediate from the propagation properties of Lagrangian distributions. Condition (7.79) implies that $WF(u_0)$ is contained in a conic neighbourhood of the bicharacteristic for p_0 through $(0; (0, 0, 1))$:

$$x_2 = x_1^2, \; x_3 = \frac{2}{3} x_2^3, \; \xi_1 = 0, \; \xi_2 = -x_1, \; \xi_3 = 1.$$

Thus by smoothly cutting off of the initial data (which produces C^∞ errors) we can assume that $u_0 \in C^\infty(\mathbb{R}_{x_2}; \mathcal{E}'(\mathbb{R}^2_{x_1,x_3}))$ and consequently that the same is true for ψw, where $\psi \in C^\infty(\mathbb{R}_{x_2}, C^\infty_0(\mathbb{R}_{x_1,x_2}))$, $\psi \equiv 1$ near $\pi(\mathcal{F}_0(\Gamma_0))$. We also note that $(1 - \psi) w \in C^\infty(\mathbb{R}^3)$. Taking the Fourier transform $\mathcal{F}$ in x_1 and x_3 we obtain

$$(D^2_{x_2} - \xi_3(\xi_3 x_2 + \xi_1)) \widehat{\psi w}(\xi_1, x_2, \xi_3) = \mathcal{F}\left(-[P_0, \psi] w\right)(\xi_1, x_2, \xi_3) \in C^\infty(\mathbb{R}_{x_2}; \mathcal{S}(\mathbb{R}^2_{\xi_1,\xi_3})),$$

and since $\widehat{\psi w}$ is tempered in (ξ_1, ξ_3) we conclude that

$$\phi(\xi_3) \widehat{\psi w}(\xi_1, x_2, \xi_3) = \phi(\xi_3) Ai(-\xi_3^{-1/3}(\xi_1 + \xi_3 x_2)) \widehat{w}_0(\xi_1, \xi_3),$$

where we can neglect the term ϕ in the left hand side (see (7.57)) as $\mathcal{F}^{-1}((1 - \phi)\widehat{\psi w}) \in C^\infty$. If Γ_0 is small enough, then $x_2 > \delta_1 > 2 \max_{\xi \in \Gamma_0} |\xi_1 / \xi_3|$ in $(\mathbb{R}^3)_- \cap \mathrm{sing\ supp} u_0$.

Thus, for $(x,\xi) \in \mathcal{F}_0(\Gamma_0) \cap \pi^{-1}(X_- \cap \text{sing supp}\, u_0)$

$$\phi(\xi_3) Ai(-\xi_3^{-1/3}(\xi_1 + \xi_3 x_2)) = \xi_3^{-1/6}(x_2 + \xi_1/\xi_3)^{-1/4}\phi(\xi_3) \times \\ \left[e^{i\frac{2}{3}\xi_3(x_2+\xi_1/\xi_3)^{3/2}} a_+(\xi_3^{2/3}(x_2 + \xi_1\xi_3)) + e^{-i\frac{2}{3}\xi_3(x_2+\xi_1/\xi_3)^{3/2}} a_-(\xi_3^{2/3}(x_2+\xi_1\xi_3)) \right],$$

$a_\pm \in S^0_{\text{phg}}(\mathbb{R}), a_\pm(1) \neq 0$. Since the phases are smooth then, we conclude that

$$WF\left(\mathcal{F}^{-1}\left(\exp(\pm\frac{2}{3}i\xi_3(x_2+\xi_1/\xi_3)^{3/2})a_\pm \widehat{w}_0\right)\restriction_{\{x_2>\delta_1, |x|<\delta_2\}}\right) \subset \\ \mathcal{F}_0(WF(w_0)) \cap \{x_2 > \delta_1, \pm x_1 < 0\}.$$

Hence, up to an error in $C^\infty(\mathbb{R}_{x_2}; \mathcal{S}(\mathbb{R}^2_{\xi_1,\xi_3}))$

$$\widehat{u}_0(\xi_1, x_2, \xi_3) = \xi_3^{-\frac{1}{6}}(x_2+\xi_1/\xi_3)^{-1/4}\phi(\xi_3) e^{i\frac{2}{3}\xi_3(x_2+\xi_1/\xi_3)^{3/2}} a_+(\xi_3^{2/3}(x_2+\xi_1\xi_3))\widehat{w}_0(\xi_1,\xi_3).$$

If $u_0 \in L^2_{\text{loc}}(\mathbb{R}_{x_2}; L^2(\mathbb{R}^2_{x_1,x_2}))$ we conclude that $\widehat{w}_0 \in \langle\xi_3\rangle^{1/6} L^2(\mathbb{R}^2)$, that is, $w_0 \in H_{-\frac{1}{6}}$. The desired conclusion is equivalent to the stability condition

$$(\xi_3 D_{\xi_1})^{k_1}(\xi_3 D_{\xi_3})^{k_2}\widehat{w}_0 \in \langle\xi\rangle^{1/6}L^2(\mathbb{R}^2), \quad k_1 + k_2 \leq k,$$

while we know that

$$(-\xi_3 D_{\xi_3} + 2x_2 D_{x_2} - \xi_1 D_{\xi_1})^{k_0}(D_{x_2} - \xi_3 D_{\xi_1})^{k_1}\widehat{u}_0 \in L^2_{\text{loc}}(\mathbb{R}_{x_2}; L^2(\mathbb{R}^2_\xi)).$$

