PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS WITH FIBRED BOUNDARIES

RAFE MAZZEO† AND RICHARD B. MELROSE*

Respectfully dedicate to Professor M. Sato on the occasion of his 70th birthday

Abstract. Let X be a compact manifold with boundary. Suppose that the boundary is fibred, $\phi : \partial X \longrightarrow Y$, and let $x \in \mathcal{C}^\infty(X)$ be a boundary defining function. This data fixes the space of 'fibred cusp' vector fields, consisting of those vector fields V on X satisfying $Vx = O(x^2)$ and which are tangent to the fibres of ϕ; it is a Lie algebra and $\mathcal{C}^\infty(X)$ module. This Lie algebra is quantized to the 'small calculus' of pseudodifferential operators $\Psi^*_\Phi(X)$. Mapping properties including boundedness, regularity, Fredholm condition and symbolic maps are discussed for this calculus. The spectrum of the Laplacian of an 'exact fibred cusp' metric is analyzed as is the wavefront set associated to the calculus.

Introduction

Algebras of pseudodifferential operators can be used to investigate local regularity of solutions to partial differential equations and to relate such local matters to more global properties. On a compact manifold with boundary there are a number of different natural algebras of pseudodifferential operators which generalize the 'standard' algebra of pseudodifferential operators on a compact manifold without boundary. Amongst these are the calculus of b-pseudodifferential operators [11] (b=boundary), [14], the scattering calculus [15] and the uniformly degenerate (or zero) calculus [7] and [9]. The distinction between the terms 'calculus' and 'algebra' is not great here. The former is preferred because all of the algebras we discuss have natural, and useful, extensions to somewhat larger spaces of operators in which not every pair of elements can be composed. If the manifold has more structure, for example if its boundary admits a fibration, then there are other possibilities, such as the edge calculus [8] which interpolates between the b and uniformly degenerate calculi. In this paper we shall discuss another algebra of this general type; it is associated to a fibration of the boundary and a choice of boundary defining function up to second order at the boundary, or more precisely to a trivialization of the conormal bundle to the boundary over each fibre. The extreme cases, in terms of the fibre dimension of the fibration, of this algebra correspond to the 'cusp' algebra, of operators naturally associated to (finite volume) hyperbolic cusps, and the scattering algebra, of operators associated to Euclidean scattering theory.

The purpose of this paper is to give a concise yet complete treatment of this 'fibred-cusp' algebra, along with a few of the most basic consequences. More sophisticated applications will be taken up elsewhere. In this introduction we shall

---

This research was supported in part by the National Science Foundation under an NYI and grants DMS-9626382 (†) and DMS-9306389 (§). Manuscript available from http://www-math.mit.edu/∼rbm/rbm-home.html.

give an outline of some of the salient features of the algebra which will be proved in full later in the paper.

Let X be a compact $\mathcal{C}^\infty$ manifold with boundary and suppose that the boundary has a smooth fibration

$$\phi : \partial X \longrightarrow Y, \tag{1}$$

where Y is the space of fibres. Suppose also that $x \in \mathcal{C}^\infty(X)$ is a choice of boundary defining function, i.e. $x \geq 0$, $\partial X = \{x = 0\}$ and $dx \neq 0$ at ∂X. In particular, x fixes a trivialization of the conormal bundle to the boundary. Associated with this structure is the space of *fibred cusp* vector fields

$$\begin{aligned} \mathcal{V}_\Phi(X) = \big\{V \in \mathcal{C}^\infty(X;TX); Vx \in x^2\mathcal{C}^\infty(X) \text{ and } V_p \\ \text{is tangent to } \phi^{-1}(\phi(p)) \ \forall\ p \in \partial X\big\}. \end{aligned} \tag{2}$$

As shown below, $\mathcal{V}_\Phi(X)$ is a Lie algebra and $\mathcal{C}^\infty(X)$ module which is projective in the sense that there is a $\mathcal{C}^\infty$ vector bundle ${}^\Phi TX$ over X with natural vector bundle map $\iota_\Phi : {}^\Phi TX \longrightarrow TX$, which is an isomorphism over $X^\circ = X \setminus \partial X$, and is such that

$$\mathcal{C}^\infty(X; {}^\Phi TX) = \iota_\Phi \circ \mathcal{V}_\Phi(X).$$

That is, $\mathcal{V}_\Phi(X)$ can be naturally identified with $\mathcal{C}^\infty(X; {}^\Phi TX)$. The identifier '$\Phi$' will be used to denote objects which are naturally associated to $\mathcal{V}_\Phi(X)$. Note that $\mathcal{V}_\Phi(X)$ determines the map ϕ, but does not completely determine the defining function x.

There are two extreme cases to keep in mind as a guide to this discussion, occurring when ϕ is one of the 'trivial' (or universal) fibrations. The first is when $Y = \{\mathrm{pt}\}$ and the second when $Y = \partial X$. In the former case, $\mathcal{V}_\Phi(X)$ determines, and is determined by, the defining function x up to the equivalence $x' \sim x$ if $x' = cx + x^2 g$, where $c > 0$ is constant and $g \in \mathcal{C}^\infty(X)$. This will be called the *cusp algebra.* In the latter case, the Lie algebra is independent of the choice of x and is called the *scattering algebra.* The algebra of pseudodifferential operators associated to it is discussed in [15] and [19] and in local form on $\mathbb{R}^n$ goes back at least to Shubin [22]. When X is the upper half-sphere, the interior of which may be identified with $\mathbb{R}^n$ via stereographic compactification $\mathbb{R}^n \hookrightarrow \mathbb{S}^n_+$, the scattering algebra is generated by the translation-invariant vector fields.

Since $\mathcal{V}_\Phi(X)$ is a Lie algebra and $\mathcal{C}^\infty(X)$ module it is natural to consider the enveloping algebra, $\mathrm{Diff}^*_\Phi(X)$, consisting of those operators on $\mathcal{C}^\infty(X)$ which can be written as finite sums of products of elements of $\mathcal{V}_\Phi(X)$ and $\mathcal{C}^\infty(X)$. It is filtered by the subspaces $\mathrm{Diff}^k_\Phi(X)$ which have elements expressible as sums of products involving at most k factors from $\mathcal{V}_\Phi(X)$. Let ${}^\Phi T^*X$ be the dual bundle to ${}^\Phi TX$ and let $P^k({}^\Phi T^*X) \subset \mathcal{C}^\infty({}^\Phi T^*X)$ be the space of functions which are homogeneous polynomials of degree k on the fibres. The principal symbol map extends from the interior to $\sigma_{\Phi,k} : \mathrm{Diff}^k_\Phi(X) \longrightarrow P^k({}^\Phi T^*X)$. This map is multiplicative and gives a short exact sequence delineating the filtration

$$0 \longrightarrow \mathrm{Diff}^{k-1}_\Phi(X) \hookrightarrow \mathrm{Diff}^k_\Phi(X) \xrightarrow{\sigma_{\Phi,k}} P^k({}^\Phi T^*X) \longrightarrow 0. \tag{3}$$

We *microlocalize* this algebra of differential operators to obtain the filtered algebra of fibred-cusp, or Φ-, pseudodifferential operators

$$\mathrm{Diff}^k_\Phi(X) \subset \Psi^k_\Phi(X)$$

where $\Psi^m_\Phi(X)$ is defined for each $m \in \mathbb{R}$. Again there is a multiplicative symbol map delineating the filtration

$$0 \longrightarrow \Psi^{m-1}_\Phi(X) \hookrightarrow \Psi^m_\Phi(X) \xrightarrow{\sigma_{\Phi,m}} S^m({}^\Phi T^*X)/S^{m-1}({}^\Phi T^*X) \longrightarrow 0 \tag{4}$$

where $S^m(E)$, for any vector bundle E, is the space of symbols of order m. The construction of $\Psi^m_\Phi(X)$ is effected geometrically. More specifically, these spaces of operators are characterized by the regularity properties of their Schwartz' kernels. These, in turn, are defined as conormal distributions on a space, X^2_Φ, which is a resolution of X^2. This resolution is obtained from the ordinary 'double space' through a sequence of blow-ups. One of the main facts about $\Psi^*_\Phi(X)$, that it is closed under composition, is proved using a resolution X^3_Φ of the ordinary triple space X^3, as we shall explain later.

Whether a particular element in $\Psi^*_\Phi(X)$ acts as a Fredholm operator, say on L^2, is no longer determined solely by the invertibility of its image under the symbol map (3) or (4). In fact, there is a second symbol map, the range of which is in general no longer a commutative algebra. To introduce this *normal operator*, we first describe the space of operators in which it lies.

If F is any compact manifold without boundary and W is a real vector space, then the space $\Psi^m(F \times W)$ of all pseudodifferential operators on the $\mathcal{C}^\infty$ manifold $F \times W$ is well defined. This is not an algebra because we have imposed no growth restrictions on the kernels. A special subclass consists of those elements which are invariant under translation in W, and therefore loosely speaking act by convolution in the W factor and as ordinary pseudodifferential operators in F. Now consider

$$\Psi^m_{\mathrm{sus}(W)}(F) \subset \Psi^m(F \times W) \tag{5}$$

consisting of those translation invariant operators with convolution kernels on $F^2 \times W$ which are rapidly decreasing with all derivatives at infinity. These spaces form a filtered algebra in the usual way and we call them the 'W-suspended pseudodifferential operators on F', even though they act on functions on $F \times W$. They are invariant under diffeomorphisms of F and linear transformation of W. This means that we can define $\Psi^m_{\mathrm{sus}(W)-\phi}(X'; W)$, where $\phi : X' \longrightarrow Y$ is any fibration, $W \longrightarrow Y$ a vector bundle, and $G = X' \times_Y W$ the fibre product, where elements are defined as in (5) on the fibres of G and depend smoothly on the base variable in Y.

If $\phi : \partial X \longrightarrow Y$ is the fibration (1), and $\iota_\Phi : {}^\Phi TX \longrightarrow TX$ is the natural inclusion map, set

$$ {}^\Phi N_p \partial X = \left\{ v \in {}^\Phi T_p X,\ p \in \partial X; \iota_\Phi(v) = 0 \right\}. \tag{6}$$

Although this is defined as a bundle over ∂X, in fact it is the lift to ∂X of a bundle, ${}^\Phi NY$, over Y,

$$ {}^\Phi N \partial X = \phi^*({}^\Phi NY), \tag{7}$$

and hence is of the form just described. The *normal homomorphism*, which we will define later, takes values in the corresponding space of suspended operators, and there is a multiplicative short exact sequence

$$0 \longrightarrow x\Psi^m_\Phi(X) \hookrightarrow \Psi^m_\Phi(X) \xrightarrow{N_\Phi} \Psi^m_{\mathrm{sus}({}^\Phi NY)-\phi}(\partial X) \longrightarrow 0.$$

The symbol and normal operator together are sufficient to capture the Fredholm property for these differential or pseudodifferential operators.

Theorem 1. *An element $P \in \Psi^0_\Phi(X)$ is Fredholm as an operator on $L^2(X)$ if and only if it is fully elliptic in the sense that its symbol $\sigma_{\Phi,0}$ is invertible and in addition its normal operator $N_\Phi(P)$ is invertible as an element of $\Psi^m_{\mathrm{sus}({}^\Phi NY)-\phi}(\partial X)$.*

We will state and prove a more general result for pseudodifferential operators of any order acting on sections of a vector bundle. This raises the following fundamental

Problem 1. *Find an explicit index formula for fully elliptic Φ-pseudodifferential operators in terms of the symbol and normal operator.*

This has been done in full generality in only one case, where $Y = \partial X$, i.e. for the scattering calculus. This is discussed briefly in [17], where it is reduced to the Atiyah-Singer theorem. In the other extreme case, where $Y = \{\mathrm{pt}\}$, the calculus is essentially that of manifolds with cylindrical ends. The index theorem in this setting for Dirac operators is that of Atiyah, Patodi and Singer [1]. There is a somewhat non-explicit index formula for general fully elliptic pseudodifferential operators here due to Piazza [20]. In [16] a definition of the eta invariant in this context is given, and [18] contains an index formula in terms of it.

Beyond these index questions, another reason for developing these calculi of operators is to analyze the regularity of solutions to related differential equations. We formalize this process using the notion of a *wavefront set*, which is defined by microlocal invertibility properties of Φ-pseudodifferential operators. In the analytic category the wavefront set (singular spectrum) was introduced by Sato, see [21]; in the $\mathcal{C}^\infty$ category it is due to Hörmander [6].

To describe this consider again the structure bundle ${}^\Phi TX$ and its dual ${}^\Phi T^*X$. The stereographic compactification of a vector space to a ball, or half-sphere, is linearly covariant, and so we can define the fibrewise compactification any vector bundle. Since X is a manifold with boundary the compactification ${}^\Phi\overline{T}^*X$ is a manifold with corners up to codimension two. The restriction to the boundary of the bundle ${}^\Phi T^*X$ has as quotient ${}^\Phi N^*\partial X$ which is, as noted above, naturally the lift of a bundle ${}^\Phi N^*Y$ over the base Y. This is the parameter space for the normal operator. The disjoint union of the part 'at infinity' of the bundle ${}^\Phi\overline{T}^*X$ and the compactification, ${}^\Phi\overline{N}^*Y$,

$$C_\Phi = {}^\Phi S^*X \sqcup {}^\Phi\overline{N}^*Y,$$

is the carrier of the Φ-wavefront set

$$\mathrm{WF}_\Phi(u) = \mathrm{WF}^\sigma_\Phi \sqcup \mathrm{WF}^\partial_\Phi \subset C_\Phi.$$

It has properties and utility similar to the usual wavefront set.

The authors thank András Vasy for helpful comments and Michael Singer for a careful reading of the manuscript.

1. Fibred cusp algebras

We begin our more detailed discussion by analyzing the space of vector fields defined by (2). Thus, X is a compact $\mathcal{C}^\infty$ manifold with boundary and as in (1), ϕ is a fibration of the boundary. If the boundary is not connected we denote by $M_1(X)$ the set of boundary components. Then each boundary hypersurface $H \in M_1(X)$ has a specified fibration $\phi_H : H \longrightarrow Y_H$. There need be no relationship between these fibrations. For the most part we shall simplify the discussion by supposing that ∂X is connected, but when confusion might arise in the general case we make a precise statement.

In addition to the fibration, we also suppose that a boundary defining function $x \in \mathcal{C}^\infty(X)$ is given. As will be discussed shortly, the structure we describe does not depend on all the information in x. Consider $\mathcal{V}_\Phi(X)$ defined by (2) which should now be written more carefully as

$$\mathcal{V}_\Phi(X) = \big\{V \in \mathcal{C}^\infty(X;TX); Vx \in x^2\mathcal{C}^\infty(X) \text{ and } V_p \\ \text{is tangent to } \phi_H^{-1}(\phi(p)) \;\forall\; p \in H, \;\forall\; H \in M_1(X)\big\}.$$

Lemma 1. *Suppose $p \in \partial X$ and $y_1, \dots, y_\ell$ are local coordinates in Y near $\phi(p)$. Let $\widetilde{y}_1, \dots, \widetilde{y}_\ell \in \mathcal{C}^\infty(X)$ be functions satisfying $\widetilde{y}_j = \phi^*(y_j)$ on ∂X near p and choose $k = n - \ell - 1$ functions $z_1, \dots, z_k$ such that $x, \widetilde{y}_j, z_i$ give local coordinates in X. Then near p, $\mathcal{V}_\Phi(X)$ is spanned by*

$$x^2\frac{\partial}{\partial x},\; x\frac{\partial}{\partial \tilde{y}_j},\; \frac{\partial}{\partial z_i}. \tag{1.1}$$

Proof. Since the differentials of x and the $\widetilde{y}_j$ must be independent at p there do indeed exists functions z_i completing them to a coordinate system. A general vector field on X is locally

$$V = a\frac{\partial}{\partial x} + \sum_{j=1}^{\ell} b_j\frac{\partial}{\partial \tilde{y}_j} + \sum_{i=1}^{k} c_i\frac{\partial}{\partial z_i}$$

for $\mathcal{C}^\infty$ coefficients a, b_j, c_i. Then $Vx = a$, so the first condition on V in (2) is that $a = O(x^2)$, i.e. $a = x^2a'$ where a' is $\mathcal{C}^\infty$ near p. Locally the fibres of ϕ are the surfaces $\tilde{y}$ =const, in $x = 0$. Thus if $V \in \mathcal{V}_\Phi(X)$ then $b_j = xb'_j$. This shows that the elements in (1.1) span $\mathcal{V}_\Phi(X)$ locally over $\mathcal{C}^\infty(X)$. □

Lemma 1 actually shows that $\mathcal{V}_\Phi(X)$ is projective, and this means that we can interpret this space of vector fields as the full set of sections of some vector bundle. For any $p \in X$ let $\mathcal{I}_p(X) \subset \mathcal{C}^\infty(X)$ be the ideal of functions vanishing at p. Then denote by $\mathcal{I}_p \cdot \mathcal{V}_\Phi(X) \subset \mathcal{V}_\Phi(X)$ the finite linear span of products, aV, for $a \in \mathcal{I}_p(X)$ and $V \in \mathcal{V}_\Phi(X)$, and set

$${}^\Phi T_pX = \mathcal{V}_\Phi(X)/\mathcal{I}_p \cdot \mathcal{V}_\Phi(X).$$

Lemma 2. *For each $p \in X$, ${}^\Phi T_pX$ is a vector space of dimension* $\dim X$, *and the disjoint union*

$${}^\Phi TX = \bigsqcup_{p \in X} {}^\Phi T_pX$$

has a natural structure as a smooth vector bundle over X. There is a natural linear map $\iota_p : {}^\Phi T_pX \longrightarrow T_pX$ which is an isomorphism when $p \in X^\circ = X \backslash \partial X$; these maps define a smooth bundle map $\iota : {}^\Phi TX \longrightarrow TX$ with the property that for every $\tilde{V} \in \mathcal{C}^\infty(X; {}^\Phi TX)$ there is a unique $V \in \mathcal{V}_\Phi(X) \subset \mathcal{C}^\infty(X;TX)$ such that

$$\iota_p\tilde{V}_p = V_p \;\forall\; p \in X^\circ.$$

Conversely, each $V \in \mathcal{V}_\Phi(X)$ defines a section $\tilde{V} \in \mathcal{C}^\infty(X; {}^\Phi TX)$.