However,

$$(D_{x_2} - \xi_3 D_{\xi_1})\widehat{u}_0 = \\ \xi_3^{-1/6}(x_2+\xi_1/\xi_3)^{-1/4}\phi(\xi_3)e^{i\frac{2}{3}\xi_3(x_2+\xi_1/\xi_3)^{3/2}} a_+(\xi_3^{2/3}(x_2+\xi_1\xi_3))(-\xi_3 D_{\xi_1})\widehat{w}_0, \\ (-\xi_3 D_{\xi_3} + 2x_2 D_{x_2} - \xi_1 D_{\xi_1})\widehat{u}_0 \equiv \xi_3^{-1/6}(x_2+\xi_1/\xi_3)^{-1/4}\phi(\xi_3) \times \\ e^{i\frac{2}{3}\xi_3(x_2+\xi_1/\xi_3)^{3/2}} a_+(\xi_3^{2/3}(x_2+\xi_1\xi_3))(-3\xi_3 D_{\xi_3} - \xi_1 D_{\xi_1})\widehat{w}_0,$$

which together with the wave-front assumption (in particular, $|\xi_1| \leq \xi_3$) concludes the proof. □

With the notation of Proposition 7.4, we want to show that

$$S^\sharp(W(\mathcal{A}iw_0)) \in I_k L^2_{\nu^+_{\frac{1}{2}}}(Z_+, \tilde{\tilde{\Xi}}_0) + I_k L^2_{\nu^+_{\frac{1}{2}}}(Z_+, \tilde{\tilde{\Xi}}_2),$$

as then Lemma 7.4 will provide the desired conclusion. Again, we consider the elliptic and hyperbolic components separately and start with the former:

$$S_e(W(\mathcal{A}iw_0)) = S_e(\phi(-D_x)\widetilde{\mathcal{A}}iW(w_0)), \quad \widetilde{\mathcal{A}}i \circ W \equiv W \circ \mathcal{A}i.$$

Since $Ai(-\xi) = \mathcal{O}(\langle\xi\rangle^{-N})$ for all N if $\xi \leq 0$, we easily see that $\phi(-\xi)Ai(-\xi)\widehat{Ww_0}(\lambda,\xi) \in \mathcal{S}^{1/3}_k L^2(Z)$. Thus we can use the proof of Lemma 7.6 to conclude that

$$S_e(\widetilde{\mathcal{A}}iW(w_0)) \in I_k L^2_{\nu^+_{\frac{1}{2}}}(Z_+, \tilde{\tilde{\Xi}}_2).$$

In fact, a much stronger conclusion holds:

$$(\lambda D_\lambda)^{k_0} D_x^{k_1} x^{k_2} D_y^{k_3} y^{k_4} S_e(\widetilde{\mathcal{A}}iW(w_0)) \in \lambda^{1/3}L^2([1,\infty)\times\mathbb{R}; H_{(1/4)}(\mathbb{R})) \subset \lambda^{1/2}L^2(Z_+).$$

Turning to the hyperbolic component we observe that by writing $Ai = -\omega A_+ - \bar{\omega} A_-$, $\omega = \exp(2\pi i/3)$, we obtain

$$S_h(\widetilde{\mathcal{A}}i(Ww_0)) = -\omega G W w_0 - \bar{\omega} G \left(\widetilde{\left(\frac{\mathcal{A}_-}{\mathcal{A}_+}\right)} W w_0 \right).$$

Straightforward analogues of Lemmas 7.8 and 7.9 conclude the proof. □

We should remark that a minimal amount of additional care would remove the assumption that k is even, but that is irrelevant to us as indicated in the following

Theorem 7.3. *If the assumptions of Proposition 7.7 are satisfied with $k \in \mathbb{N}_0$, then there exists a continuous map*

$$E : I_k L^2(X;\Lambda_F) \longrightarrow J_k L^2_c(\widetilde{X}, H)$$

such that $E(u_0)\restriction_X = u$.

Proof. We only need to combine Proposition 7.7 with Proposition 5.3, Theorem 6.1 and an interpolation argument as in the proof of Theorem 7.2. □

8. Proof of the main theorem

To give the proof of the propagation theorem for the pseudo-conormal space given by Definition 3.2, two preparatory facts are still needed.

Lemma 8.1. *If $g \in J_s L^2{}_{\mathrm{loc}}(\widetilde{X}, H)$ then*

$$WF^{(s)}(g) \subset N^*\widetilde{F} \cup N^*\widetilde{R} \cup N^*\widetilde{S}_+ \cup T^*_\Gamma \widetilde{X} \setminus 0 \cup N^*L \cup N^*B_+ \cup N^*D_+ \cup N^*H. \tag{8.1}$$

Proof. The statement follows easily from Definition 3.1 if $s \in \mathbb{N}_0$. We need to see that it is preserved when the space $J_k L^2_{\mathrm{loc}}(\widetilde{X}, H)$ is interpolated. For convenience we shall denote the union on the right hand side of (8.1) by $\widetilde{\mathcal{L}}$.

Let $\Omega \subset T^*\widetilde{X} \setminus 0$ be an open conic set such that $\overline{\Omega}$ is disjoint from $\widetilde{\mathcal{L}}$. The condition $WF^{(s)}(g) \subset \widetilde{\mathcal{L}}$ is equivalent to saying that for every such Ω, if $a \in S^0_{\mathrm{hg}}(T^*X \setminus 0)$ is elliptic in Ω and supp $a \cap \widetilde{\mathcal{L}} = \emptyset$, then $a(x,D)\langle D\rangle^s g \in L^2_{\mathrm{loc}}(X)$. Fixing a with that property and defining

$$W_s(\widetilde{X}, \Omega) = \{u \in L^2_{\mathrm{loc}}(X) : a(x,D)\langle D\rangle^s g \in L^2_{\mathrm{loc}}(\widetilde{X})\}$$

we see that W_s is an interpolation space in s and that $WF^{(s)}(g) \subset \widetilde{\mathcal{L}}$ if and only if for all Ω above $g \in W_s(\widetilde{X}, \Omega)$. Fixing Ω we now see that

$$\begin{array}{ccc} J_{k+1} L^2_{\mathrm{loc}}(\widetilde{X}, H) & \longrightarrow & W_{k+1}(\widetilde{X}, \Omega) \\ \big\downarrow & & \big\downarrow \\ J_k L^2_{\mathrm{loc}}(\widetilde{X}, H) & \longrightarrow & W_k(\widetilde{X}, \Omega) \end{array}$$

and the complex interpolation finishes the proof. □

To guarantee the singular support condition in Definition 3.2 of $J_s L^2_{\mathrm{loc}}(X)$ we need the following crude propagation property:

Lemma 8.2. *Let u satisfy*

$$Pu = f \quad in \quad X, \;\; u\restriction_{\partial X} = 0, \;\; u\restriction_{X_-} = 0 \tag{8.2}$$

with $f = \tilde{f}\restriction_X$, where $WF^{(s)}(\tilde{f})$ is contained in the right hand side of (8.1) *If*

$$\text{sing supp}^{(s)}(f) \cap (\widetilde{R} \setminus R \cup \widetilde{F} \setminus F \cup \widetilde{S}_+ \setminus S_+) = \emptyset$$

then

$$\text{sing supp}^{(s+\frac{1}{2})}(u) \cap (\widetilde{R} \setminus R \cup \widetilde{F} \setminus F \cup \widetilde{S}_+ \setminus S_+) = \emptyset.$$

Proof. We construct the solution u by starting with the interior problem:

$$\widetilde{P}\tilde{u} = \widetilde{f} \;\text{ in }\; \widetilde{X}, \;\;\; \tilde{u}\restriction_{\widetilde{X}_-} = 0,$$

where we choose *any* strictly hyperbolic extension of $P\restriction_X$ to $\widetilde{X}\setminus X$ (a fixed extension P was introduced in Sect.2). If we denote the right-hand side of (8.1) by $\widetilde{\mathcal{L}}$ and introduce

$$\mathcal{L} = \widetilde{\mathcal{L}} \setminus \pi^{-1}[(\widetilde{R} \setminus R \cup \widetilde{F} \setminus F \cup \widetilde{S}_+ \setminus S_+) \cap X],$$

then the assumption on f shows that $WF^{(s)}(\tilde{f}) \subset \mathcal{L}$. If $\widetilde{\mathcal{L}}_1$ is the $\widetilde{P}$–flow-out of $\widetilde{\mathcal{L}} \cap \{\sigma_2(\widetilde{P}) = 0\}$, then the standard propagation result (see [14], Sect.26.1) gives

$$WF^{(s+1)}(\tilde{u}) \subset \mathcal{L} \cup \widetilde{\mathcal{L}}_1.$$

By choosing $\widetilde{P}$ in $\widetilde{X} \setminus X$ appropriately, it can be arranged (for instance, by decreasing the speed of propagation in $\widetilde{X} \setminus X$) that

$$(\widetilde{\mathcal{L}}_1 \setminus \mathcal{L}) \cap (\widetilde{\mathcal{L}} \setminus \mathcal{L}) = \emptyset, \tag{8.3}$$

and this will be crucial later. At the boundary, all the terms in $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}_1$ are disjoint except over Γ where all of $T^*_\Gamma\widetilde{X} \setminus 0$ is included. Thus

$$WF^{(s+\frac{1}{2})}(-\tilde{u}\restriction_{\partial X}) \subset \widetilde{\mathcal{L}}_1^{\partial},$$

where $\widetilde{\mathcal{L}}_1^{\partial}$ comes from the projection of each term in $\widetilde{\mathcal{L}}_1$ to $T^*\partial X \setminus 0$. Hence when we solve

$$Pv = 0 \;\text{ in }\; X, \;\; v\restriction_{\partial X} = -\tilde{u}\restriction_{\partial X}, \;\; v\restriction_{X_-} = 0,$$

the propagation of singularities theorem for the diffractive Dirichlet problem (see [37] and [14], Sect.24.4) shows that

$$v \in \mathcal{N}(X), \quad WF_b^{(s+\frac{1}{2})}(v) \subset \jmath(\widetilde{\mathcal{L}}_1\restriction_{T^*X\setminus 0}),$$

where $\jmath : T^*X \setminus X \to {}^bT^*X \setminus 0$. Putting $u = \tilde{u}\restriction_X + v$ we obtain the solution to (8.2) and it is independent of the extension $\widetilde{P}$ chosen. Thus we can take either the fixed extension P (as in Sect.2 and consequently in the definition the extended fronts and $\widetilde{\mathcal{L}}$) or $\widetilde{P}$ such that (8.3) holds. This shows that

$$\text{sing supp}^{(s+\frac{1}{2})}(u) \subset \pi(\widetilde{\mathcal{L}})\restriction_X \cap \pi(\widetilde{\mathcal{L}}_1)\restriction_X \subset \pi(\mathcal{L})\restriction_X$$

concluding the proof. □

We can finally give the long promised

Theorem 8.1. *If* $f\restriction_{X_-}=0$ *and* $u\in L^2_{loc}(X)$ *satisfies*

$$Pu=f \quad in \quad X, \quad u\restriction_{\partial X}=0, \quad u\restriction_{X_-}=0,$$

then

$$f\in J_sL^2{}_{\mathrm{loc}}(X)\Longrightarrow u\in J_{s+\frac{1}{2}}L^2_{\mathrm{loc}}(X).$$

Proof. Definition 3.2 guarantees that for any $H\in\mathcal{R}$ there exists $\tilde f\in J_sL^2_{\mathrm{loc}}(\widetilde X,H)$ such that $f=\tilde f\restriction_X$. If we apply Theorem 5.1 to the problem

$$Pu_1=\tilde f \;\text{ in }\; \widetilde X, \quad u_1\restriction_{\widetilde X_-}=0,$$

we conclude that $u_1\in J^1_sL^2_{\mathrm{loc}}(\widetilde X,H)$. Theorem 7.1 then implies that

$$-u_1\restriction_{\partial X}\in J_{s+\frac{1}{2}}(\partial X,H)$$

and we solve the Dirichlet problem with that boundary data:

$$Pu_2=0 \;\text{ in }\; X, \quad u_2\restriction_{X_-}=0, \quad u_2\restriction_{\partial X}=-u_1\restriction_{\partial X}.$$