Proof. Over the interior of X the elements of $\mathcal{V}_\Phi(X)$ are unconstrained, and so ${}^\Phi T_pX \equiv T_pX$ for $p \in X^\circ$. We write this identification as $\iota_p : {}^\Phi T_pX \longrightarrow T_pX$. Near a boundary point p we have shown that $V \in \mathcal{V}_\Phi(X)$ has a unique smooth decomposition in terms of the vector fields (1.1). Thus $V' \in \mathcal{I}_p \cdot \mathcal{V}_\Phi(X)$ if and only if its decomposition has coefficients vanishing at p. This means that (the residue classes of) $x^2\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial \tilde{y}_j}$ and $\frac{\partial}{\partial z_i}$ give a basis of ${}^\Phi T_pX$, and therefore this vector space has dimension $\dim X$. In fact, these sections clearly give ${}^\Phi TX$ the structure of a vector bundle near p, where any smooth section is (locally) given by an element of $\mathcal{V}_\Phi(X)$ and conversely. It remains only to show that this vector bundle structure is independent of the choice of the local coordinates. This follows simply by inserting the change of coordinate formula for vector fields into the basis (1.1). ☐

Generally we shall ignore the map ι and identify $\mathcal{V}_\Phi(X)$ with $\mathcal{C}^\infty(X; {}^\Phi TX)$ as this lemma permits us to do. As noted in the introduction

Lemma 3. *The space $\mathcal{V}_\Phi(X)$ is a Lie subalgebra of $\mathcal{C}^\infty(X; TX)$.*

Proof. If V, $W \in \mathcal{V}_\Phi(X)$, then by definition they are tangent to the fibres of ϕ in ∂X. Because tangency to a submanifold persists for commutators, $[V, W]$ also has this property. Similarly, since $Vx = x^2a$ and $Wx = x^2b$ for some functions $a, b \in \mathcal{C}^\infty(X)$,

$$\begin{aligned}[V,W]x &= V(Wx) - W(Vx) = V(x^2a) - W(x^2b)\\ &= x^2(Va - Wb) + 2x(a-b)(V-W)(x) = x^2(Va - Wb) + 2x^3(a-b)^2.\end{aligned}$$

☐

As noted above, the algebra $\mathcal{V}_\Phi(X)$ is determined by ϕ and the choice of a boundary defining function. Conversely, $\mathcal{V}_\Phi$ determines ϕ but it does not completely determine x. In fact two boundary defining functions x and x' determine the same Lie algebra $\mathcal{V}_\Phi(X)$, and hence the same 'boundary structure' relative to ϕ, if and only if $x' = \alpha x$, with $\alpha|_{\partial X} \in \phi^*\mathcal{C}^\infty(Y)$. Thus if we let $\mathcal{C}^\infty_\phi(X) \subset \mathcal{C}^\infty(X)$ be the space of smooth functions on X which are constant on each leaf of ϕ at the boundary then this means $x' \in x\mathcal{C}^\infty_\phi(X)$.

The Lie algebra $\mathcal{V}_\Phi(X)$ has a natural ideal, consisting of those elements which vanish at the boundary as vector fields in the usual sense. In terms of the basis (1.1), it is spanned by $x^2\partial_x$ and $x\partial_{\tilde{y}_j}$ and $x\partial_{z_i}$ near each boundary point and is unconstrained in the interior. Over the boundary it spans the subbundle (6). This ideal is the span over $\mathcal{C}^\infty(X)$ of a smaller subalgebra

$$\mathcal{W}_\Phi(X) = \{V \in \mathcal{V}_\Phi(X); V \in x\mathcal{C}^\infty(X, TX) \text{ and } Vx \in x\mathcal{C}^\infty_\phi(X)\}. \tag{1.2}$$

This latter condition is clearly independent of the choice of x defining $\mathcal{V}_\Phi(X)$, i.e. $\mathcal{W}_\Phi(X)$ *is* an invariantly defined subspace of the latter. Since $\mathcal{W}_\Phi(X)$ is a $\mathcal{C}^\infty_\phi(X)$-module (and not a $\mathcal{C}^\infty(X)$ module) the subbundle of ${}^\Phi T_{\partial X}X$ it defines is naturally the lift of a bundle from Y. This is the bundle ${}^\Phi NY$ in (7).

There is a direct representation of the fibre ${}^\Phi T^*_pX$, $p \in \partial X$, of the dual bundle which is useful later. For $p \in \partial X$ let $I_p(X) \subset \mathcal{C}^\infty_\phi(X)$ be the ideal of functions vanishing on the fibre through p and $J_p(X) \subset I_p(X)$ the smaller ideal of functions with restriction to $\phi^{-1}(\phi(p))$ vanishing to second order at p. Note that $I_p(X)$ only depends on the fibre through p but $J_p(X)$ depends also on the location of p within this fibre. If x is an admissible boundary defining function then there is a canonical isomorphism

$$^{\Phi}T_p^*X \equiv x^{-1}\mathcal{C}^\infty_\phi(X)/x^{-1}J_p(X)$$

given by applying $V \in \mathcal{V}_\Phi(X)$ and evaluating at p; this follows from (1.1).

Let $\mathrm{Diff}^m_\Phi(X)$ be the space of operators on $\mathcal{C}^\infty(X)$ generated by $\mathcal{C}^\infty(X)$ and products of up to m elements of $\mathcal{V}_\Phi(X)$. The local structure of these operators is easy to determine.

Lemma 4. *In the local coordinates near a boundary point described in Lemma 1, any $P \in \mathrm{Diff}^m_\Phi(X)$ may be written*

$$P = \sum_{|\alpha|+|\beta|+q\le m} p_{\alpha,\beta,q}(x,y,z)(x^2D_x)^q(xD_y)^\beta D_z^\alpha, \qquad D_t = \frac{1}{i}\frac{\partial}{\partial t}. \tag{1.3}$$

Conversely, if $P \in \mathrm{Diff}^m(X)$ and this holds in a neighbourhood of each boundary point, then $P \in \mathrm{Diff}^m_\Phi(X)$.

Proof. This follows by induction on m. Certainly (1.3) holds when $m = 0$. In general, $\mathrm{Diff}^{m+1}_\Phi(X)$ is the span of $\mathcal{V}_\Phi(X)\cdot\mathrm{Diff}^m_\Phi(X)$ and $\mathrm{Diff}^m_\Phi(X)$. Using the local representation of $\mathcal{V}_\Phi(X)$ given by (1.1) and the representation (1.3) for $\mathrm{Diff}^m_\Phi(X)$, the same result follows directly for $\mathrm{Diff}^{m+1}_\Phi(X)$. □

Note that the order of the various factors in (1.3) is immaterial, because changing it would just change the coefficients slightly. The main properties of this space of Φ-differential operators will be discussed in more detail once we have defined the space of Φ-pseudodifferential operators.

2. Φ-PSEUDODIFFERENTIAL OPERATORS

We now turn to the definition of the 'small' calculus of Φ-pseudodifferential operator. These can be thought of as 'symbolic' functions of the vector fields in $\mathcal{V}_\Phi(X)$ in the same sense that, by (1.3), the Φ-differential operators are polynomial functions in these vector fields. Our definition of this calculus is quite geometric; this has the virtue that many of the main properties we need to develop, in particular the fact that this space of operators is closed under composition, may be proved directly and also quite geometrically.

Following a general 'microlocalization' principle for algebras of this type, the operators in $\mathcal{V}_\Phi(X)$ will be characterized by the lifts of their Schwartz kernels from X^2 to a space X^2_Φ which is obtained by a resolution process, more specifically by blowing up a sequence of p-submanifolds in X^2. Here the p-submanifolds (for 'product') are those around which the manifold with corners has a product decomposition, they may be thought of as properly embedded. The point of this geometric resolution is that it encodes the approximate local homogeneities of Φ-differential operators, and so it is natural to define the Φ-pseudodifferential operators by requiring that their Schwartz kernels also have the same approximate local homogeneities, i.e. lift to well-behaved distributions on X^2_Φ. We refer to [3], [12], [9] and [4] for a discussion of the process of blowing up a p-submanifold in a manifold with corners.

As already noted, all our constructions proceed independently at each boundary hypersurface of X, and so it is sufficient to suppose that $\partial X \equiv H$ is connected.

The Schwartz kernel of any operator on $\mathcal{C}^\infty(X)$ is a distribution on X^2. Of course, we are particularly interested in the behaviour of these operators, and hence kernels, near the boundary. We use the notation

$$L(H) = H \times X,\ R(H) = X \times H$$

or simply L, R when H is understood. For any manifold with corners Z, let $M_k(Z)$ denote set of boundary components of codimension k. In particular $\{L, R\} = M_1(X^2)$. Because X is a manifold with boundary, X^2 has boundary components only up to codimension two. Amongst these, only the faces $B(H) \in M_2(X^2)$,

$$B = B(H) = H \times H \subset X^2,$$

which are the ones intersecting the diagonal, are of interest to us. The other manifold of primary importance in this discussion is the diagonal

$$D = \{(z, z) \in X^2\}.$$

An important feature of this geometry is that these submanifolds do not intersect normally. We resolve this by blowing up B to get the b-double space

$$X^2_b = [X^2; B];\ \beta^2_b :\ X^2_b \longrightarrow X^2. \tag{2.1}$$

This compact manifold with corners is obtained by taking the disjoint union of $X^2 \setminus B$ and the inward-pointing spherical normal bundle of B and endowing this set with the (unique) minimal $\mathcal{C}^\infty$ structure for which smooth functions on X^2 and polar coordinates in X^2 around B all lift to be smooth.

We next describe the lifts of the submanifolds of X^2; we shall use the same letters to denote the lifts but add a subscript 'b' when necessary to distinguish between a manifold and its lift.

First, the 'front face' of X^2_b, which is produced by the blow-up of B is denoted

$$B_b = (\beta^2_b)^{-1}(B) = SN^+B.$$

By definition it is a quarter circle bundle over B. In fact, since the fibred-cusp structure specifies the 1-jet of a defining function x for the boundary H of X, this bundle is naturally trivial over B

$$B_b = B \times [-1, 1]_s. \tag{2.2}$$

To see this, note that if x and x' are the lifts to X^2 of the given boundary defining function from the left and right factors of X, respectively, then NB is spanned by ∂_x and $\partial_{x'}$. The interior normal bundle N^+B is therefore $\{(p, a\partial_x + a'\partial_{x'}); a, a' \geq 0\}$ and it is then easy to check that (2.2) follows, if we use $s = \frac{a-a'}{a+a'}$.

Next consider L and R; the inverse images of these boundary faces under β^2_b contain B_b. We define instead their *lifts* to exclude the interior of the front face:

$$L_b = \mathrm{cl}\left((\beta^2_b)^{-1}(L)\backslash B_b\right) = \mathrm{cl}\left((\beta^2_b)^{-1}(L\backslash B)\right)$$

$$R_b = \mathrm{cl}\left((\beta^2_b)^{-1}(R)\backslash B_b\right) = \mathrm{cl}\left((\beta^2_b)^{-1}(R\backslash B)\right)$$

where cl denotes the closure. These are boundary hypersurfaces of X^2_b and all of the boundary hypersurfaces of X^2_b have been enumerated, so that

$$M_1(X^2_b) = \bigcup_{H \in M_1(X)} \{L_b(H), R_b(H), B_b(H)\}.$$

We also define the lifted diagonal

$$D_b = \mathrm{cl}\left((\beta_b^2)^{-1}(D\backslash D \cap B)\right) \subset X_b^2.$$

As noted earlier, the diagonal itself does not intersect the boundary normally. However, D_b is a closed embedded p-submanifold, and the only boundary hypersurfaces it meets are the diagonal front faces $B_b(H)$. In this sense the blow-up of X^2 to X_b^2 resolves the 'geometry' on X^2 consisting of the boundary faces and the diagonal. This blow-up is the basis for the direct definition of the b-calculus, see [14].

There are however further degeneracies, associated to the fibred-cusp algebra, which need to be resolved. These occur along the fibre diagonal of the front face $B = H \times H$, given by

$$\{(h, h') \in B;\ \phi(h) = \phi(h') \text{ in } Y\}.$$

Using the product decomposition (2.2) this lifts to the submanifold

$$\Phi = \Phi(H) \equiv \{(h, h', 0) \in B_b = B \times [-1, 1]_s; \phi(h) = \phi(h')\}, \tag{2.3}$$

which is an embedded, closed submanifold in the interior of B_b, hence is a p-submanifold of X_b^2. For any manifold with corners Z we denote by $\mathcal{V}_b(Z)$ the Lie algebra of smooth vector fields which are tangent to each of the boundary faces.

Proposition 1. *The Lie algebra $\mathcal{V}_b(X)$ lifts to X_b^2 from either factor to a Lie subalgebra of $\mathcal{V}_b(X_b^2)$ transversal to D_b. In each case the elements of the lift of $\mathcal{V}_\Phi(X)$ constitute the subset of the lift of $\mathcal{V}_b(X)$ consisting of those vector fields which are tangent to Φ.*

Proof. Using local coordinates x, y, z, x', y', z', as in Lemma 1, on both the left and right factors of X, gives coordinates

$$x', s = \frac{x - x'}{x + x'},\ \tilde{y}, \tilde{y}', z, z' \text{ in } X_b^2, \tag{2.4}$$

on X_b^2 valid near $s = 0$. The vector fields in (1.1) lift to

$$\frac{x'}{2}(1+s)^2\partial_s, x'\frac{1+s}{1-s}\frac{\partial}{\partial \tilde{y}_j}, \frac{\partial}{\partial z_j},$$

and these are clearly tangent to $\Phi = \{x' = 0,\ s = 0, \tilde{y} = \tilde{y}'\}$. On the other hand, the basic generating set of vector fields on $\mathcal{V}_b(X)$ is

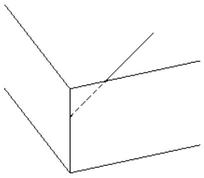
$$x\partial_x,\ \partial_{\tilde{y}_j},\ \partial_{z_i}; \tag{2.5}$$

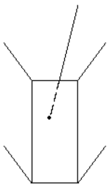
these lift to

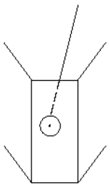
$$\frac{1}{2}(1 - s^2)\partial_s,\ \partial_{\tilde{y}_j},\ \partial_{z_j}$$

which are clearly transversal to $D_b = \{s = 0, \tilde{y} = \tilde{y}', z = z'\}$. If such a lift is tangent to Φ then it is easily seen to be the lift of an element of $\mathcal{V}_\Phi(X)$. ☐

It follows from (2.4) that Φ is the flow out of ∂D_b under the lifts of $\mathcal{V}_\Phi(X)$ from the left and right factors. It is therefore the minimal submanifold to which these lifted vector fields are tangent.