Theorem 7.2 shows that $u_2\in J_{s+\frac{1}{2}}L^2_{\mathrm{loc}}(\widetilde X,H)\restriction_X$ and $u=\tilde u\restriction_X$, $\tilde u=u_1+u_2\in J_{s+\frac{1}{2}}L^2_{\mathrm{loc}}(\widetilde X,H)$. Since H varies freely in $\mathcal{R}$ we conclude the proof by observing that the singular support condition required by Definition 3.2 is easily furnished by Lemmas 8.1 and 8.2. □

The proof of the main theorem on the conormal regularity for semi-linear diffractive mixed problems is now an equally easy consequence of Theorems 3.1, 7.3 and 8.1. We recall that F is a C^∞ characteristic surface satisfying $F\cap\bar X_-\cap\partial X=\emptyset$ and the pseudo-conormal space $J_kL^2_{\mathrm{loc}}(X)$ is given by Definition 3.2.

Theorem 8.2. *Let* $u\in L^\infty_{\mathrm{loc}}(X)$ *be the solution of the semi-linear mixed problem:*

$$Pu=f(x,u) \quad in \quad X, \quad u\restriction_{\partial X}=0, \quad u\restriction_{X_-}=u_0$$

where $f\in C^\infty(\mathbb{C})$ *and* $u_0\in I_kL^2_{loc}(X_-,F)$. *Then* $u\in J_kL^2_{\mathrm{loc}}(X)$.

Proof. To apply the standard procedure based on the algebra property (Theorem 3.1) and the propagation property (Theorem 8.1) (see [32, 34] and references given there) we only need to check that

$$Pw=0,\; w\restriction_{\partial X}=0,\; w\restriction_{X_-}=u_0 \;\Longrightarrow\; w\in J_kL^2_{\mathrm{loc}}(X).$$

That however is easy now as Theorem 7.3 shows that $w\in J_kL^2_{\mathrm{loc}}(\widetilde X,H)\restriction_X$, for any $H\in\mathcal{R}$, and the singular support statement follows from propagation of singularities for the diffractive mixed problem (cf. [35] and [14], Sect.24.4). □

The results presented in Sect.1 are easy consequences of Theorem 8.2 and the definitions.

Appendix A. Glancing hypersurfaces and b-geometry

The purpose of this appendix is to present some refinements of the equivalence of glancing hypersurfaces (see Sect.2 for definitions and [24, 35] for a detailed discussion) and of the construction of solutions to the diffractive transport equations.

We will use the notation similar to that in Sect.4: $\mathbb{R}^{n+1}_+ = \{(x,y) : x > 0, y \in \mathbb{R}^n\}$ and denote the coordinates in $T^*\mathbb{R}^{n+1}\backslash 0$ and ${}^bT^*\bar{\mathbb{R}}^{n+1}_+\backslash 0$ by $(x,y;\xi,\eta)$ and $(x,y;\lambda,\eta)$ respectively, so that $\jmath : T^*\bar{\mathbb{R}}^{n+1}_+\backslash 0 \to {}^bT^*\bar{\mathbb{R}}^{n+1}_+\backslash 0$ takes the form $\jmath(x,y;\xi,\eta) = (x,y;x\xi,\eta)$.

Our starting point is the following theorem from [35]:

Proposition A.1. *If P and Q in $T^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0$ are given by*

$$Q = \{x = 0\} \quad P = \{p = 0\}, \ p = \xi^2 + 2a(x,y,\eta)\xi + b(x,y,\eta), \ a \in S^1_{\mathrm{hg}}, \ b \in S^2_{\mathrm{hg}},$$

and are glancing at $m_0 = (0;(0,1,0,\cdots,0))$, then there exists a conic neighbourhood of $\jmath(m_0)$, $\Gamma \subset {}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0$ and a $b-$canonical transformation

$${}^b\chi : \Gamma \longrightarrow {}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0, \quad {}^b\chi(\jmath(m_0)) = \jmath(m_0)$$

such that

$${}^b\chi(\{\lambda^2 + \epsilon x^3\eta_1^2 - x^2\eta_1\eta_n = 0\} \cap \Gamma) \ \subset \ \overline{\jmath(P)}, \quad \epsilon = \mathrm{sgn}(\partial_x b(0,0)).$$

We remark that ${}^b\chi(\{x = 0\} \cap \Gamma) \subset \{x = 0\}$ is immediately satisfied and that a comparison with the general discussion in Sect.2 shows that $\epsilon = -1$ and $\epsilon = 1$ in the diffractive and gliding cases respectively. It is also important to remember that ${}^b\chi$ is essentially obtained by extending an appropriately chosen equivalence of glancing hypersurfaces (or rather its restriction to the boundary, χ_∂):

$$\chi : \Gamma_0 \longrightarrow T^*\mathbb{R}^{n+1} \setminus 0, \ \Gamma_0 \subset T^*\mathbb{R}^{n+1} \setminus 0, \ \jmath(\chi(m)) = {}^b\chi(\jmath(m)).$$

For χ we immediately have $\chi^*p = a(x,y;\xi,\eta)(\xi^2 + \epsilon x\eta_1^2 - \eta_1\eta_n)$, where a is homogeneous of degree 0 and non-zero in Γ_0. The corresponding statement for ${}^b\chi$ does not however follow immediately from Proposition A.1. Nevertheless we have

Proposition A.2. *Let $p \in S^{2,2}_{\mathrm{hg}}({}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0)$ satisfy*

$$x^2 p|_{\lambda^2 - x^3\eta_1^2 - x^2\eta_n\eta_1 = 0}, \quad d\jmath^*p(m_0) \neq 0.$$