In the second (and final) stage of the fibred boundary blow-up we define

$$X_\Phi^2 = [X_b^2;\Phi], \qquad \beta_{\Phi-b} : X_\Phi^2 \longrightarrow X_b^2. \tag{2.6}$$

There is also a full blow-down map

$$\beta_\Phi = \beta_{\Phi-b} \circ \beta_b : X_\Phi^2 \longrightarrow X^2.$$

Lemma 5. *The Lie algebra $\mathcal{V}_\Phi(X)$ lifts, from either the left or right factor of X to a Lie subalgebra of $\mathcal{V}_b(X_\Phi^2)$. The diagonal D lifts to a closed embedded p-submanifold*

$$D_\Phi = \operatorname{cl} \beta_\Phi^{-1}(D \cap (X^2)^\circ) \tag{2.7}$$

and the lifted algebra is transversal to the lifted diagonal.

Proof. These statements are all trivial in the interior of X_Φ^2, which is diffeomorphic to the interior of X^2. Since all constructions are local near fibres of ϕ it suffices to consider the model product fibred-cusp structure

$$X = [0,1)_x \times Y_y \times F_z.$$

Near its front face, X_b^2 is also a product

$$X_b^2 \simeq [-1,1]_s \times [0,1)_{x+x'} \times Y^2 \times F^2. \tag{2.8}$$

The boundary fibre diagonal is the submanifold

$$\Phi = \{0\} \times \{0\} \times \mathrm{Diag}_Y \times F^2.$$

The second blow-up occurs only in the first three factors of (2.8). This effectively reduces the problem to the case $F = \{\mathrm{pt}\}$. In the first blow-up, (2.1), if $r = x + x'$ then

$$x = \frac{1}{2}(1+s)r,\ x' = \frac{1}{2}(1-s)r,\ s = \frac{x-x'}{x+x'}.$$

As noted in Proposition 1, the basis fields (1.1) lift smoothly to X_b^2 as

$$x^2\partial_x \longmapsto \frac{1}{4}(1+s)^2r^2\partial_r + \frac{1}{4}(1+s)^2(1-s)r\partial_s$$
$$x\partial_{\tilde{y}_j} \longmapsto \frac{1}{2}(1+s)r\partial_{\tilde{y}_j}.$$

Since the lifts of smooth coefficients are smooth, we can instead consider near Φ the simpler basis

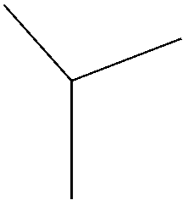
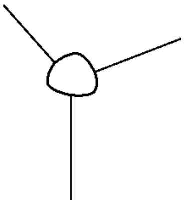
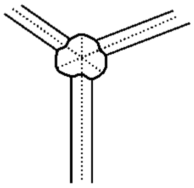
$$r(r\partial_r + (1-s)\partial_s),\ r\partial_{\tilde{y}_j}, \tag{2.9}$$

which also spans the lift of $\mathcal{V}_\Phi(X)$ over $\mathcal{C}^\infty(X_b^2)$.

Since these vector fields are tangent to $\Phi = \{r = 0, s = 0, y = \tilde{y}\}$, they lift smoothly under the blow-up of Φ. Near the lifted diagonal the variables r, $S = s/r$, $Y_j = (\tilde{y}_j - \tilde{y}'_j)/r$ give local coordinates and in terms of these the vector fields in (2.9) become

$$r^2\partial_r - rS\partial_S - rY\cdot\partial_Y + (1 - rS)\partial_S,\ \partial_{Y_j}.$$

Since the lifted diagonal itself is $\{S = 0,\ Y = 0\}$, this shows that the lift of $\mathcal{V}_\Phi(X)$ from the left factor (and hence by symmetry also from the right factor) is transversal to the lifted diagonal D_Φ. □

Notice that $D_\Phi \simeq X$, with the diffeomorphism given by $\pi \circ \beta^2_\Phi$ where π is either the left of the right projection from X^2 to X. The transversality in Lemma 5 shows that there are natural isomorphisms from the normal and conormal bundles of D_Φ in X^2_Φ to the Φ-tangent and cotangent bundles of X covering this identification,

$${}^\Phi TX \simeq ND_\Phi,\ {}^\Phi T^*X \simeq N^*D_\Phi.$$

Now that X^2_Φ has been defined, we consider the structure of the lifts of the Schwartz kernels of Φ-differential operators to this space. The fundamental case to understand is the identity operator. In local coordinates its Schwartz kernel (on X^2) is

$$\begin{aligned} K_{\mathrm{Id}} &= \delta(x - x')\delta(y - y')\delta(z - z')\,dx'dy'dz' \\ &\Longrightarrow u(x, y, z) = \int K_{\mathrm{Id}}u(x', y', z'). \end{aligned}$$

From now on we identify this distribution with the operator $K_{\mathrm{Id}} = \mathrm{Id}$. It is more convenient to write it in terms of a 'Φ-density' $(x')^{-(\ell+2)}dx'dy'dz'$ where $\ell = \dim Y$:

$$\begin{aligned} \mathrm{Id} &= (x')^{\ell+2}\delta(x - x')\delta(y - y')\delta(z - z')\nu'_\Phi \\ \nu'_\Phi &= (x')^{-\ell-2}dx'dy'dz'. \end{aligned} \tag{2.10}$$

Consider its behaviour when lifted from the interior of X^2 to X^2_b. It is supported on the diagonal D, so it suffices to consider a neighbourhood of $\{s = 0\}$ in (2.7). Since $x - x' = rs$ and $x + x' = r$, (2.10) becomes

$$\mathrm{Id} = r^{\ell+1}(1 - s)^{\ell+2}\delta(s)\delta(\tilde{y} - \tilde{y}')\delta(z - z')\beta^*_b(\nu'_\Phi).$$

Because $(1 - s)\delta(s) = \delta(s)$, the factor $(1 - s)^{\ell+2}$ may be dropped.

Next, consider the lift from X^2_b to X^2_Φ. Of course, the support of Id is contained in the Φ-diagonal D_Φ. In terms of the coordinates r, $S = s/r$, $Y = (\tilde{y} - \tilde{y}')/r$ valid near D_Φ (along with $\tilde{y}', z, z'$) we have

$$\mathrm{Id} = \delta(S)\delta(Y)\delta(z - z')\beta^*_\Phi(\nu'_\Phi). \tag{2.11}$$

The density factor is simply a smooth, non vanishing, section of the lift from the right factor of the density bundle $x^{-\ell-2}dx\,dy\,dz$; that is of the Φ density bundle.

For any embedded p-submanifold M in a manifold with corners X, the smooth δ-functions on M are the elements of a space

$$\mathcal{D}^0(M) = \mathcal{C}^\infty(X) \cdot \mu$$

where μ is any non-vanishing δ-function with smooth coefficients, as in (2.11). The delta functions of order at most k are obtained by differentiation

$$\mathcal{D}^k(M) = \operatorname{Diff}^k(X) \cdot \mu.$$

In fact it is only necessary to differentiate across M. Thus if $\mathcal{V}$ is any Lie algebra of smooth vector fields which is transversal to M, in the sense that for any section of $T_M X \big/ TM = NM$ there is an element of $\mathcal{V}$ which projects to it along M, then

$$\mathcal{D}^k(M) = \sum_{j \le k} \mathcal{V}^j \cdot \mu.$$

By Lemma 5, $\mathcal{V}_\Phi$ lifts to X^2_Φ to give such a Lie algebra transversal to D_Φ. Thus

$$\mathcal{D}^k(D_\Phi) \otimes \nu'_\Phi = \operatorname{Diff}^k_\Phi \cdot \operatorname{Id}. \tag{2.12}$$

The choice of initial smooth density on M is of course irrelevant; moreover, since it is in the right factor, the differentiations on the right in (2.12) do not affect it. The space on the right here is, by definition, the space of Schwartz kernels of $\operatorname{Diff}^k_\Phi(X)$, and so we conclude

Proposition 2. *The Schwartz' kernels of the elements of* $\operatorname{Diff}^k_\Phi(X)$ *lift to* X^2_Φ *to be precisely the space* $\mathcal{D}^k(D_\Phi) \cdot \nu'_\Phi$ *of all smooth* δ*-functions on* D_Φ *up to order* k *with a right* Φ*-density factor.*

This result can be extended directly to operators on other bundles. Thus if E and F are vector bundles over X then $\operatorname{Diff}^k_\Phi(X; E, F)$ consists of all the differential operators from sections of E to sections of F which are given by matrices with elements in $\operatorname{Diff}^k_\Phi(X)$ in local trivializations. It follows that

$$\operatorname{Diff}^k_\Phi(X; E, F) = \mathcal{C}^\infty(X; F) \cdot \operatorname{Diff}^k_\Phi(X) \cdot \mathcal{C}^\infty(X; E^*).$$

The space on the right here is the finite linear span of (ordered) products of elements from each of the three component spaces, hence is simply the tensor product over $\mathcal{C}^\infty(X)$. If $\operatorname{Hom}(E, F)$ is the bundle over X^2 with fibre $\hom(E_p, F_{p'}) = E^*_p \otimes F_{p'}$ at (p, p') then it is also true that

$$\begin{aligned} \operatorname{Diff}^k_\Phi(X; E, F) &= \mathcal{D}^k(D_\Phi) \cdot \mathcal{C}^\infty(X^2_\Phi; \beta^*_\Phi \operatorname{Hom}(E, F) \otimes {}^\Phi\Omega') \\ &= \mathcal{D}^k(D_\Phi) \cdot \beta^*_\Phi \mathcal{C}^\infty(X^2_\Phi; \operatorname{Hom}(E, F) \otimes {}^\Phi\Omega'), \end{aligned} \tag{2.13}$$

where ${}^\Phi\Omega'$ is the lift of the Φ-density bundle from the right factor.

This gives the following normalization.

Corollary 1. *The lifts to* X^2_Φ *of Schwartz kernels of elements of* $\operatorname{Diff}^k_\Phi(X; E, F)$ *coincides with the space*

$$\mathcal{D}^k(D_\Phi) \cdot \mathcal{C}^\infty\left(X^2; \operatorname{Hom}(E; F) \otimes {}^\Phi\Omega'\right). \tag{2.14}$$

On a manifold without boundary this identification of the kernels of differential operators becomes

$$\operatorname{Diff}^k(X'; E.F) \equiv \mathcal{D}^k(D) \cdot \mathcal{C}^\infty\left(X^2; \operatorname{Hom}(E; F) \otimes \Omega'\right),$$

where again Ω' the density bundle lifted from the right factor. To obtain the space of pseudodifferential operators in the boundaryless case, one replaces $\mathcal{D}^k(D)$, which is the space of polynomials in all smooth vector fields, by $I^k(X^2, D)$, the space of conormal distributions, which may be thought of as symbolic functions of these vector fields. Clearly

$$\mathcal{D}^k(D) \subset I^k(X^2, D),$$

and in fact, $\mathcal{D}^k(D)$ may be characterized as that subspace of conormal distributions, whose elements have supports contained in D.

Although conormal distributions are initially defined with respect to submanifolds of the interior, which do not intersect the boundary, we may define conormal distributions with respect to any interior p-submanifold, simply by requiring that they extend across the boundary as conormal distributions for some (hence any) extension of the submanifold. Thus $I^k(X^2_\Phi, D_\Phi; G)$ is defined for any vector bundle G over X^2_Φ, and its elements are smooth outside D_Φ. Letting $\equiv$ denote equality in Taylor series we define the *microlocalization* of $\operatorname{Diff}^k_\Phi(X)$ (or $\mathcal{V}_\Phi(x)$), to be the following space of Φ-pseudodifferential operators:

Definition 1. For any $m \in \mathbb{R}$ the space of Φ-pseudodifferential operators of order m (in the small calculus) is

$$\begin{aligned}(2.15)\quad \Psi^m_\Phi(X; E, F) = \big\{K \in I^m(X^2_\Phi, D_\Phi; \beta^*_\Phi\left(\operatorname{Hom}(E; F)\right) \otimes {}^\Phi\Omega'\big) ;\\ K \equiv 0 \text{ at } \partial X^2_\Phi \setminus \operatorname{ff}(X^2_\Phi)\big\},\end{aligned}$$

where $\operatorname{ff}(X^2_\Phi)$ is the front face produced by the blow up (2.5). There is some ambiguity in the definition of I^m, depending on whether symbols of type $1, 0$ or the smaller space of 1-step polyhomogeneous (i.e. classical) symbols are used. When absolutely necessary, we shall denote the polyhomogeneous space by

$$\Psi^m_{p\Phi}(X; E, F) \subset \Psi^m_\Phi(X; E, F).$$

Generally the statements we make are valid with either interpretation of I^m.

3. Action of Φ-pseudodifferential operators

Combining Corollary 1 and Definition 1 we have

$$(3.1)\qquad \operatorname{Diff}^k_\Phi(X; E, F) \subset \Psi^k_\Phi(X; E, F)\ \forall\ k \in \mathbb{N}$$

as spaces of kernels. We wish to interpret these Schwartz kernels as operators so that (3.1) still holds.

For simplicity take $E = F = \mathbb{C}$. Then the spaces in (2.14) and (2.15) can be rewritten as

$$\mathcal{D}^k(D_\Phi) \cdot \dot{\mathcal{C}}^\infty_{\operatorname{ff}}(X^2; \beta^*_\Phi {}^\Phi\Omega'))$$
$$I^m(X^2_\Phi, D_\Phi) \cdot \dot{\mathcal{C}}^\infty_{\operatorname{ff}}(X^2; \beta^*_\Phi {}^\Phi\Omega'),$$

respectively, where $\dot{\mathcal{C}}^\infty_{\operatorname{ff}}(X^2; \beta^*_\Phi {}^\Phi\Omega')$ is the space of sections vanishing to infinite order at all boundary faces except $\operatorname{ff}(X^2_\Phi)$.

Consider the lift of a non-vanishing density from X to the left factor of X^2 and then to X^2_Φ. Using the diffeomorphism which exchanges factors, the computation leading to (2.14) shows that the tensor product identification $\rho^*_L\Omega \otimes \rho^*_R {}^\Phi\Omega \equiv \Omega$

extends from the interior of X^2, and so of X^2_Φ, to give an isomorphism of spaces of sections

$$\dot{\mathcal{C}}^\infty_{\mathrm{ff}}\left(X^2_\Phi;\beta^*_\Phi(\rho^*_L\Omega\otimes\rho^*_R{}^\Phi\Omega)\right)\equiv\dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X^2_\Phi;\Omega). \tag{3.2}$$

That is, the singular Jacobian factors arising all occur at faces other than $\mathrm{ff}(X^2_\Phi)$ and have finite order singularities, which are absorbed by the infinite order vanishing at these faces.

Fixing any $0<v\in\mathcal{C}^\infty(X;\Omega)$, the action of $P\in\mathrm{Diff}^k_\Phi(X)$ on $u\in\mathcal{C}^\infty(X)$ can then be written

$$Pu\cdot v=(\pi_L)_*(P\cdot\pi^*_Lv\cdot\pi^*_Ru)$$

where (3.2) is used to identify the product on the right as a density on X^2_Φ and $\pi_L=\rho_L\circ\beta_\Phi$. Generalizing from this, we see that in order to define the action of Φ-pseudodifferential operators it suffices to establish the following result about push-forward:

Lemma 6. *Push-forward to the left factor defines a continuous linear map*

$$(\pi_L)_*:I^m(X^2_\Phi,D_\Phi)\cdot\dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X^2_\Phi;\Omega)\longrightarrow\mathcal{C}^\infty(X;\Omega).$$

and hence using (3.2)

$$\begin{aligned}(\pi_L)_*:I^m(X^2_\Phi,D_\Phi)\cdot\dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X^2_\Phi;\beta^*_\Phi\pi^*_R{}^\Phi\Omega)&\longrightarrow\mathcal{C}^\infty(X),\\ (\pi_L)_*(P\cdot\pi^*_Lv)&=(\pi_L)_*(P)\cdot v.\end{aligned} \tag{3.3}$$

Proof. This is a special case of the push-forward theorems discussed in [13]. To apply these theorems, we need to know that π_L and π_R are b-fibrations, and this is established below. The singularities of the kernel of D_Φ are integrated out since π_L is a smooth map which is transversal to the lifted diagonal. This transversality follows from Lemma 5 which shows that the lift of $\mathcal{V}_\Phi(X)$ from the right factor is transversal to D_Φ and spans the null space of the differential of π_L. From the general properties of conormal distributions

$$(\pi_L)_*:I^m(X^2_\Phi,D_\Phi)\cdot\mathcal{C}^\infty_c(U;\Omega)\longrightarrow\mathcal{C}^\infty(X)$$

if U is a small neighbourhood of the diagonal. For such a neighbourhood $\mathcal{C}^\infty_c(U)\subset\dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X^2_\Phi)$ so it suffices to consider the case $m=-\infty$, i.e. to show that

$$(\pi_L)_*:\dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X^2_\Phi;\Omega)\longrightarrow\mathcal{C}^\infty(X;\Omega). \tag{3.4}$$

Notice that (3.4) is not quite trivial since it is *not* the case that $(\pi_L)_*$ maps $\mathcal{C}^\infty(X^2_\Phi;\Omega)$ into $\mathcal{C}^\infty(X;\Omega)$. As discussed in [13] a result such as (3.4) follows from two facts

$$\begin{gathered}\pi_L\text{ is a b-fibration and if }H\in M_1(X)\text{ then}\\ \pi_L^{-1}(H)\cap\mathrm{ff}\ \text{ is a boundary hypersurface of }X^2_\Phi.\end{gathered}$$

The second condition here just means that $f\in\dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X^2_\Phi)$ vanishes to infinite order on all of $\pi_L^{-1}(H)$ except for the one boundary hypersurface, which is the front face corresponding to H in (2.5).

Thus it is only necessary to show that $\pi_L=\rho_L\circ\beta_\Phi$ *is* a b-fibration (for the definition of this and other terms here we refer to [13]). Both ρ_L and β_Φ are surjective b-maps and b-submersions, and so π_L is also a b-submersion. It remains only to see that no boundary hypersurface of X^2_Φ is mapped into a boundary face of codimension two or more in X, but since X does not have any such faces this is automatically the case. □

Tensoring with the general coefficient bundle we deduce the elementary mapping properties of Φ-pseudodifferential operators.

Proposition 3. *Using the identification* (3.2)*, each element* $A \in \Psi^m_\Phi(X;E,F)$ *defines a continuous linear operator*

$$A : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;F) \tag{3.5}$$

which restricts to $A : \dot{\mathcal{C}}^\infty(X;E) \longrightarrow \dot{\mathcal{C}}^\infty(X;F)$ *and extends by continuity in the distributional topologies to*

$$\begin{gathered} A : \mathcal{C}^{-\infty}(X;E) \longrightarrow \mathcal{C}^{-\infty}(X;F) \text{ and} \\ A : \dot{\mathcal{C}}^{-\infty}(X;E) \longrightarrow \dot{\mathcal{C}}^{-\infty}(X;F). \end{gathered} \tag{3.6}$$

These actions are consistent with the inclusion (3.1)*.*

Proof. The discussion above proves (3.5), since any element can be decomposed as a finite sum of products

$$\Psi^m_\Phi(X;E.F) \equiv \mathcal{C}^\infty(X;F) \cdot \Psi^m_\Phi(X) \cdot \mathcal{C}^\infty(X;E^*).$$

That $Au \in \dot{\mathcal{C}}^\infty(X;F)$ if $u \in \dot{\mathcal{C}}^\infty(X;E)$ follows from the observation that if $\phi \in \mathcal{C}^\infty(X)$ and $u \in \mathcal{C}^\infty(X;E)$ then

$$P(\phi u) = (\pi_R^*\phi \cdot P)u$$

where, directly from Definition 1, $\pi_R^*\phi \cdot P \in \Psi^m_\Phi(X;E,F)$ if $P \in \Psi^m_\Phi(X;E,F)$. Now, if $u \in \dot{\mathcal{C}}^\infty(X;E)$ it can be written as a finite sum, $u = \sum_j \phi_j u_j, u_j \in \mathcal{C}^\infty(X;E)$ and $\phi_j \in \dot{\mathcal{C}}^\infty(X)$.

If $\phi \in \dot{\mathcal{C}}^\infty(X)$ then

$$\pi_R^*\phi \cdot \dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X^2_\Phi) \subset \dot{\mathcal{C}}^\infty(X^2_\Phi);$$

the extra vanishing at $\mathrm{ff}(X^2_\Phi)$ comes from the first factor. From this it follows that

$$\pi_L^*(x^{-k}) \cdot \pi_R^*\phi P \in \Psi^m_\Phi(X;E,F) \ \forall\ k \in \mathbb{N}, \quad \forall\ \phi \in \dot{\mathcal{C}}^\infty(X),$$

hence $x^{-k}Pu \in \mathcal{C}^\infty(X;F)$, i.e. $Pu \in \dot{\mathcal{C}}^\infty(X;F)$.

Again from Definition 1, the formal adjoint of $P \in \Psi^m_\Phi(X;E,F)$ with respect to smooth inner products on E and F and a density on X is an element of $\Psi^m_\Phi(X;F,E)$. Thus the mapping properties (3.6) follow by duality. □

The singular function x/x' on X^2, where $x \in \mathcal{C}^\infty(X)$ is a boundary defining function on the left factor and x' is the same function on the right factor, lifts to be $\mathcal{C}^\infty$ up to the interior of the front face of X^2_b, and hence up to the front face of X^2_Φ. Since it has only a finite order singularity at the other boundary hypersurfaces, $\beta^*_\Phi(x/x')$ is a multiplier on $\Psi^m_\Phi(X;E,F)$. This means in particular that

$$\begin{gathered} P_\partial : \mathcal{C}^\infty(\partial X;E) \longrightarrow \mathcal{C}^\infty(\partial X;F) \\ P_\partial u = P\tilde{u}\big|_{\partial X}, \tilde{u} \in \mathcal{C}^\infty(X;E) \text{ with } u = \tilde{u}\big|_{\partial X} \end{gathered} \tag{3.7}$$

is well defined, regardless of the extension $\tilde{u}$ of u. This corresponds to the map obtained by restricting an element of $\mathcal{V}_\Phi(X)$ to the boundary. Below it is augmented appropriately to define the normal operator, which is the boundary symbol in this context. Before doing this, however, we first discuss the ordinary symbol map for Φ-pseudodifferential operators.

For conormal distributions the symbol map

$$\begin{gathered} I^m(X;G) \xrightarrow{\sigma_m} S^{[M]}(N^*G;\Omega^{\frac12}(N^*G)\otimes\pi^*(\Omega^{\frac12}X)),\\ M = m + \frac14\dim X - \frac12\dim G \end{gathered} \tag{3.8}$$

was introduced by Hörmander. It is normalized on half-densities. Here

$$S^{[M]}(\Lambda) = S^M(\Lambda)/S^{M-1}(\Lambda)$$

for any conic manifold Λ, is the quotient. For the case $G = D_\Phi \subset X^2_\Phi$ it has already been shown in Lemma 5 that $N^*D_\Phi \simeq {}^\Phi T^*X$. The (singular) symplectic form on ${}^\Phi T^*X$ trivializes the bundle ${}^\Phi\Omega = x^{-\ell-2}\Omega$ so (3.8) leads to the desired map

$$\sigma_{\Phi,m} : \Psi^m_\Phi(X;E,F) \longrightarrow S^{[m]}({}^\Phi T^*X;\pi^*\hom(E,F)). \tag{3.9}$$

This generalizes the symbol map for differential operators obtained by taking the leading part of (1.3) as a polynomial on ${}^\Phi T^*X$. It gives a short exact sequence

$$0 \longrightarrow \Psi^{m-1}_\Phi(X;E,F) \hookrightarrow \Psi^m_\Phi(X;E,F) \xrightarrow{\sigma_{\Phi,m}} S^{[m]}({}^\Phi T^*X;\pi^*\hom(E,F)).$$

For the polyhomogeneous spaces, $\Psi^m_{p\Phi}(X;E,F)$ the symbol becomes a homogeneous section of $\hom(E,F)$ lifted to ${}^\Phi T^*X\backslash 0$. Letting $Z = {}^\Phi S^*X$ be the boundary 'at infinity' of the radial compactification ${}^\Phi\overline{T}^*X$ this allows the symbol map to be written

$$\sigma_{p\Phi,m} : \Psi^m_{p\Phi}(X;E,F) \longrightarrow \mathcal{C}^\infty(Z;(N^*Z)^m\otimes\pi^*\hom(E,F)),\ Z = {}^\Phi S^*X.$$

Next let us note how the action of Φ-pseudodifferential operators can be written locally.

Proposition 4. *If $\chi \in \mathcal{C}^\infty(X)$ has support in a coordinate patch, U, based at a boundary point $p \in \partial X$ with coordinates $x, \tilde y, z$ as in Lemma 1 then the localized action of $P \in \Psi^m_\Phi(X)$ on $u \in \mathcal{C}^\infty_c(U)$ takes the form*

$$\chi P u = \int P_\chi(x,\tilde y,z,S,Y,z-z')\tilde v(x(1+xS),\tilde y - xY, z')dS\,dY\,dz' \tag{3.10}$$

where $\tilde v(x,y,z)$ is the coordinate representation of u and the kernel P_χ is the restriction to $U\times\mathbb{R}^N$ of a distribution on $\mathbb{R}^n\times\mathbb{R}^{n-k}\times\mathbb{R}^k$ which has compact support in the first and third variables, is conormal to $\{S=0,Y=0\}\times\{z=z'\}$ (which is the origin in the second two factors) and is rapidly decreasing with all derivatives as $|(S,Y)|\longrightarrow\infty$.

Proof. The kernel of the localized operator can be taken to be $\chi P\chi$. Any part of the kernel, on X^2_Φ, away from D_Φ and $\mathrm{ff}(X^2_\Phi)$ is smooth as a function on X^2 and vanishes rapidly at the boundary. Localizing on X^2_Φ this gives a smooth section of $\mathrm{Hom}(E,F)\otimes\pi_R^*\omega$ over X^2 vanishing rapidly at both boundaries; such a term can be written in the form (3.10) with P_χ both $\mathcal{C}^\infty$ and rapidly decreasing in S and Y.

Thus we can suppose that the kernel has support in a small neighbourhood of $D_\Phi\cup\mathrm{ff}(X^2_\Phi)$. The part in the interior has a conormal singularity at the diagonal and, since $x, x' \neq 0$, can again easily be written in the form (3.10). Thus we can suppose that the kernel has support in a small neighbourhood of $\mathrm{ff}(X^2_\Phi)$. Suppose initially that its support only meets the interior of $\mathrm{ff}(X^2_\Phi)$. In this region $x, \tilde y, z, S = \frac{x-x'}{x^2}, Y = \frac{\tilde y-\tilde y}{x}$ and z' gives coordinates on X^2_Φ. Thus (3.10) results by introducing

$x' = x(1 + Sx), y = \tilde{y}' - xY$. The kernel has compact support and only a conormal singularity at $S = 0$, $Y = 0$, $z = z'$ so (3.10) results.

The final term then is a smooth contribution to P supported near $\mathrm{ff}(X^2_\phi)$ and vanishing to infinite order at the other boundary faces nearby. Although the coordinates $x, \tilde{y}, z, S, Y, z - z'$ are *not* valid up to these adjacent boundaries a smooth function vanishing in Taylor series in this sense just corresponds to a Schwartz function in the variables S, Y, i.e. rapidly decreasing with all derivatives as $|(S, Y)| \longrightarrow \infty$. This proves the local representation (3.10). $\square$

This proposition does not quite give a complete local description of the action of P. However, if p, $p' \in \partial X$ lie in the same fibre of ϕ then $\phi(p) = \phi(p')$ lie in some coordinates patch in Y. Thus one can take 'consistent' coordinates near p and p' given by $x, \tilde{y}, z$ and $x, \tilde{y}, \tilde{z}$ respectively. The same argument as in the proposition gives a representation

$$\chi P u = \int \tilde{P}_\chi(x, y, z, S, Y, \tilde{z}')\tilde{v}(x(1 + xS), y - xY, \tilde{z}')dS\ dY\ d\tilde{z}' \tag{3.11}$$

where $\chi \in \mathcal{C}^\infty_c(X)$ has support in the coordinate patch near p and v has support in the coordinate patch near p'. The localized kernel $\tilde{P}_\chi$ is smooth in all variables, compactly supported in x, y, z and $\tilde{z}'$, and is rapidly decreasing with all derivatives as $|(S, Y)| \longrightarrow \infty$.

Other pieces of the kernel correspond either to points p, p' in different fibres over the boundary or where either, or both, of the pair lie in the interior. In these regions the localization of the kernel is a smooth section of X^2, except for a conormal singularity at the diagonal, and with rapid vanishing at any boundary.

The front fact of X^2_Φ is a bundle over ∂X with fibre $\phi^{-1}(\bar{y})^2 \times {}^\Phi\overline{N}_{\bar{y}}T$ over $\bar{y}$. The singular variables $Y = (\tilde{y} - \tilde{y}')/x$ and $S = (x - x')/x^2$ introduced above give linear coordinates in ${}^\Phi N_{\bar{y}}Y$, depending on the choice of admissible coordinates. Under a change of such coordinates Y and S transform linearly at $x = 0$, as a bundle transform on ${}^\Phi NY$, and as Taylor series at $x = 0$ vary polynomially:

$$(Y, S) \longrightarrow A(\tilde{y}') \cdot (Y, S) + \sum_{j \geq 1} x^j P_j(z, z', \tilde{y}', Y, S)$$

where the P_j smooth and are polynomials (without constant terms) in the variables Y, S.

4. Normal operator

Using the representations (3.10) and (3.11), we see that the restriction map in (3.7) is locally represented by

$$\begin{aligned} P_\partial u &= \int P_\partial(y, z, z - z')u(y, z')dz' \\ P_\partial(y, z, z - z') &= \iint P_\chi(0, y, z, S, Y, z - z')dS\, dY. \end{aligned} \tag{4.1}$$

This shows

Lemma 7. *The map* $P \longmapsto P_\partial$ *in* (3.7) *gives a surjective map*

$$\Psi^m_\Phi(X; E, F) \twoheadrightarrow \Psi^m_\phi(\partial X; E, F) \tag{4.2}$$

where $\Psi^m_\phi(\partial X; E, F)$ *is the space of pseudodifferential operators acting on the fibres of* $\phi : \partial X \longrightarrow Y$ *and depending smoothly on the base point.*

It is important to note that the null space of (4.2) consists of those elements for which the integral in (4.1) vanishes for all $y \in Y$ (and $z, z' \in F$). This is closely related to the question of determining which Φ-pseudodifferential operators are compact as operators on $L^2(X; E, F)$. For example, as will be seen below, the most obvious class of residual operators, the elements of $\Psi_\Phi^{-\infty}(X; E, F)$, are not all compact. The operators P for which $P_\partial = 0$ are also not, in general, compact. In fact, such an operator is compact only when the whole of the restriction of its kernel to $\mathrm{ff}(X^2_\Phi)$ vanishes, not just its fibre average as in (4.1).

We examine this issue by means of 'oscillatory testing'. To do this, fix a point $p \in \partial X$, and suppose $f \in \mathcal{C}^\infty(Y)$ is real-valued and has $df(\phi(p)) \neq 0$. Choose $\tilde f \in \mathcal{C}^\infty(X)$, also real-valued, with $\tilde f \restriction \partial X = \phi^* f$. Finally, take $\chi \in \mathcal{C}^\infty(X)$ such that

$$\chi \equiv 1 \quad \text{near} \quad \phi^{-1}(\phi(p))$$
$$d\tilde f \neq 0 \quad \text{on} \quad \phi(\operatorname{supp}\chi \cap \partial X)$$

and consider the 'oscillatory test section'

$$u_f = e^{i\tilde f/x}\chi u, \;\; u \in \mathcal{C}^\infty(X; E).$$

Lemma 8. *For an 'oscillatory test section' of this form, and for any operator* $P \in \Psi_\Phi^m(X; E, F)$,

$$P(e^{i\tilde f/x}\chi u) = e^{i\tilde f/x}\widetilde{P}u \tag{4.3}$$

with $\widetilde{P} \in \Psi_\Phi^m(X; E, F)$.