Then, for some conic neighbourhood of $m_0 = (0,(0,1,0,\cdots,0))$, $\Gamma_1 \subset T^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0$, there exists a $b-$canonical transformation

$${}^b\chi_1 : \Gamma_1 \longrightarrow {}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0$$

such that

$${}^b\chi_1^* p = c(\xi^2 - x\eta_1^2 - \eta_n\eta_1), \quad c \in C^\infty({}^bT^*\mathbb{R}^{n+1}_+ \setminus 0), \quad c \neq 0 \text{ in } \Gamma_1. \tag{A.1}$$

Proof. By the Malgrange preparation theorem (see [14], Sect.7.5) we can write

$$x^2 p = r_0(x,y,\eta) + r_1(x,y,\eta)\lambda + (\lambda^2 - x^3\eta_1 - x^2\eta_1\eta_n)s(x,y,\lambda,\eta), \tag{A.2}$$

for $(x,y;\lambda,\eta)$ in some conic neighbourhood of $\jmath(m_0)$. On the other hand, since $p \in S^{2,2}_{hg}$,

$$x^2 p = x^2 s_2(x,y,\lambda,\eta) + \lambda x s_1(x,y,\lambda,\eta) + \lambda^2 s_0(x,y,\lambda,\eta).$$

The differentiation of (A.2) in x and putting $x = \lambda = 0$ shows that $\partial_x r_0(0,y,\eta) = r_0(0,y\eta) = 0$. The comparison of the two expressions for $x^2 p$ gives $r_1(0,y,\eta)\lambda + \lambda^2 s(0,y,\lambda,\eta) = \lambda^2 s_0(0,y,\eta,\lambda)$ and consequently $r_1(0,y,\eta) \equiv 0$. Since $r_0 + r_1\lambda$ vanishes identically when $\lambda^2 = x^2(x\eta_1^2 + \eta_1\eta_n)$, it also follows that r_0 and r_1 vanish identically when $x\eta_1 + \eta_n \geq 0$ ($\eta_1 > 0$ near $\jmath(m_0)$). The assumption on p implies that $s(m_0) \neq 0$ and thus, in view of the above discussion, we can assume that sufficiently near m_0

$$p = \xi^2 - x\eta_1^2 - \eta_n\eta_1 + \rho_0(x, y, \eta) + \xi\rho_1(x, y, \eta)$$

where ρ_0 and ρ_1 vanish identically in $\eta_n + x\eta_1 \geq 0$.

To eliminate the last two terms in p, we construct a $b-$canonical transformation by the homotopy method. Thus we define

$$p_s = \xi^2 - x\eta_1^2 - \eta_n\eta_1 + s(\rho_0(x, y, \eta) + \xi\rho_1(x, y, \eta))$$

and we want to find a_s and b_s homogeneous of degree 0 in (λ, η) and such that

$$\frac{d}{ds}\left(\exp H_{b_s}\right)^*(a_s p)) = 0,$$

which is the same as

$$\left\{\frac{db_s}{ds}, a_s p_s\right\} + \frac{d}{ds}(a_s p_s) = 0.$$

By introducing $\alpha_s = \log a_s$ (the solution a_s is required to be strictly positive) this reduces to

$$\left\{\frac{db_s}{ds}, p_s\right\} + \frac{dp_s}{ds} = \left(-\left\{\frac{db_s}{ds}, \alpha_s\right\} - \frac{d\alpha_s}{ds}\right) p_s. \tag{A.3}$$

We easily see that, with $\eta_1 = 1$ for simplicity,

$$\begin{aligned} \jmath_* H_{p_s} &= 2\xi\left(\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial\lambda}\right) + x\frac{\partial}{\partial\lambda} - 2x\frac{\partial}{\partial y_1} \\ &\quad - sx\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\lambda} + s\rho_1\left(\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial\lambda}\right) + \widetilde{H}_{-\eta_1\eta_n + s(\rho_0 + \xi\rho_1)} \end{aligned}$$

where $\widetilde{H}$ is the tangential Hamilton vector field. From this we compute for $q_s \in C^\infty({}^bT^*\mathbb{R}^{n+1}_+ \setminus 0)$

$$\begin{aligned} -x\{q_s, p_s\} &= \left(\lambda\frac{\partial}{\partial x} + 2(\eta_n - s\rho_0(0, y, \eta))x\frac{\partial}{\partial\lambda} + s\rho_1(0, y, \eta)\left(x\frac{\partial}{\partial x} - \lambda\frac{\partial}{\partial\lambda}\right)\right) q_s \\ &\quad + p_s x\frac{\partial q_s}{\partial\lambda} + x\widetilde{H}_{-\eta_1\eta_n + s(\rho_0 + \xi\rho_1)} q_s + W_s q_s, \end{aligned}$$

where W_s is a vectorfield with coefficients vanishing to second order at $x = \lambda = 0$. With $q_s = db_s/ds$ in mind (see (A.3) we want to solve

$$-x\{q_s, p_s\} = x\frac{dp_s}{ds} + x\mu_s p_s \tag{A.4}$$

for some smooth $\mu_s = \mu_s(x, y, \lambda, \eta)$ so that q_s and μ_s vanish in $x\eta_1 + \eta_n \geq 0$.

To solve (A.4) we first construct the Taylor series of q_s and μ_s at $x = \lambda = 0$ and for that we need the following

Lemma A.1. *Let H_l be the space of real homogeneous polynomials of degree l in x, λ and let $A(t)$ be defined as*

$$A(t) = \lambda\frac{\partial}{\partial x} + tx\frac{\partial}{\partial\lambda} + f(t)\left(\lambda\frac{\partial}{\partial\lambda} - x\frac{\partial}{\partial x}\right),$$

with $f \in C^\infty(\mathbb{R}; \mathbb{R})$, $f(t) = \mathcal{O}(t^N)$, for all N as $t \to 0$.