Proof. The kernel of $\widetilde{P}$ is $e^{-i\tilde f/x}Pe^{i\tilde f'/x'}\chi'$, using the obvious notation for variables and functions lifted from the left and right, respectively. It will suffice to show that

$$\begin{gathered}\text{the lift of } e^{-i\tilde f/x}\chi e^{i\tilde f'/x'}\chi' \text{ is } \mathcal{C}^\infty \text{ on the union of } (X^2)^\circ \\ \text{and } (\mathrm{ff}(X^2_\Phi))^\circ \text{ and multiplication by it preserves } \dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X^2_\Phi).\end{gathered} \tag{4.4}$$

Recall that the space in the final statement here consists of the smooth functions on X^2_Φ which vanish to infinite order at all boundary faces except $\mathrm{ff}(X^2_\Phi)$. The main point is to demonstrate the smoothness up to the interior of $\mathrm{ff}(X^2_\Phi)$. First set $x' = (1+s)x$, corresponding to the blow up (2.1), so that

$$\frac{\tilde f}{x} - \frac{\tilde f'}{x'} = \frac{f(\tilde y)}{x} + g(x, \tilde y, z) - \frac{f(\tilde y')}{x'} - \frac{g(x', \tilde y', z')}{1+s}.$$

Clearly we may restrict attention to the singular part

$$\frac{f(\tilde y)}{x} - \frac{f(\tilde y')}{x'} = \frac{f(\tilde y)}{x} - \frac{f(\tilde y - xY)}{(1+xS)x}.$$

Using a Taylor expansion and the fact that $d_y f \neq 0$ on $\operatorname{supp}\chi$, we see that this is $\mathcal{C}^\infty$ up to $x = 0$ as a function of $(S, Y) \in \mathbb{R}^{1+k}$. This proves the first part of (4.4); it also shows that this function has singularities only of finite order at all boundaries of X^2_Φ besides $\mathrm{ff}(X^2_\Phi)$. It is therefore a multiplier on $\dot{\mathcal{C}}^\infty_{\mathrm{ff}}(X^2_\Phi)$. In fact, this shows that

$$e^{-i\tilde f/x}\chi e^{i\tilde f'/x'}\chi' \text{ is a multiplier on } \Psi_\Phi^m(X; E, F),$$

and this proves the lemma. $\square$

The operator $\widetilde{P}$ in (4.3) depends not only on f, but also its extension $\tilde{f}$ and the cut off function χ. However, the restriction $\widetilde{P}_\partial$ depends only on f and χ. This follows from (4.1), for if we assume that χ is supported in a coordinate patch, then

$$\widetilde{P}_\partial(y,z,z-z') = \iint \exp(if'(y)\cdot Y + if(y)S) \\ P_\chi(0,y,z,S,Y,z-z')dS\,dY. \tag{4.5}$$

As noted in Lemma 7, the y variable enters here only as a parameter. A similar formula may be obtained from (3.11), and so we conclude that if $\chi(y)=1$ on the fibre $\phi^{-1}(y)$ then for that value of y, $\widetilde{P}_\partial(y,z,z-z') \in \Psi^m(\phi^{-1}(y);E,F)$ depends only on $f(y)$, $df(y)$ and P.

Lemma 9. *If we fix a point $(y,\eta)\in T^*Y$ and a constant $\tau\in\mathbb{R}$, then the indicial operator*

$$\widehat{P}(y,\tau,\eta)\in\Psi^m(\phi^{-1}(y);E,F)$$

is well-defined as the restriction to that fibre of $\widetilde{P}_\partial$, where $\widetilde{P}$ is defined by Lemma 8 with f chosen so that $f(y)=\tau$ and $d_yf(y)=\eta$. If $\widehat{P}(y,\tau,\eta)=0$ for every y,τ,η, then $P\in x\Psi^m_\Phi(X;E,F)$.

Proof. Only the last statement needs to be checked. From (4.5) it follows that if $\widehat{P}(y,\tau,\eta)\equiv 0$ then the Fourier transform of the kernel on each fibre of $\mathrm{ff}(X^2_\phi)$ over Y vanishes, hence $P \restriction \mathrm{ff}(X^2_\Phi)\equiv 0$ and this is equivalent to the existence of $Q\in\Psi^m_\phi(X;E,F)$ such that $P=xQ$ (or equivalently, the existence of some Q' with $P=Q'x'$). □

As is clear from (4.5), (3.10) and (3.11) the information carried in the operators $\widehat{P}$ as we let $y\in Y$, $f(y)=\tau$ and $\eta=df(y)$ vary determines the restriction of the kernel of P to $\mathrm{ff}(X^2_\Phi)$. We shall reorganize these individual operators into the family of *normal operators.* Before we may do this, however, we must first describe the algebra in which the normal family takes values.

For any compact manifold without boundary, M, and real vector space, V, $M\times V$ is a $\mathcal{C}^\infty$ manifold so the spaces $\Psi^m(M\times V)$ of pseudodifferential operators on $M\times V$ are well defined. These do not compose since the growth of the kernels is unrestricted at infinity in V. We consider the subspace

$$\Psi^m_{\mathrm{sus}(V)}(M)\subset\Psi^m(M\times V)$$

consisting of the translation-invariant elements with V-convolution kernels vanishing rapidly, with all derivatives at infinity. Thus If $A\in\Psi^m(M\times V)$ then $A\in\Psi^m_{\mathrm{sus}(V)}(M)$ if

$$\begin{aligned} &AT_v^*u = T_v^*Au\ \forall\ u\in\mathcal{C}^\infty_c(M\times V),\ v\in V\\ &\text{and } A:\mathcal{C}^{-\infty}_c(M\times V)\longrightarrow\mathcal{C}^{-\infty}_c(M\times V)+\mathcal{S}(M\times V).\end{aligned}\tag{4.6}$$

Here $T_v(m,w)=(m,w-v)$ is translation by v and $\mathcal{S}(M\times V)$ is the Schwartz space. The translation-invariance means that the kernel is of the form

$$A(m,m',v-v')\in\mathcal{C}^{-\infty}(M^2\times V^2;\Omega_R).$$

Then, with some abuse of notation in which A also stands for the V-convolution kernel, the second condition in (4.6) means that

$$A \in \mathcal{C}_c^{-\infty}(M^2 \times V; \Omega_R(M \times V)) + \mathcal{S}(M^2 \times V; \Omega_R(M \times V)) \tag{4.7}$$

where $\Omega_R(M \times V) = \pi_R^* \Omega$, $\pi_R : \ M^2 \times V \longrightarrow M \times V$ being projection onto the right factor of M.

For the case $V = \mathbb{R}$ this is the 'suspended algebra' considered in [18]. From (4.7) and the general properties of pseudodifferential operators it follows that $\Psi^*_{\mathrm{sus}(V)}(M)$ is an order-filtered algebra of operators

$$A : \ \mathcal{S}(M \times V) \longrightarrow \mathcal{S}(M \times V).$$

The notation here, $\Psi^*_{\mathrm{sus}(V)}(M)$, is to indicate that the algebra can be thought of as the 'V-suspended algebra of pseudodifferential operators on M.' In this sense the primary object is M. To have the corresponding algebra of operators acting on a vector bundle, the vector bundle should be defined over M and pulled back to $M \times V$. Thus if E is a bundle over M then

$$\Psi^*_{\mathrm{sus}(V)}(M; E) = \Psi^*_{\mathrm{sus}(V)}(M) \otimes_{\mathcal{C}^\infty(M^2)} \mathcal{C}^\infty(M^2; \mathrm{Hom}(E))$$

defines the algebra of operators

$$A : \ \mathcal{S}(M \times V; E) \longrightarrow \mathcal{S}(M \times V; E).$$

Directly from the definition, $\Psi^m_{\mathrm{sus}(V)}(M; E)$ is invariant under arbitrary diffeomorphism of M and linear transformations of V, as well as bundle transformations of E over M. This allows us to define the more general object we need.

Definition 2. Let $\phi : X' \longrightarrow Y$ be a fibration of compact manifolds, $E \longrightarrow X'$ a vector bundle and $V \longrightarrow Y$ a real vector bundle. Then the algebra of V-suspended fibre pseudodifferential operators on X', $\Psi^m_{\mathrm{sus}(V)-\phi}(X'; E)$ is the space of operators

$$A : \ \mathcal{S}(X' \times_Y V; E) \longrightarrow \mathcal{S}(X' \times_Y V; E)$$

which are local in Y and for any open set $O \subset Y$ over which ϕ and V are trivial reduce to a smoothly O-parametrized element of $\Psi^m_{\mathrm{sus}(V_y)}(\phi^{-1}(y); E)$.

Thus an element $A \in \Psi^m_{\mathrm{sus}(V)_\phi}(X'; E)$ has Schwartz kernel of the form

$$\begin{aligned} A(y, z, z', v) \in\ & \mathcal{C}_c^{-\infty}(X' \times_Y X' \times_Y V; \mathrm{Hom}(E) \otimes \Omega_R) \\ & + \mathcal{S}(X' \times_Y X' \times_Y V; \mathrm{Hom}(E) \otimes \Omega_R) \end{aligned}$$

where A is conormal with respect to the submanifold

$$D_\Phi \times \{0\} = \{(y, z, z, 0)\} \subset X' \times_Y X \times_Y V$$

which is the fibre diagonal. The action of A is given explicitly by

$$Au(y, z, v) = \int A(y, z, z', v - v') u(y, z', v') dz' dv'$$

since Ω_R is the lift from the right factor of the fibre density bundle of $X' \times_Y V$ as a fibration over Y.

Since the kernel is essentially a density on the fibres of V when all the variables are held fixed its Fourier transform is well defined and is a smooth function of the dual variables

$$\hat{A}(y, z, z', w^*) = \int e^{-iw^* \cdot w} A(y, z, z', w) dV.$$

For each $w^* \in V_y^*$ it is a pseudodifferential operator on the fibre $\phi^{-1}(y)$. This corresponds to the indicial operator in Lemma 9. In fact

Proposition 5. *For a Φ-structure on a compact manifold with boundary X, the indicial operators of Lemma 9 combine to give the Fourier transform of an element of $\Psi^m_{\mathrm{sus}(V)-\phi}(\partial X; E, F)$ where $V = {}^{\Phi}NY$ is the null bundle, on Y, of the restriction ${}^{\Phi}T_{\partial X}X \longrightarrow T\partial X$ and the resulting map, defining the normal operator, gives a short exact sequence*

$$0 \longrightarrow x\Psi^m_\Phi(X; E, F) \longrightarrow \Psi^m_\Phi(X; E, F) \xrightarrow{N_\Phi} \Psi^m_{\mathrm{sus}(V)-\phi}(\partial X; E, F) \longrightarrow 0. \tag{4.8}$$

Proof. From (4.5) we know that $\widehat{P}$ is the Fourier transform of the restriction of the kernel of P to the front fact, $\mathrm{ff}(X^2_\Phi)$. Thus, at the level of kernels, the map N_Φ is just restriction to $\mathrm{ff}(X^2_\Phi)$. This shows that the null space of N_Φ acting on $\Psi^m_\Phi(X; E, F)$ is precisely $x\Psi^m_\Phi(X; E.F)$ and directly from Definition 2, N_Φ is surjective as is (4.8). □

When we consider composition below it will be apparent that (4.8) is multiplicative.

5. Composition

It is relatively straightforward, if tedious, to check that the space $\Psi^*_\Phi(X; E)$ is an algebra by using the local representations (3.10) and (3.11). Instead we use a more conceptual approach that has the virtue of applying in rather general circumstances [10] and in the present circumstances to more general operators (i.e. 'larger calculi' with non-trivial boundary behaviour).

Thus our approach is to define a 'triple Φ product' X^3_Φ with maps back to the double product X^2_Φ defined in (2.6). The definition of X^3_Φ from X^3 proceeds by a chain of five blow ups. These are carried out independently at each of the boundary faces of X, so for simplicity we generally assume that $\partial X = H$ is connected. We shall use a notation for the boundary faces of X^3 similar to that used above for X^2. Namely if $H \in M(X)$ then set

$$L(H) = H \times X^2,\ M(H) = X \times H \times X,\ R(H) = X^2 \times H.$$

Thus in general

$$M_1(X^3) = \bigcup_{H \in M_1(X)} \{L(H),\ M(H),\ R(H)\}.$$

For the codimension two boundary faces we are only interested in those meeting the diagonal; we use the notation

$$S(H) = H \times H \times X,\ C(H) = H \times X \times H,\ F(H) = X \times H \times H.$$

Here 'S = second', 'C = composite' and 'F = first' arise from the relationship to the composition of operators. The only codimension three boundary faces meeting the diagonal are

$$T(H) = H^3 \subset M_3(X^3),$$

the 'triple' faces. In general we drop the reference to H.

The two stage blow up leading to X^3_b resolves the intersection of T, S, C and F:

$$X^3_b = [X^3; T; S; C; F]. \tag{5.1}$$

Although there are in principle four blow ups here, after the blow up of T the lifts of S, C and F are disjoint p-submanifolds so can be blown up in any order.

The remaining stages in the definition of X^3_Φ involve the blow up of various Φ-diagonal submanifolds. To see how these arise, consider the product

$$X \times X^2_b = [X^3; F].$$

The submanifold $\Phi \subset X^2_b$ defined in (2.3) therefore defines a submanifold we denote

$$\Phi_F = X \times \Phi \subset X \times X^2_b. \tag{5.2}$$

Now $T \subset F$ so, by the commutativity of blow-up in this setting (see [10], [3]), the order of blow ups can be exchanged to obtain a natural isomorphism

$$[X^3; F; T] \simeq [X^3; T; F].$$

The product structure in (5.2) and the fact that T lifts to $[X^3; F]$ to be

$$T' = H \times B_b(H) \subset [X^3; F] = X \times X^2_b,$$

shows that Φ_F has a common product decomposition with T'. The inverse image of Φ_F in $[X^3; F; T]$ is therefore the union of two p-submanifolds which we denote

$$\begin{cases} \widetilde{\Phi}_F = \beta^*_{T'}(\Phi_F) = \mathrm{cl}((\beta_{T'})^{-1}(\Phi_F \backslash T')) \text{ and} \\ \widetilde{\Phi}_{FT} = (\beta_{T'})^{-1}(\Phi_F \cap T'). \end{cases}$$

Neither of these p-submanifolds meets the lifts of S or C to $[X^3; F, T]$ so they equally well define submanifolds

$$\widetilde{\Phi}_F, \widetilde{\Phi}_{FT} \subset X^3_b \equiv [X^3; F; T; S; C].$$

Of course from the basic symmetry of the set up we have similar submanifolds

$$\widetilde{\Phi}_S, \widetilde{\Phi}_{ST}, \widetilde{\Phi}_C, \widetilde{\Phi}_{CT} \subset X^3_b.$$

Notice that $\widetilde{\Phi}_O \subset O_b \subset X^3_b$, $O = F, S, C$ when O_b denotes the front face produced by the blow up of O in defining X^3_b. On the other hand $\widetilde{\Phi}_{OT} \subset T_b$ for $O = F, S, C$.

Lemma 10. *The intersection of any pair of* $\widetilde{\Phi}_{ST}$, $\widetilde{\Phi}_{FT}$ *and* $\widetilde{\Phi}_{CT}$ *is the submanifold*

$$\widetilde{\Phi}_T = \widetilde{\Phi}_{ST} \cap \widetilde{\Phi}_{FT} \cap \widetilde{\Phi}_{CT}$$

which is contained in the interior of T_b.

Proof. Let us examine these definitions more closely. Since we only need to consider the operations near each boundary, X can be replaced by $[0,1)_x \times H$, so X^2_b is given by (2.8) and in this representation

$$\Phi = \{0\} \times \{0\} \times D_\Phi$$

where $D_\Phi \subset H \times H$ is the fibre diagonal. Now, $X^3 \simeq [0,1)^3 \times H^3$. So near the new faces

$$[X^3; T] \simeq [0,1) \times G \times H^3$$
$$X^3_b \simeq [0,1) \times G_b \times H^3.$$

Here, $G \subset \mathbb{R}^2$ is an equilateral triangle with centre the origin and G_b is obtained by blowing up each corner of it. Thus G_b can be embedded in $\mathbb{R}^2$ as a regular hexagon with centre the origin. The sides of this hexagon are alternately the front faces and original boundaries, i.e. $C_b, R_b, F_b, M_b, S_b, L_b$.

The lifts of the Φ diagonals are easily identified, thus

$$\widetilde{\Phi}_F \simeq [0,1) \times \{p_F\} \times H \times D_\Phi \tag{5.3}$$

where $p_F \in G_b$ is the midpoint of the side corresponding to F_b and $D_\Phi \subset H^2$ is the ϕ-fibre diagonal. Similarly

$$\widetilde{\Phi}_{FT} = \{0\} \times \ell_F \times H \times D_\Phi$$

where $\ell_F \subset G_b$ is the line through p_F, the origin and the midpoint of the side representing L_b.

This proves the lemma with

$$\widetilde{\Phi}_T = \{0\} \times \{0\} \times T_\Phi,\ T_\Phi \subset H^3 \text{ the triple } \Phi\text{-diagonal.} \tag{5.4}$$

Now we complete the definition of the triple Φ-space by three more (levels of) blow up

$$X^3_\Phi = [X^3_b; \widetilde{\Phi}_T; \widetilde{\Phi}_{FT}; \widetilde{\Phi}_{ST}; \widetilde{\Phi}_{CT}; \widetilde{\Phi}_F; \widetilde{\Phi}_S; \widetilde{\Phi}_C]. \tag{5.5}$$

From (5.3) and (5.4) it follows that $\widetilde{\Phi}_{FT}, \widetilde{\Phi}_{ST}; \widetilde{\Phi}_{CT}$ lift to be disjoint after the blow up of $\widetilde{\Phi}_T$ so the orders of these three blow ups, and the last three, are immaterial. However, the order between the last three blow ups and the preceding three is important and cannot be arbitrarily rearranged, since for instance $\widetilde{\Phi}_{FT}$ and $\widetilde{\Phi}_F$ intersect but not transversally, nor is one contained in the other. This space is mainly useful for the maps defined on it. $\square$

Proposition 6. *For $O = F, S, C$ there is a b-fibration $\pi^3_{\Phi,0} : X^3_\Phi \longrightarrow X^2_\Phi$ fixed by the demand that it give a commutative diagramme with the corresponding projection*

$$\begin{tikzcd} X^3 \arrow[r, "\psi^3"] \arrow[d, "\pi^3_{\Phi,0}"] & X^3_b \arrow[r, "\beta^3_b"] \arrow[d, "\pi^3_{b,0}"] & X^3 \arrow[d, "\pi^3_0"] \\ X^2_\Phi \arrow[r, "\psi^2"'] & X^2_b \arrow[r, "\beta^2_b"'] & X^2. \end{tikzcd} \tag{5.6}$$

Proof. To define these maps we start with the corresponding maps for the b-calculus; the middle maps in (5.6). These can be constructed using the commutability of blow ups for $O \supset T$, we shall take $O = F$ for the sake of definiteness. Then

$$X^3_b = [X^3; T; F; S, C] = [X^3; F; T; S, C] = \big[[X^3; F]; T; S, C\big]. \tag{5.7}$$

Now $[X^3; F] = X \times X^2_b$ so there is a commutative diagram with vertical projections.

$$\begin{tikzcd} X \times X^2_b \arrow[r] \arrow[d, "\gamma_F"] & X^2 \arrow[d, "\pi^3_F"] \\ X^2_b \arrow[r, "\beta^2_b"] & X^2. \end{tikzcd}$$

Then $\pi^3_{b,F} = \gamma_F \circ \tilde{\beta}$, $\tilde{\beta} : \big[[X^3; F]; T; S; C\big] \longrightarrow [X^3; F]$ being the blow down map. Thus $\pi^3_{b,F}$ is defined and is automatically a b-map. We need to show that it is a b-submersion and finally a b-fibration. Certainly it is surjective.

A b-fibration, f, remains a b-submersion when composed with the blow down map for blow up of some p-submanifold, M, if, for each point p of the submanifold the induced map

$$f : M \longrightarrow \mathrm{Fa}(f(p)) \tag{5.8}$$

is a b-submersion. Here $\mathrm{Fa}(q)$ is the smallest boundary face of the range space containing $f(p)$. For any boundary face, M, this condition is automatically satisfied. This 'blown up' b-fibration is again a b-fibration, rather than just a b-submersion, if $f(M)$ is a boundary hypersurface of the range space, which is to say it is not contained in a boundary face of codimension 2. Since this is immediately clear for the blow ups is the definition of $\tilde{\beta}$, and hence $\pi^3_{b,F}$, the latter map is a b-fibration.

Now that we have fixed the central vertical maps in (5.6) we proceed to the definition of the $\pi^3_{\Phi,O}$, again taking $O = F$ for definiteness sake. In (5.5) the submanifolds $\widetilde{\Phi}_{ST}$, $\widetilde{\Phi}_{CT}$ and $\widetilde{\Phi}_F$ are disjoint, so the order can be changed to

$$X^3_\Phi = [X^3_b; \widetilde{\Phi}_T; \widetilde{\Phi}_{FT}; \widetilde{\Phi}_F; \widetilde{\Phi}'],\ \ \widetilde{\Phi}' = \widetilde{\Phi}_{ST}; \widetilde{\Phi}_{CT}; \widetilde{\Phi}_S; \widetilde{\Phi}_C.$$

Similarly $\widetilde{\Phi}_T \subset \widetilde{\Phi}_{FT}$ and $\widetilde{\Phi}_T$ is disjoint from $\widetilde{\Phi}_F$ so

$$X^3_\Phi = [X^3_b; \widetilde{\Phi}_{FT}; \widetilde{\Phi}_F; \widetilde{\Phi}''],\ \ \widetilde{\Phi}'' = \widetilde{\Phi}_T; \widetilde{\Phi}'. \tag{5.9}$$

Consider again the definition, (5.1), of X^3_b, reorganized as in (5.7). The submanifold S and C are disjoint from $\widetilde{\Phi}_{FT}$ and $\widetilde{\Phi}_F$ so (5.9) can be written

$$X^3_\Phi = [X \times X^2_b; T; \widetilde{\Phi}_{FT}; \widetilde{\Phi}_F; R],\ \ R = S; C; \widetilde{\Phi}''. \tag{5.10}$$

In $X \times X^2_b$ the submanifold $X \times \Phi$ lifts to $\widetilde{\Phi}_F$ under the blow up of T and $\widetilde{\Phi}_{FT}$ is the lift, in fact preimage, of $\phi(X \times \widetilde{\Phi}) \cap T$ under blow up of T. Thus (5.10) can be commuted to

$$X^3_\Phi = \big[X \times X^2_b; (T \cap (X \times \Phi)); T; \widetilde{\Phi}_F; R\big].$$

The second and third blow up are disjoint so in fact

$$X^3_\Phi = \big[X \times X^2_b; (X \times \Phi);\ T \cap (X \times \Phi); T; R\big] \tag{5.11}$$

The final rearrangement here is of two cleanly intersecting submanifolds with are blown up with there intersection, this can be accomplished by blowing up either of them first, then the intersection, then the other, with the same final result.

The first blow up in (5.11) is the definition of X^2_Φ so

$$X^3_\Phi = [X \times X^2_\Phi; T \cap (X \times \Phi); T; R] \tag{5.12}$$

allows the blown up projection in (5.6) to be defined by

$$\pi^3_{\Phi,F} = \tilde{\gamma}_F \cdot \tilde{\psi}, \tilde{\gamma}_F : X \times X^2_\Phi \longrightarrow X^2_\Phi$$

being the projection, with $\tilde{\psi}$ the collective blow up of R in (5.12).

To show that $\pi^3_{\Phi,F}$, and hence by symmetry each of the $\pi^3_{\Phi,O}$, is a b-fibration it is only necessary to check the two conditions involving (5.8) for each of the blow ups in $\tilde{\psi}$. In fact, using the product structure in (5.3), etc., this is straightforward so the details are omitted. Suffice it to say that the fibration can be eliminated directly and the case $\phi = \mathrm{Id}$ is then simpler to analyze. □

We further augment Proposition 6 by considering the relationship between these maps and the lifted diagonals.

Lemma 11. *The lifted diagonals, defined as the closures in* X^3_Φ *of the diagonal* $D \subset X^\circ \times X^\circ$ *in each of the three possible positions, are p-submanifolds* $D_{\Phi,F}$, $D_{\Phi,S}$, $D_{\Phi,C}$ *as is the lifted triple diagonal* $D_{\Phi,T}$. *Each of the maps* $\pi^3_{\Phi,O}$ *is transversal to* $D_{\Phi,O'}$, *for* $O' \neq O$ *and maps*

$$D_{\Phi,T} = D_{\Phi,O_1} \cap D_\Phi, O_2, O_1 \neq O_2$$

diffeomorphically onto $D_\Phi \in X^2_\Phi$.

Proof. These results are immediate away from any boundaries. The transversality of $\pi^3_{\Phi,F}$, say, to $D_{\Phi,S}$ follows by lifting $\mathcal{V}_\Phi(X)$ from the left factor. This is in the null space of the differential of $\pi^3_{\Phi,F}$ and lifts to be transversal to $D_{\Phi,S}$, essentially by Lemma 5. Thus $\pi^3_{\Phi,F}$ maps $D_{\Phi,S}$ diffeomorphically onto X^2_Φ and hence embeds the submanifold $D_{\Phi,T} \subset D_{\Phi,S}$ as $D_\phi \subset X^2_\Phi$. ☐

With these maps and transversality results available the composition formula is now straightforward.

Theorem 2. *For any vector bundles* E, F, G *over a compact manifold with boundary* X, *and fibred boundary structure* Φ,

$$\Psi^m_\Phi(X;F,G) \circ \Psi^{m'}_\Phi(X;E,F) \subset \Psi^{m+m'}_\Phi(X;E,G)$$

and both the symbol map (3.9) *and normal operators*

$$\Psi^m_\Phi(X;E.F) \longrightarrow \Psi^m_{\mathrm{sus}(V)-\phi}(\partial X;E,F),$$

$V = {}^\Phi NY$, *of Proposition 5, are multiplicative.*

Proof. The composition is well defined by Proposition 3. ☐

6. Mapping properties

To deduce the L^2 boundedness of the operators of order zero we shall use an argument due to Hörmander [5] which depends on the existence, within the calculus, of an approximate square root of a positive elliptic element.

Proposition 7. *If* $B \in \Psi^0_\Phi(X)$ *is formally self-adjoint, for some smooth positive density on* X, *then for* $C > 0$ *sufficiently large*

$$C + B = A^*A + R,$$

for some $A \in \Psi^0_\Phi(X)$ *and* $R \in x^\infty \Psi^{-\infty}_\Phi(X)$.

Proof. Since B is formally self-adjoint with respect to the density, ν, the indicial family $\widehat{B}(\tau,\eta)$ consists of operators which are self-adjoint with respect to the boundary density, defined by $\nu = dx \otimes \nu_0$ for an admissible defining function x. Thus, for $C > 0$ sufficiently large

$$(C + \widehat{B}(\tau,\eta))^{\frac{1}{2}} \in \Psi^0_\phi(\partial X)$$

and from the uniqueness of this positive square root it is the indicial family of some $A_0 \in \Psi^0_\Phi(X)$. Again for C large enough A_0 can be chosen to have

$$\sigma_0(A_0) = (C + \sigma_0(B))^{\frac{1}{2}}$$

as well. Thus, replacing A_0 by $\frac{1}{2}(A_0 + A_0^*)$ we find

$$C + B - A_0^2 \in x\Psi^{-1}_\Phi(X).$$

Proceeding by induction, as in the standard case, one can suppose that $A_{(k-1)} \in \Psi^0_\Phi(X)$ has been constructed such that $A^*_{(k-1)} = A_{(k-1)}$ and

$$C + B - A^2_{(k-1)} = R_k \in x^k \Psi^{-k}_\Phi(X).$$

Adding an unknown $A_k \in x^k \Psi^{-k}_\Phi(X)$ to $A_{(k-1)}$ gives

$$\begin{aligned} C + B - (A_{(k-1)} + A_k)^2 &= R_k - A_{(k-1)}A_k - A_k A_{(k-1)} - A_k^2 \\ &\equiv R_k - A_{(k-1)}A_k - A_k A_{(k-1)} \end{aligned}$$

modulo $x^{k+1}\Psi^{-k-1}_\Phi(X)$. Thus if $A_k = x^k G_k$ is chosen to satisfy

$$N(A_0)N(G_k) + N(G_k)N(A_0) = N(F_k),\ F_k = x^{-k}R_k \tag{6.1}$$

then $A_{(k)} = A_{(k-1)} + A_k$ satisfies the inductive hypothesis at the next level. Notice that, at the level of the indicial families, (6.1) is indeed solvable, as the linearization of the definition of the square root

$$(\widehat{A_0}(\tau,\eta) + \widehat{G_k}(\tau,\eta))^2 = \widehat{A_0}(\tau,\eta)^2 + \widehat{F_k}(\tau,\eta),$$

$\widehat{A_0}(\tau,\eta)$ being a positive operator for all τ,η. Finally then A can be taken as an asymptotic sum of the series defined by the A_k. □

Theorem 3. *Each element $P \in \Psi^0_\Phi(X;E)$ defines a bounded linear operator on $L^2(X;E)$, defined with respect to a positive smooth density on X.*

Proof. Since X is compact, boundedness on L^2 is a local property of operators, so it suffices to consider the case $E = \mathbb{C}$ by local trivialization. Then applying Proposition 7 with $B = -P^*P$ shows that, for all $u \in \dot{\mathcal{C}}^\infty(X)$,

$$\|Pu\|^2 = C\|u\|^2 - \|Au\|^2 + \langle Ru, u\rangle \le C\|u\|^2 + |\langle Ru, u\rangle| \le C'\|u\|^2,$$

where the fact that elements of $x^\infty \Psi^{-\infty}_\Phi(X)$, being smoothing operators, are L^2 bounded has been used. □

Just as the construction of an approximate square root proceeds as in the boundaryless case, with some extra care needed to handle the normal operator, so the existence of parametrices for 'fully elliptic' operators is straightforward.

Proposition 8. *If $P \in \Psi^m_\Phi(X;E,F)$ is* fully elliptic *in the sense that its symbol is everywhere invertible and its normal operator is invertible on each fibre of ϕ, then there exits $Q \in \Psi^{-m}_\Phi(X;F,E)$ satisfying*

$$P \circ Q - \mathrm{Id} \in x^\infty \Psi^{-\infty}_\Phi(X;F) \text{ and } Q \circ P - \mathrm{Id} \in x^\infty \Psi^{-\infty}_\Phi(X;E).$$

Proof. Using the symbol calculus, $Q_0 \in \Psi^{-m}_\Phi(X;F,E)$ can be chosen to have

$$\sigma_{-m}(Q_0) = (\sigma_m(P))^{-1},\ N(Q) = N(P)^{-1}.$$

This ensures that $P \circ Q_0 = \mathrm{Id} - R_1$, with $R_1 \in x\Psi^{-1}_\Phi(X;F)$. Proceeding inductively it can be supposed that $Q_j \in x^j \Psi^{-m-j}_\Phi(X;F,E)$ have been constructed so that

$$P \circ (\sum_{j=0}^{k-1} Q_j) = \mathrm{Id} - R_k x^k,\ R_k \in \Psi^{-k}_\Phi(X;F).$$

Adding $Q_k = T_k x^k \in x^k \Psi^{-m-k}_\Phi(X;F,E)$ where $\sigma_{-m-k}(T_k) = (\sigma_m(P))^{-1}\sigma_{-k}(R_k)$ and $N(T_k) = N(P)^{-1}N(T_k)$ gives the next inductive step. Then Q can be taken to be an asymptotic sum of the Q_k. □

As in the boundaryless case these basic results easily lead to continuity, compactness and Fredholm properties on Sobolev spaces. For positive real number m, and any $l \in \mathbb{R}$ set

$$\begin{aligned} x^l H^m_\Phi(X;E) &= \left\{u \in x^l L^2(X;E); Pu \in L^2(X;E) \ \forall \ P \in \Psi^m_\Phi(X;E)\right\} \\ x^l H^{-m}_\Phi(X;E)& \\ &= \left\{ u \in \mathcal{C}^{-\infty}(X;E); u = \sum_{i=1}^N P_i u_i, \ u_i \in x^l L^2(X;E), \ P_i \in \Psi^m_\Phi(X;E) \right\} \end{aligned}$$

Lemma 12. *For these Φ-Sobolev spaces*

$$x^l H^m_\Phi(X;E) \subset x^{l'} H^{m'}_\Phi(X;E) \Longleftrightarrow l \geq l' \text{ and } m \geq m'$$

with the inclusion then continuous. The inclusion is compact if and only if $l > l'$ and $m > m'$ and each $P \in \Psi^m_\Phi(X;E,F)$ defines a continuous linear map

$$P : x^l H^{m'}_\Phi(X;E) \longrightarrow x^l H^{m'-m}_\Phi(X;F) \tag{6.2}$$

for all l and m'.

Proposition 9. *Each fully elliptic element, $P \in \Psi^m_\Phi(X;E,F)$, is Fredholm as a map* (6.2) *and conversely this condition characterizes the fully elliptic elements. The null space of such an operator is contained in $\dot{\mathcal{C}}^\infty(X;E)$ and there is a complement to the range in $\dot{\mathcal{C}}^\infty(X;F)$.*

Proposition 10. *If $P \in \Psi^m_\Phi(X;E,F)$ is fully elliptic then P^*P+1 has a two-sided inverse in $\Psi^{-2m}_\Phi(X;E)$.*

7. WAVEFRONT SET

There is a natural notion of wavefront set associated to the calculus of operators $\Psi^*_\Phi(X;E)$. In fact in a certain sense there are two such notions, one associated to regularity and the other associated to growth at the boundary. In each case we first consider the corresponding notion of microlocal support, or operator wavefront set, for the operators before examining the wavefront set of distributions.

For an embedded submanifold Y of a manifold X the conormal distributions introduced by Hörmander, $I(X,Y)$, have wavefront set a closed conic subset of the conormal bundle to Y in X. Let SN^*Y be the boundary of the compactification of this bundle, i.e. the quotient of $N^*Y \setminus 0$ by the $\mathbb{R}^+$-action. Then

$$\mathrm{WF}(u) \subset SN^*Y, \ u \in I^*(X,Y)$$

can also be identified with the cone support of the symbol obtained by transverse Fourier transformation of u. This second definition extends directly to the case of an interior p-submanifold of a manifold with corners. In particular it applies to the lifted diagonal in X^2_Φ. This allows us to define the 'symbolic' part of the Φ-wavefront set by

$$\begin{gathered} \mathrm{WF}'_{\Phi,\sigma}(A) = \mathrm{WF}(A) \subset SN^*(\mathrm{Diag}_\phi) = {}^\Phi S^*X, \\ \mathrm{WF}'_{\Phi,\sigma}(A) = \emptyset \Longleftrightarrow A \in \Psi^{-\infty}_\Phi(X;E). \end{gathered}$$

The elliptic subset, $\mathrm{Ell}^m_\sigma(A) \subset \mathrm{WF}'_{\Phi,\sigma}(A)$ is the open subset of ${}^\Phi S^*X$ on which the symbol of order m has an inverse of order $-m$. Here we have used the identification of the conormal bundle to the lifted diagonal with ${}^\Phi T^*X$.