Then for $t \neq 0$ and $k \in \mathbb{N}_0$

i) The linear transformation $A(t)|_{H_{2k+1}}$ is invertible, $\|(A(t)|_{H_{2k+1}})^{-1}\| = \mathcal{O}(|t|^{-1})$.

ii) Every $u \in H_{2k}$ can be written as

$$u = A(t)v(t) + c(t)(\lambda^2 - tx^2 - 2f(t)x\lambda)^k, \ c(t) \in \mathbb{R},$$

$\|(d/dt)^m v\| = \mathcal{O}(|t|^{-1-m})\|u\|$, $(d/dt)^m c = \mathcal{O}(|t|^{-1-m})\|u\|$, *for any fixed norms* $\| \bullet \| = \| \bullet \|_l$ *on* H_l.

Proof. To keep notation simple we will only consider the case $t < 0$. It is convenient to introduce the following change of variables

$$F(t)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{|t|}x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix},$$

so that

$$(F(t)^{-1})^* A(t) F(t)^* = \sqrt{|t|}\left[\lambda_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial \lambda_1} + \frac{f(t)}{\sqrt{|t|}}\left(\lambda_1 \frac{\partial}{\partial \lambda_1} - x_1 \frac{\partial}{\partial x_1}\right)\right]. \tag{A.5}$$

Let $\langle \bullet, \bullet \rangle_l$ denote the inner product on H_l obtained by restricting to $x^2 + \lambda^2 = 1$ and taking the L^2 inner product. We introduce a t-dependent inner product on H_l:

$$\langle F(t)^* v, F(t)^* w \rangle_{t,l} = \langle v, w \rangle_l.$$

It depends smoothly on t for $t < 0$ and satisfies

$$\langle v, w \rangle_l / C \le \langle v, w \rangle_{t,l} \le C|t|^{-1}\langle v, w \rangle_l, \ \ (\frac{d}{dt})^m \langle v, v \rangle_{t,l} = \mathcal{O}(|t|^{-1-m})\langle v, v \rangle_l.$$

Thus, we only need to consider $A_1(t)$ given by (A.5). If $l = 2k+1$, $A_1(t)|_{H_l}$ is invertible with the norm bounded by $|t|^{-1/2}(1 + o(1))$. Consequently, $A(t)|_{H_l}$ is invertible with the norm bounded by $|t|^{-1}(1 + o(1))$. When $l = 2k$, $A_1(t)|_{H_l}$ has a one dimensional kernel spanned by $(\lambda_1 + x_1^2 - 2f(t)/\sqrt{|t|}\lambda_1 x_1)^k$. Thus we can take the inverse of $A_1(t)$ restricted to the orthogonal complement of the kernel with respect to $\langle \bullet, \bullet \rangle_l$. Translating back to $A(t)$ that gives ii) with $c = \langle u, (\lambda^2 - tx^2 - 2f(t)x\lambda)^k \rangle_{l,t}$. ☐

Since $sdp_s/ds = x\rho_0 + \lambda\rho_1$ we can solve (A.4) in Taylor series:

$$q_s(x, \lambda; x, \eta) \sim \sum_{l=1}^{\infty} q_s^{(l)}(x, \lambda; y, \eta), \ \ q_s^{(l)}(\bullet; y, \eta) \in H_l, \ q_s^{(l)} \equiv 0 \text{ in } \eta_n \ge 0, \tag{A.6}$$

$$\mu_s(x, \lambda; x, \eta) \sim \sum_{l=1}^{\infty} \mu_s^{(l)}(x, \lambda; y, \eta), \ \ \mu_s^{(l)}(\bullet; y, \eta) \in H_l, \ \mu_s^{(l)} \equiv 0 \text{ in } \eta_n \ge 0. \tag{A.7}$$

In fact, we apply Lemma A.1 with $t = \eta_n - s\rho_0(0, y, \eta)$ and $f(t) = s\rho_1(0, y, \eta)/2$ (treating η_n as the only variable and the remaining ones as parameters), set $q_s^{(0)} = 0$ and

$$q_s^{(1)}(x, \lambda; y, \eta) = (2A(t)|H_1)^{-1}(x\rho_0(0, 0; y, \eta) + \lambda\rho_1(0, 0; y, \eta)), \ \ \mu_s^0 = \partial_\lambda q_s^{(1)}.$$

We then continue solving

$$\begin{aligned} 2A(t)q_s^{(2k)} &= \gamma_s^{(2k)} + c_s^{(2k)}(\lambda^2 - tx^2 - 2f(t)x\lambda)^k, \\ &\qquad \langle \gamma_s^{(2k)}, (\lambda^2 - tx^2 - 2f(t)x\lambda)^k \rangle_{t,2k} = 0, \\ 2A(t)q_s^{(2k+1)} &= \gamma_s^{(2k+1)}, \ \ c_s^{(2k+1)} = 0. \end{aligned}$$

where $\gamma_s^{(l)}$ are obtained from $q_s^{(m)}$ with $m < l$. The corresponding $\mu_s^{(l)}$ are given by

$$\mu_s^{(l)} = c_s^{(l)} x(\lambda^2 - tx^2 - 2f(t)x\lambda)^{\frac{l-2}{2}} + \partial_\lambda q_s^{(l+1)}. \tag{A.8}$$

The crucial observations are that

$$\lambda^2 - tx^2 - 2f(t)x\lambda = x^2 p_s + \mathcal{O}(|x|^3 + |\lambda|^3),$$

and that the last term in the right hand side of (A.8) does not contribute to $\gamma_s^{(\bullet)}$. The norm estimates in Lemma A.1 show that $q_s^{(l)}, \mu_s^{(l)}$ are C^∞ in (y, η) and the vanishing in $\eta_n \geq 0$ is immediate. The Borel lemma now gives q_s, μ_s, C^∞ in $(x, y; \lambda, \eta)$ and s, vanishing identically in $\eta_n + \eta_1 x \geq 0$ and such that (A.6) holds. Thus,

$$-x\{q_s, p_s\} = x\frac{dp_s}{ds} + xp_s\mu_s + \nu_s,$$

where $\nu_s = \nu_s(x, y; \lambda, \eta)$ vanishes in $\eta_n + \eta_1 x$ and to infinite order at $x = \lambda = 0$. But then $\nu_s/(x^2 p_s) \in C^\infty({}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0)$ and we have solved (A.4) with μ_s replaced by $\mu_s + x(\nu_s/(x^2 p_s))$.

Going back to (A.3) we now solve for b_s and α_s

$$\frac{db_s}{ds} = q_s, \quad b_0 = 0$$
$$\frac{d\alpha_s}{ds} - \{q_s, \alpha_s\} = -\mu_s, \quad \alpha_0 = 0.$$

Putting $a = \exp(\alpha_1)$ and ${}^b\chi_1 = \exp(H_{b_1})$ we obtain the desired $b-$canonical transformation:

$${}^b\chi_1^*(ap) = \xi^2 - x\eta_1^2 - \eta_n\eta_1.$$

□

Remark A.1. Any $b-$canonical transformation ${}^b\chi$ satisfies

$${}^b\chi^* x = ax, \quad {}^b\chi^*\lambda = b\lambda + cx, \quad a, b \neq 0$$

and thus induces a canonical transformation on $T^*\mathbb{R}^n \setminus 0$ (or a conic subset of it):

$$\chi_\partial(y; \eta) = (y', \eta') \iff {}^b\chi(0, y; 0, \eta) = (0, y'; 0, \eta').$$

Since the construction in the proof of Proposition A.2 gave $q_s(0, y; 0, \eta) \equiv 0$ and consequently $b_s(0, y; 0, \eta) \equiv 0$, ${}^b\chi_1$ satisfying (A.1) can be chosen so that

$$(\chi_1)_\partial = Id|_{\Gamma_0}, \quad \Gamma_0 = \{(y, \eta) \in T^*\mathbb{R}^n \setminus 0 : (0, y; 0, \eta) \in \Gamma_1\}.$$

This observation is very convenient in applications presented in Sect.7.

Combining the two propositions we obtain:

Theorem A.1. *The $b-$canonical transformation ${}^b\chi$ in Proposition A.1 can be chosen so that*

$${}^b\chi^* p = c(\xi^2 - x\eta_1^2 - \eta_1\eta_n),$$

with $c \in S^0_{\mathrm{hg}}({}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0)$, $c \neq 0$ in Γ.

We now want to consider transport equations and again we start by recalling some facts already contained, in a slightly different form, in Sect.4.4 of [35]:

Lemma A.2. *If $\Gamma_2 \subset \mathbb{R}^{n+1}_{x,y} \times \mathbb{R}^n_\eta$ is a conic neighbourhood of $m_0 = (0;(1,0,\cdots,0))$ (with respect to the $\mathbb{R}_+$ action on the last n variables) and $A, B \in S^0(\Gamma_2)$ then there exist a conic neighbourhood of m_0, $\Gamma_3 \subset \Gamma_2$ and $g, h, e_1, e_2 \in S^0(\Gamma_2)$ such that in Γ_3:*

$$\begin{pmatrix} -\partial_{y_n} - (2+\eta_1^{-1}\eta_n)\partial_{y_1} & 1 + 2(x+\eta_1^{-1}\eta_n)\partial_x \\ 2\partial_x & -\partial_{y_n} - (2+\eta_1^{-1}\eta_n)\partial_{y_1} \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

and the solution satisfies

$$g(m_0) = 1,\ \ h|_{x=0} \equiv 0,\ \ e_i \equiv 0 \text{ in } x\eta_1 + \eta_n \geq 0,\ \ e_i = \mathcal{O}(x^N),\ \ N \in \mathbb{N},\ \ x \to 0.$$

This lemma allows to solve the transport equations when the operator is the model one: $p = \xi^2 - x\eta_1^2 - \eta_1\eta_n$. Here we are interested in the following slight refinement of the standard procedure:

Proposition A.3. *If $p = \xi^2 - x\eta_1^2 - \eta_1\eta_n$ and $r \in S^{1,1}({}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0)$ then there exist a conic neighbourhood of $(0;(0,1,0,\cdots,0))$, $\Gamma \subset {}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0$ and $a \in S^0({}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0)$, $b \in S^{-1}({}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0)$ such that*

$$H_p a = r + pb \ \text{ in } \ \jmath^{-1}(\Gamma),$$

where we identified a, b, r with their pullbacks under $\jmath : T^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0 \to {}^bT^*\bar{\mathbb{R}}^{n+1}_+ \setminus 0$.

Proof. Arguing as in the beginning of the proof of Proposition A.2 (with the argument applied to xr), we see that for $(x, y, x\xi, \eta) \in \Gamma$,

$$r = r_1(x,y,\eta) + \xi r_0(x,y,\eta) + pb^\sharp(x,y;x\xi,\eta),$$

where $r_i \in S^i(\Gamma_2)$ with $\Gamma_2 \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$, a conic neighbourhood of $(0;(0,1,0,\cdots,0)$ and $b^\sharp \in S^{-1}(\Gamma)$.