Now, the discussion above of the composition of Φ-pseudodifferential operators shows that the diagonal singularity of the composite arises from the same operation as in the interior case. In particular the standard proof of the microlocality of composition shows that

$$\mathrm{WF}'_{\Phi,\sigma}(A\circ B)\subset \mathrm{WF}'_{\Phi,\sigma}(A)\cap \mathrm{WF}'_{\Phi,\sigma}(B),\ A,B\in\Psi^*_\Phi(X;E). \tag{7.1}$$

The construction of parametrices for elliptic operators can also be microlocalized, so if $K\subset \mathrm{Ell}^m_\sigma(A)$ is closed, for a given $A\in\Psi^m_\Phi(X;E)$, then there exists $B\in\Psi^{-m}_\Phi(X;E)$ such that

$$\begin{gathered}A\circ B=\mathrm{Id}-R_L,\ B\circ A=\mathrm{Id}-R_R,\ R_L,\ R_R\in\Psi^0_\Phi(X;E)\text{ and}\\ K\cap\big(\mathrm{WF}'_{\Phi,\sigma}(R_L)\cup\mathrm{WF}'_{\Phi,\sigma}(R_R)\big)=\emptyset.\end{gathered}\tag{7.2}$$

Combining these standard results extended to the Φ-calculus leads to an alternative characterization of the operator wavefront set

Lemma 13. *For any $A\in\Psi^*_\Phi(X;E)$*

$$\big(\mathrm{WF}'_{\Phi,\sigma}(A)\big)^{\complement}=\bigcup\big\{\,\mathrm{Ell}^0_\sigma(B);B\in\Psi^0_\Phi(X;E)\text{ and }B\circ A\in\Psi^{-\infty}_\Phi(X;E)\big\}. \tag{7.3}$$

Proof. If $p\in{}^\Phi S^*X$ is in the set on the right in (7.3) then there is some $B\in\Psi^0_\Phi(X;E)$ which is elliptic at p and such that $B\circ A\in\Psi^{-\infty}_\Phi(X;E)$. Using a microlocal parametrix as in (7.2) it follows that $p\notin\mathrm{WF}'_{\Phi,\sigma}(A)$. The converse inclusion follows from the microlocality, (7.1). □

We next define the corresponding notion of support, $\mathrm{WF}_{\Phi,\sigma}(u)$, for any distribution $u\in\mathcal{C}^{-\infty}(X)$. Since operators of order $-\infty$ are ignored here we work modulo the space

$$x^{-\infty}H^\infty_\Phi(X)=\bigcup_{k\in\mathbb{Z}}x^kH^\infty_\Phi(X),$$

Indeed,

$$A\in\Psi^{-\infty}_\Phi(X)\Longrightarrow A:\mathcal{C}^{-\infty}(X)\longrightarrow x^{-\infty}H^\infty_\Phi(X).$$

Then we simply define

$$\begin{gathered}\mathrm{WF}_{\Phi,\sigma}(u)=\bigcap\big\{\mathrm{Char}_\Phi(A);A\in\Psi^0_\Phi(X),\ Au\in x^{-\infty}H^\infty_\Phi(X)\big\}\subset{}^\Phi S^*X,\\ \mathrm{Char}_\Phi(A)=(\mathrm{Ell}^0_\sigma(A))^{\complement}.\end{gathered}$$

Thus by definition, $p\notin\mathrm{WF}_{\Phi,\sigma}(u)$ if there exists $A\in\Psi^0_\Phi(X)$ which is elliptic at p and is such that $Au\in x^{-\infty}H^\infty_\Phi(X)$. As with the standard wavefront set there is an alternate characterization in terms of the essential support

$$\begin{gathered}(\mathrm{WF}_{\Phi,\sigma}(u))^{\complement}=\bigcup\{U;U\subset{}^\Phi S^*X\text{ is open s.t.}\\ A\in\Psi^0_\Phi(X),\ \mathrm{WF}'_{\Phi,\sigma}(A)\subset U\Longrightarrow Au\in x^{-\infty}H^\infty_\Phi(X)\big\}.\end{gathered}\tag{7.4}$$

This follows by use of the calculus as in the boundaryless case. From (7.4), or directly, the calculus is microlocal for this wavefront set:

$$\mathrm{WF}_{\Phi,\sigma}(Au)\subset\mathrm{WF}'_{\Phi,\sigma}(A)\cap\mathrm{WF}_{\Phi,\sigma}(u),\ A\in\Psi^\infty_\Phi(X),\ u\in\mathcal{C}^{-\infty}(X).$$

Note also that

$$u\in\mathcal{C}^{-\infty}(X)\text{ and }\mathrm{WF}_{\Phi,\sigma}(u)=\emptyset\Longrightarrow u\in x^{-\infty}H^\infty_\Phi(X).$$

Together with this extension of the usual notion of wavefront set we next consider related notions at the boundary. First consider the operator wavefront set. This will be defined as a subset of the radial compactification ${}^{\Phi}\overline{N}^*Y$ of the bundle ${}^{\Phi}N^*Y$. This 'Φ-conormal bundle' to the fibres of the boundary is the space of parameters in the normal operators; note that it is a bundle over Y, the base of the fibration, and that it is the dual of the bundle corresponding to the Lie subalgebra in (1.2). Its lift to ∂X, ${}^{\Phi}N^*\partial X = \phi^*({}^{\Phi}N^*Y)$, occurs as the quotient of the part, ${}^{\Phi}T^*_{\partial X}X$, of the dual of the structure bundle over the boundary by the subbundle

$$(7.5)\qquad {}^{\phi}T^*\partial X = \bigcup_{p\in\partial X} T^*\phi^{-1}(p) \subset {}^{\Phi}T^*_{\partial X}X,\ {}^{\phi}\pi : {}^{\Phi}T^*_{\partial X}X \longrightarrow {}^{\Phi}N^*\partial X,$$

of the fibre cotangent bundles. The inclusion here is just given by pairing with vector fields, which shows ${}^{\phi}T^*\partial X$ to be the annihilator bundle in ${}^{\Phi}T^*_{\partial X}X$ of the lift of ${}^{\Phi}NY$.

Now, let $\bar{y} \in Y$ be a point in the base of the fibration of the boundary and consider a finite point $p \in {}^{\Phi}N^*_{\bar{y}}Y$. For any admissible coordinates x, $\tilde{y}$ near $\bar{y}$, $p = d(\frac{\bar{\lambda}+\bar{\eta}\cdot(\tilde{y}-\bar{y})}{x})$ for some $\bar{\lambda}$, $\bar{\eta}$.

Then we *define*

$$(7.6)\qquad \begin{gathered} p \notin (\mathrm{WF}'_{\Phi,\partial}(A)\cap {}^{\Phi}N^*Y) \Longleftrightarrow {}^{\phi}S^*_{\bar{y}}X \cap \mathrm{WF}'_{\Phi,\sigma}(A) = \emptyset \text{ and } \exists\ \psi \in \mathcal{C}^\infty(X) \text{ s.t.} \\ \exp\left(-i\frac{\lambda+\eta\cdot(\tilde{y}-\bar{y})}{x}\right)\psi A \exp\left(i\frac{\lambda+\eta\cdot(\tilde{y}-\bar{y})}{x}\right) : \mathcal{C}^\infty(X) \longrightarrow \dot{\mathcal{C}}^\infty(X) \\ \forall\ (\lambda,\eta) \text{ in some neighbourhood of } (\bar{\lambda},\bar{\eta}), \end{gathered}$$

where $\psi \in \mathcal{C}^\infty_{\phi}(X)$, is of the form $\psi = \phi^*\psi'$ on the boundary with $\psi'(\bar{y}) = 1$ and ψ' is supported in the coordinate patch.

Thus in order that $p \notin \mathrm{WF}'_{\Phi,\partial}(A)$ we first demand that $\mathrm{WF}'_{\Phi,\sigma}(A)$ not meet ${}^{\phi}S^*_{\phi^{-1}(\bar{y})}X$. Note that the preimage of p in ${}^{\Phi}N^*_{\partial X}X$ under projection to ${}^{\Phi}N^*\partial X$ and then ${}^{\Phi}N^*Y$ meets the sphere bundle at infinity ${}^{\Phi}S^*\partial XX$ exactly in ${}^{\phi}S^*_{\phi^{-1}(\bar{y})}X$. Thus this is the condition that the part of $\mathrm{WF}'_{\Phi,\sigma}(A)$ 'lying above' p should be trivial. The second part of (7.5) implies in particular that the normal operator of A should be trivial near p. In fact, in terms of the local representation (3.10) and (3.11), it means that the Fourier transform in S and Y of the local kernel should vanish in a fixed neighbourhood of the point (λ,η) and $\tilde{y}' = \bar{y}$ as a function z, z' and in the sense of Taylor series in x. The uniformity of the neighbourhood in x is important. It follows from the remarks after (3.11), in particular the polynomial dependence of the coordinate transformation, that this condition, of vanishing, is independent of coordinates.

Thus this notion is independent of the choice of coordinates in (7.6). It is clearly multiplicative in the usual sense that

$$\mathrm{WF}'_{\Phi,\partial}(AB) \subset \mathrm{WF}'_{\Phi,\partial}(A) \cap \mathrm{WF}'_{\Phi,\partial}(B) \text{ if } A, B \in \Psi^*_{\Phi}(X).$$

There is an importance difference between operators of finite order and those of order $-\infty$ as regards $\mathrm{WF}'_{\Phi,\partial}$. Of course, for the latter the condition on $\mathrm{WF}'_{\Phi,\sigma}(A)$ in (7.6) is vacuous and then given a point $p \in {}^{\Phi}N^*Y$ with a neighbourhood U we can always decompose

$$
\begin{aligned}
(7.7)\quad \Psi_\Phi^{-\infty}(X) \ni B = B' + B'',\ B', B'' \in \Psi_\Phi^{-\infty}(X),& \\
p \notin \mathrm{WF}'_{\Phi,\partial}(B''),\ \mathrm{WF}'_{\Phi,\partial}(B') \subset U,\ p \in {}^\Phi N^*Y.&
\end{aligned}
$$

Such a decomposition is not in general possible for operators of finite order, since for instance the ellipticity of the symbol of B would imply that the indicial operator never vanishes.

To an infinite point $p \in {}^\Phi SN^*_{\bar y}Y$ there corresponds a 'preimage' $\Gamma(p) \subset {}^\Phi S^*_{\bar y}X$, consisting of the intersection

$$
\Gamma(p) = \mathrm{cl}\left({}^\phi\pi^{-1}(p')\right) \cap {}^\Phi S^*X \text{ in } {}^\Phi\overline{T}^*X. \tag{7.8}
$$

Here $p' \subset {}^\Phi N^*Y$ is the open half line corresponding to the point p on the sphere at infinity and ${}^\phi\tilde\pi$ is the composite of ${}^\phi\pi$ in (7.5) and the projection from ∂X to Y. Thus $\Gamma(p)$ is a closed half-sphere bundle of fibre dimension $\dim F + 1$ over $\phi^{-1}(\pi(p))$. We define the condition

$$
\begin{aligned}
(7.9)\quad p \notin \mathrm{WF}'_{\Phi,\partial}(A) \text{ for } p \in {}^\Phi SN^*Y \Longleftrightarrow \Gamma(p) \cap \mathrm{WF}'_{\Phi,\sigma}(A) = \emptyset \text{ and}& \\
(\gamma \cap {}^\Phi N^*Y) \cap \mathrm{WF}'_{\Phi,\partial}(A) = \emptyset \text{ for some open } \gamma \subset {}^\Phi\overline{N}^*Y \text{ with } p \in \gamma.&
\end{aligned}
$$

Note that (7.6) shows that the analogue of $\Gamma(p)$ in case $p \in {}^\Phi N^*\bar yY$ is finite is ${}^\phi S^*_{\phi^{-1}(\bar y)}X$. If $p^\Phi SN^*\bar yY$ then $\Gamma(p) \supset {}^\phi S^*_{\phi^{-1}(\bar y)}X$.

The restriction of the conjugated operator

$$
\exp\left(-i\frac{\lambda + \eta\cdot(\tilde y - \bar y)}{x}\right)\psi A \exp\left(i\frac{\lambda + \eta\cdot(\tilde y - \bar y)}{x}\right)
$$

in (7.6) to the boundary fibre above $\pi(p)$ is the indicial operator, $N(A,p)$, at p. We define ellipticity for operators of order m in this boundary sense by

$$
\begin{aligned}
(7.10)\quad \mathrm{Ell}^m_\partial(A) = \{p \in {}^\Phi N^*Y; N(A,p)^{-1} \text{ exists in } \Psi^{-m}(\phi^{-1}(\pi(p))\}& \\
\cup \{p \in {}^\Phi SN^*Y; \Gamma(p) \subset \mathrm{Ell}^m_\sigma(A)\} \subset {}^\Phi\overline{N}^*Y.&
\end{aligned}
$$

Then certainly $\mathrm{Ell}^m_\partial(A) \subset \mathrm{WF}'_{\Phi,\partial}(A)$.

Lemma 14. *For any $A \in \Psi^m_\Phi(X)$ the set $\mathrm{Ell}^m_\partial(A)$ is open in ${}^\Phi\overline{N}^*Y$.*

Proof. Certainly if $p \in \mathrm{Ell}^m_\partial(A) \cap {}^\Phi N^*Y$ then $\mathrm{Ell}^m_\partial(A)$ contains a neighbourhood of p, since the invertibility of the normal operator is an open condition. So consider $p \in {}^\Phi SN^*Y$ 'at infinity' and suppose $p \in \mathrm{Ell}^m_\partial(A)$. Since $\Gamma(p')$ is compact and depends continuously on $p' \in {}^\Phi SN^*Y$ it follows that $\Gamma(p') \subset \mathrm{Ell}_\sigma(A)$ for p' in a neighbourhood of p. Thus it remains to show that $N(A,q)^{-1} \in \Psi^{-m}(\phi^{-1}(\pi(q)))$ for $q \in \gamma' \cap {}^\Phi N^*Y$ for some neighbourhood γ' of p in ${}^\Phi\overline{N}^*Y$. Using the calculus, we may construct an operator $G \in \Psi^{-m}_\Phi(X)$ such that $G \circ A = \mathrm{Id} - E$ where $\Gamma(p) \cap \mathrm{WF}'_{\Phi,\sigma}(E) = \emptyset$. Shrinking γ' as necessary, it follows that $N(E,q)$ is in $\Psi^{-\infty}(\phi^{-1}(\pi(q)))$ for $q \in \gamma' \cap {}^\Phi N^*Y$ and is rapidly vanishing as $q \to {}^\Phi SN^*Y$ in γ'. Thus $N(A,q)^{-1} \in \Psi^{-m}(\phi^{-1}(\pi(q)))$ exists for all q in the intersection of ${}^\Phi N^*\partial X$ and some neighbourhood of p in ${}^\Phi\overline{N}^*\partial X$. Thus $\mathrm{Ell}^m_\partial(A)$ is open. $\square$

The construction in the proof of this lemma can be slightly extended to yield:

Lemma 15. *If $p \in {}^\Phi\overline{N}^*Y$ and $A \in \Psi^m_\Phi(X)$ then $p \in \mathrm{Ell}^m_\partial(A)$ if and only if there exists $G \in \Psi^{-m}_\Phi(X)$ such that $p \notin \mathrm{WF}'_{\Phi,\partial}(\mathrm{Id} - A \circ G)$, $p \notin \mathrm{WF}'_{\Phi,\partial}(\mathrm{Id} - G \circ A)$.*

Notice that in demanding that A be elliptic at a finite point $p \in {}^\Phi N^*Y$ we are requiring that A be symbolically elliptic on the whole set ${}^\phi S^*_{\pi(p)}\partial X \subset {}^\Phi S^*_{\pi(p)}\partial X$, which is the sphere of the subspace in (7.5) above the point $\pi(p)$, since $N(A,p)$ is to be invertible as a pseudodifferential operator of order m on the boundary fibre. Correspondingly if $p \in {}^\Phi N^*_{\bar y}Y$ then the parametrix G may be chosen to have $\mathrm{WF}'_{\Phi,\sigma}(G)$ concentrated near ${}^\phi S^*_{\phi^{-1}(\bar y)}\partial X \subset {}^\Phi N^*X$ whereas $\mathrm{WF}'_{\Phi,\partial}(G)$ can only be concentrated near the fibre ${}^\Phi\overline{N}^*_{\bar y}Y$. If $p \in {}^\Phi SN^*_{\bar y}Y$ then $\mathrm{WF}'_{\Phi,\sigma}(G)$ may be concentrated near $\Gamma(p)$ and again $\mathrm{WF}'_{\Phi,\partial}(G)$ may be concentrated near ${}^\Phi\overline{N}^*_{\bar y}Y$.