Let us now apply Lemma A.2 with $A = \eta_1^{-1}r_1$ and $B = r_0$ to obtain g and h. A simple computation shows that

$$H_p(g + \xi\eta_1^{-1}h) = r_1 + \xi r_0 + 2p\eta_1^{-1}\partial_x h + e_1 + \xi\eta_1^{-1}e_2.$$

Since $h|_{x=0} = 0$ we can take $a = g + \lambda h/x$, while $b = b^\sharp + 2\partial_x h + x^2[(e_1 + \lambda\eta_1^{-1}e_2/x)/(x^2p)]$, where the last term is smooth as e_i vanish identically in $\eta_1 x + \eta_n \geq 0$ and to infinite order at $x = 0$. $\square$

Appendix B. b-Sobolev spaces

In this appendix we recall the definition of the $b-$Sobolev spaces on a manifold with corners and outline the proof of some of their more useful properties. We start with the case of a manifold with boundary, for a discussion of the characteristic operators used, see [25, 28] and also [14], Sect.18.3.

Let M be a manifold with boundary and let ν be a measure on M. Then we define

$$H^b_{(s),\nu}(M) = \{u; Au \in L^2_\nu(M) \text{ for all } A \in \Psi^s_b(M)\}. \tag{B.1}$$

When the measure ν is not specified and M is locally described by a subset of $[0,\infty)_r \times \mathbb{R}^{n-1}_y$, we take $\nu = dydr/r$ and denote the corresponding L^2_ν by $L^2_b(M)$. A logarithmic change of variables $t = \log r$ induces an isomorphism

$$H^b_s([0,\infty) \times \mathbb{R}^{n-1}) \longleftrightarrow H_{(s)}(\mathbb{R}^n) \tag{B.2}$$

and thus we easily conclude that $H^b_{(s),\nu}(M)$ is an interpolation space for any smooth measure ν.

More generally if M is a manifold with corners, $H^b_{(s),\nu}(M)$ can be defined by (B.1) where $\Psi^s_b(M)$ is the space of totally characteristic operators on a manifold with corners [31] (see also [28]) or, more directly, by (B.2) and a multiple logarithmic change of variables – see [29]. As for manifolds with boundaries (see [14], Appendix B) more options for defining Sobolev spaces are available. To describe them let us assume that M has a codimension two corner and near it, is described by

$$Z_2 = [0,\infty) \times [0,\infty) \times \mathbb{R}^{n-2}, \quad \text{with coordinates } (r,x,y) \in Z_2.$$

For that manifold we define

$$\bar{H}^b_{(s)}(Z_2) = \{u;\ \text{there exists } \tilde{u} \in H^b_{(s)}([0,\infty)\times\mathbb{R}^{n-1}) \text{ such that } \tilde{u}\restriction_{x>0} = u\},$$
$$H^b_{(s,m)}(Z_2) = \{u : U \in H_{(s,m)}(\mathbb{R}\times[0,\infty)\times\mathbb{R}^{n-2}) \text{ where } U(t,x,y) = u(\log r, x, y)\},$$

where $H_{(s,m)}$ is defined in Appendix B of [14]. We easily have the following adaptation of Theorem B.2.9 to this setting

Proposition B.1. *If $P \in \mathrm{Diff}^p_b([0,\infty)\times\mathbb{R}^{n-1})$ and $\{x=0\} \subset Z_2 \times \mathbb{R}^{n-2}$ is non-characteristic for P, i.e. ${}^b\sigma_p(P)\restriction_{{}^bN^*\{x=0\}}$ does not vanish, then*

$$u \in H^b_{(s_1,m_1)}(Z_2) \text{ and } Pu \in H^b_{(s_2-p,m_2)}(Z_2) \Longrightarrow u \in H^b_{(s,m)}(Z_2)$$

for $s+m \le s_i + m_i,\ s \le s_2$.

If M is a manifold with corners and

$$\mathcal{R} = R_0 \sqcup \cdots \sqcup R_m = \left\{ \bigcup_i R_i \setminus \left(\bigcup_{i\neq j} R_i \cap R_j \right), R_i \cap R_j \setminus \left(\bigcup_k R_i \cap R_j \cap R_k \right), \cdots \right\} \tag{B.3}$$

is a variety of cleanly intersecting smooth submanifolds of M, $R_0 = \partial M$, we define the conormal space $I_k L^2_\nu(M,\mathcal{R})$ by (1.1) using all vector fields tangent to $\mathcal{R}$ for $\mathcal{V}$. By succesively blowing up all intersections (see [28]) and then the submanifolds $R_i, i>0$, we obtain the resolution M_1:

$$M_1 \xrightarrow{\gamma} M,$$

where M_1 is a manifold with corners – see Fig.13 for an example. We then clearly see that

$$I_k L^2_\nu(M,\mathcal{R}) \xleftrightarrow{\gamma_*} H^b_{(k),\nu_\gamma}(M_1), \quad \gamma_*\nu_\gamma = \nu,$$

and thus we obtain

Proposition B.2. *If M is a manifold with corners and $\mathcal{R}$ is given by* (B.3) *then $I_k L^2_\nu(M,\mathcal{R})$ are interpolation spaces in k.*

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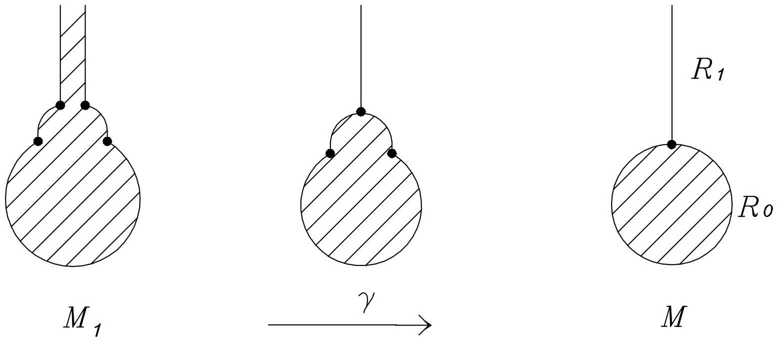

FIGURE 13. An example of the resolution $M_1 \xrightarrow{\gamma} M$

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