We now define

(7.11)
$$\mathrm{WF}_{\Phi,\partial}(u)^{\complement} = \big\{p \in {}^\Phi\overline{N}^*Y; \exists\ A \in \Psi^0_\Phi(X),\ p \in \mathrm{Ell}^0_\partial(A),\ Au = w + \sum_j B_j v_j,$$
$$w \in \dot{\mathcal{C}}^\infty(X),\ v_j \in \mathcal{C}^{-\infty}(X),\ B_j \in \Psi^{-\infty}_\Phi(X),\ p \notin \mathrm{WF}'_{\Phi,\partial}(B_j)\big\} \subset {}^\Phi\overline{N}^*Y.$$

Taking

$$\mathrm{Char}^m_\partial(A) = {}^\Phi\overline{N}^*Y \setminus \mathrm{Ell}^m_\partial(A)$$

this can also be written

(7.12)
$$\mathrm{WF}_{\Phi,\partial}(u) = \bigcap\Big\{\ \mathrm{Char}^0_\partial(A) \cup \bigcup_j \mathrm{WF}'_{\Phi,\partial}(B_j); A \in \Psi^0_\Phi(X),\ B_j \in \Psi^{-\infty}_\Phi(X) \text{ with}$$
$$Au = w + \sum_j B_j v_j, \text{ for some } w \in \dot{\mathcal{C}}^\infty(X),\ v_j \in \mathcal{C}^{-\infty}(X)\Big\}.$$

The extra finite sum of terms B_jv_j is included in (7.11), and (7.12), because of the non-localizability of $\mathrm{WF}'_{\Phi,\partial}$ for operators of finite order. Notice that if $B \in \Psi^{-\infty}_\Phi(X)$ has $\mathrm{WF}'_{\Phi,\partial}(B)$ concentrated sufficiently close to $p \notin \mathrm{WF}_{\Phi,\partial}(u)$, so $\mathrm{WF}'_{\Phi,\partial}(B_j) \cap \mathrm{WF}'_\Phi(B) = \emptyset$ for each j, then $B\sum_j B_jv_j \in \dot{\mathcal{C}}^\infty(X)$ too.

Since we are demanding that Au lie in the 'residual space' at p

$$\begin{aligned}(7.13)\quad \mathcal{R}_p(X) = \big\{&u \in \mathcal{C}^{-\infty}(X); u = u_1 + u_2,\ u_1 \in \dot{\mathcal{C}}^\infty(X),\\ &u_2 \in \{B \in \Psi^{-\infty}_\Phi(X);\ p \notin \mathrm{WF}'_{\Phi,\partial}(B)\}\cdot\mathcal{C}^{-\infty}(X)\big\} \subset x^{-\infty}H^\infty_\Phi(X),\end{aligned}$$

where the $\cdot$ means finite span; this is a considerably finer notion than $\mathrm{WF}_{\Phi,\sigma}(u)$ over the boundary.

Lemma 16. *If $p \in {}^\Phi\overline{N}^*Y$, the condition $p \notin \mathrm{WF}_{\Phi,\partial}(u)$ given by* (7.11) *is equivalent to the existence of $u_1 \in \dot{\mathcal{C}}^\infty(X)$, $C \in \Psi^0(X)$ with $p \notin \mathrm{WF}'_{\Phi,\partial}(C)$, $u'_j \in \mathcal{C}^{-\infty}(X)$ and $B_j \in \Psi^{-\infty}_\Phi(X)$ for $j = 1, \dots, J$ with $p \notin \mathrm{WF}'_{\Phi,\partial}(B_j)$ such that*

$$u = u_1 + \sum_j B_j u'_j + Cu. \tag{7.14}$$

Proof. The form (7.14) for u follows by applying the parametrix G of Lemma 15 to the defining relation in (7.11).

Conversely, if (7.14) holds for $p \in {}^\Phi N^*Y$ and $A \in \Psi^0_\Phi(X)$ is elliptic at p and has $\mathrm{WF}'_{\Phi,\sigma}(A)$ in a small neighbourhood of ${}^\phi S^*_{\pi(p)}\partial X$, so that $\mathrm{WF}'_{\Phi,\sigma}(A) \cap \mathrm{WF}'_{\Phi,\sigma}(C) =$

$\emptyset$ then

$$Au = Au_1 + \sum_j AB_j u'_j + ACu \in \mathcal{R}_p(X),$$

since $\mathrm{WF}'_{\Phi,\sigma}(AC) = \emptyset$. This gives (7.11). A similar argument applies if $p \in {}^{\Phi}SN^*Y$. □

As already noted, the subtlety with the definition of $\mathrm{WF}_{\Phi}(u)$ above arises from the non-localizability of the normal operators. In the particular case of the scattering calculus, considered in [15] and [19], there is no such difficulty. It is useful to relate the general case to this scattering case.

Lemma 17. *If $\psi \in \mathcal{C}^\infty(X)$ has support sufficiently close to $\phi^{-1}(\bar{y}) \subset \partial X$ for some point $\bar{y} \in Y$ then there is an open product neighbourhood of* $\operatorname{supp}(\psi)$ *of the form*

$$[0, \epsilon)_x \times Y' \times F, \ Y' \subset Y,$$

consistent with the fibration of the boundary and then for any $A \in \Psi_{\Phi}^{-\infty}(X)$, $\psi A \psi$ is a smooth right density on $F \times F$ with values in the scattering calculus on $X' = [0,1] \times Y$, that is $\Psi_{sc}^{-\infty}(X')$. Furthermore, this product decomposition allows ${}^{\Phi}N^*Y$ to be identified with ${}^{sc}\overline{T}^*_Y X'$ and if $B\Psi^0_{sc}(X')$ is supported sufficiently close to the boundary and has $\mathrm{WF}'_{sc}(B) \cap {}^{sc}\overline{T}^*_Y X' \cap \mathrm{WF}'_{\Phi,\partial}(C) = \emptyset$, where $C \in \Psi^0_{\Phi}(X)$ is supported in the product neighbourhood then $B \circ C \in \rho^\infty \Psi_{\Phi}^{-\infty}(X')$.

Proof. The first part follows directly from the definitions of the algebras in terms of their kernels on the blown up spaces since locally, in Y, the blow up defining the stretched product for the fibred cusp calculus is just the blow up for the scattering calculus (i.e. the case that the fibres in the boundary are points) with the fibres $F \times F$ as factors.

The composition statement in the second part follows directly from the local normal forms (3.10), (3.11). □

Despite the complexity of its definition, we may now see that this notion of wavefront set has many of the familiar properties.

Proposition 11. *The set $\mathrm{WF}_{\Phi}(u) = \mathrm{WF}_{\Phi,\sigma}(u) \cup \mathrm{WF}_{\Phi,\partial}(u) \subset {}^{\Phi}S^*X \cup {}^{\Phi}\overline{N}^*Y$ is closed, is empty only for elements of $\dot{\mathcal{C}}^\infty(X)$, satisfies*

$$\mathrm{WF}_{\Phi,\partial}(u_1 + u_2) \subset \mathrm{WF}_{\Phi}(u_1) \cup \mathrm{WF}_{\Phi}(u_2)$$

and is reduced by the application of pseudodifferential operators, $A \in \Psi^*_{\Phi}X$, in the sense that

$$\mathrm{WF}_{\Phi}(Au) \subset \mathrm{WF}'_{\Phi}(A) \cap \mathrm{WF}_{\Phi}(u), \ \mathrm{WF}'_{\Phi}(A) = \mathrm{WF}'_{\Phi,\sigma}(A) \cup \mathrm{WF}'_{\Phi,\partial}(A).$$

Proof. That $\mathrm{WF}_{\Phi}(u)$ is closed follows directly from the openness of the elliptic sets. The microlocality of pseudodifferential operators, (11), follows directly for the interior part of the wavefront set and from (7.14) for the boundary part. Thus, if $B \in \Psi^*_{\Phi}X$ and $p \notin \mathrm{WF}_{\Phi,\partial}(u)$ then first applying A to (7.14) and then applying $Q \in \Psi^0_{\Phi}X$ which is elliptic at p but has small support (see the discussion following Lemma 15) gives

$$QAu = QAu_1 + \sum_j QAB_j u_j + QACu.$$

Here in the last term, $QAC \in \Psi_{\Phi}^{-\infty}X$ if $\mathrm{WF}'_{\Phi,\sigma}(Q)$ is chosen sufficiently small and $p \notin \mathrm{WF}'_{\Phi,\partial}(QAC)$. Thus it can be absorbed as an extra term in the sum and deduce that $p \notin \mathrm{WF}'_{\Phi,\partial}(Au)$ by (7.12). The other components of (11) are simpler.

It remains to show that if $\mathrm{WF}'_{\Phi}(u) = \emptyset$ then $u \in \dot{\mathcal{C}}^{\infty}(X)$. From $\mathrm{WF}_{\Phi,\sigma}(u) = \emptyset$ it follows that $u \in x^{-\infty}H_{\Phi}^{\infty}(X)$; in particular it is smooth in the interior of X. We may localize the support of u to a small set near a boundary point, using the microlocality just discussed; thus we may assume that u has small support, in which the fibration has a product decomposition. Thus $u(x,\tilde{y},z)$ is a smooth function of z with values in a fixed space $x^{-N}H_{\mathrm{sc}}^{-N}(X')$, $X' = [0,1)_x \times Y$ as in Lemma 17. Applying the second half Lemma 17, it follows that if $A \in \Psi_{\mathrm{sc}}^{0}X'$ has wavefront set concentrated near any point $p \in {}^{\mathrm{sc}}\overline{T}_Y^*X'$ then, applying it to (7.14) $Au(x,y,z)$ is $\mathcal{C}^{\infty}$ in z with values in $\dot{\mathcal{C}}^{\infty}(X')$, and hence in $\dot{\mathcal{C}}^{\infty}(X' \times F)$. Applying this to a partition of unity in the scattering calculus it follows that $u \in \dot{\mathcal{C}}^{\infty}(X)$. □

Remark 1. The somewhat global (at least on the fibre) condition in (7.11), coming in turn from (7.10), is necessitated by the fact, mentioned above, that one cannot freely localize the indicial family. Thus, if $A \in \Psi_{\Phi}^{0}(X)$ has indicial family invertible, in the calculus, at any one point $p \in {}^{\Phi}N_yY$ its indicial family cannot be zero at any other point in that fibre, that is,

$$p \in {}^{\Phi}N_{\tilde{y}}^*X,\ p \in \mathrm{Ell}_{\partial}^{0}(A) \Longrightarrow {}^{\Phi}\overline{N}_{\tilde{y}}^* \subset \mathrm{WF}'_{\Phi,\partial}(A).$$

8. Fibred cusp metrics

As an application of the discussion above we shall examine the spectrum of the Laplacian for a metric of 'exact Φ-type'. By this we mean any Riemann metric on the interior of X, a manifold with a fibred boundary as in (1), which takes the form

$$g = \frac{dx^2}{x^4} + \frac{h'}{x^2} + g' \tag{8.1}$$

for some product decomposition near the boundary $X \subset [0,\epsilon)_x \times \partial X$ with g_Y a smooth symmetric 2-cotensor on $[0,\epsilon) \times Y$ which is positive definite when restricted to $\{0\} \times Y$ (with restriction h) and g' is a smooth symmetric 2-cotensor on X which is positive definite when restricted to each fibre over the boundary. The fibration ϕ and the boundary defining function x in (8.1) together determine a Φ structure on X. Moreover

Proposition 12. *The Laplacian of a metric* (8.1) *is a Φ-differential operator on functions or acting on sections of the Φ exterior bundle.*

The metric g is a positive definite metric on the bundle ${}^{\Phi}TX$, smooth and non-degenerate up to the boundary. This allows ${}^{\phi}T^*\partial X$ to be identified with the orthocomplement of ${}^{\Phi}N^*\partial X$ in ${}^{\Phi}T^*_{\partial X}X$. Furthermore, the boundary defining function x in (8.1) defines a natural section dx/x^2 of ${}^{\Phi}N^*\partial X$ the orthocomplement of which can be identified with the lift of T^*Y, by identifying $\eta \cdot dy$ with $\frac{\eta \cdot dy}{x}$. For each $y \in \partial X$ let Δ_y be the Laplacian on the fibres $\phi^{-1}(y)$ fixed by the metric g'. Let $\lambda_j(y)$ be the eigenvalues of Δ_y arranged in increasing order, repeated with multiplicity.

Theorem 4. *If* $u \in \mathcal{C}^{-\infty}(X; \Lambda^k)$ *satisfies* $\varDelta u - \lambda u \in \dot{\mathcal{C}}^\infty(X)$, *with* $\lambda \in \mathbb{C}$ *then*

$$\lambda \notin [0,\infty) \Longrightarrow u \in \dot{\mathcal{C}}^\infty(X), \tag{8.2}$$

$$\begin{aligned}\lambda \in [0,\infty) \Longrightarrow \mathrm{WF}_\Phi(u) \subset \{q \in {}^\Phi N_y\partial X;\ \exists\ \lambda_j(y) \le \lambda\ s.t.\\ q = s\frac{dx}{x^2} + \frac{\eta}{x}\ with\ s^2 + |\eta|_h^2 = \lambda - \lambda_j(y)\}.\end{aligned} \tag{8.3}$$

References

[1] M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Camb. Phil. Soc **77** (1975), 43–69.

[2] J. Brüning and V.W. Guillemin (Editors), *Fourier integral operators*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1994.

[3] C.L. Epstein, R.B. Melrose, and G. Mendoza, *Resolvent of the Laplacian on strictly pseudo-convex domains*, Acta Math **167** (1991), 1–106.

[4] A. Hassell, R. Mazzeo, and R.B. Melrose, *Analytic surgery and the accumulation of eigenvalues*, Comm. Anal. Geom. **3** (1995), 115–222.

[5] L. Hörmander, *Pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 501–517.

[6] ———, *Fourier integral operators, I*, Acta Math. **127** (1971), 79–183, See also [2].

[7] R. Mazzeo, *The Hodge cohomology of a conformally compact metric*, J. Diff. Geom. **28** (1988), 309–339.

[8] ———, *Elliptic theory of differential edge operators I*, Comm. in P.D.E. **16** (1991), 1615–1664.

[9] R. Mazzeo and R.B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), 260–310.

[10] R.B. Melrose, *Differential analysis on manifolds with corners*, In preparation.

[11] ———, *Transformation of boundary problems*, Acta Math. **147** (1981), 149–236.

[12] ———, *Pseudodifferential operators, corners and singular limits*, Proc. I.C.M., 1990, pp. 217–234.

[13] ———, *Calculus of conormal distributions on manifolds with corners*, Int. Math. Res. Notes **3** (1992), 51–61.

[14] ———, *The Atiyah-Patodi-Singer index theorem*, A K Peters, Wellesley, Mass, 1993.

[15] ———, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*, Spectral and scattering theory (Sanda, 1992) (M. Ikawa, ed.), Marcel Dekker, 1994, pp. 85–130.

[16] ———, *The eta invariant and families of pseudodifferential operators*, Math. Res. Letters **2** (1995), no. 5, 541–561.

[17] ———, *Geometric scattering*, Cambridge University Press, 1995.

[18] R.B. Melrose and V. Nistor, *Homology of pseudodifferential operators on manifolds with corners I. Manifolds with boundary*, Preprint, May 1996.

[19] R.B. Melrose and M. Zworski, *Scattering metrics and geodesic flow at infinity*, Invent. Math. **124** (1996), 389–436.

[20] P. Piazza, *On the index of elliptic operators on manifolds with boundary*, J. Funct. Anal. **117** (1993), 308–359.

[21] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara, *Microfunctions and pseudo-differential equations*, Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau) (Berlin), Springer, Berlin, 1973, pp. 265–529. Lecture Notes in Math., Vol. 287.

[22] M.A. Shubin, *Pseudodifferential operators on* $\mathbb{R}^n$, Sov. Math. Dokl. **12** (1971), 147–151.

Department of Mathematics, Stanford University
E-mail address: `mazzeo@math.stanford.edu`

Department of Mathematics, MIT
E-mail address: `rbm@math.mit.edu`