Isospectral sets of drumheads are compact in C^∞

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* This work was supported in part by the National Science Foundation under grant MCS 8006521

2. <u>Heat invariants</u>. It follows from invariant theory, or a computation of the type described below, that each of the coefficients a_k , $k \geq 1$, b_k , $k \geq 0$, in (1.1) can be written as an integral with respect to arclength of a polynomial in the curvature and its derivatives:

$$(2.1) \qquad b_k = \int_0^L q_k(x_0, x_1, \ldots, x_p)\, ds \quad k \geq 0 \; .$$

The polynomials q_k are not uniquely determined by (2.1) so it is convenient to demand the normalization condition:

$$(2.2) \qquad \begin{cases} q_k(x_0,x_1,\ldots,x_p) = \sum_{\beta\in B} c_\beta x^\beta \text{ with } c_\beta \neq 0, \forall \beta \in B \text{ and if} \\ P(\beta) = \max_p(\beta_p \neq 0) \text{ then } \beta \in B,\ p = P(\beta) \geq 1 \text{ implies that } \beta_p \geq 2. \end{cases}$$

That is, in each term of q_k the highest derivative occurs at least quadratically. The existence of q_k satisfying (2.1), (2.2) is easily established by integration by parts; its uniqueness can also be shown but this will not be needed here.

There is a natural transformation which can be used to deduce a homogeneity property of the b_k . Namely, consider radial expansion from an interior point of the domain by a factor R. Under such an expansion

$$(2.3) \qquad L' = RL \; , \quad \kappa'(s) = R^{-1}\kappa(R^{-1}s).$$

Moreover, the eigenvalues of the Dirichlet problem are transformed according to $\lambda_k' = R^{-2}\lambda_k$. Thus,

$$(2.4) \qquad H'(t) = \sum_k \exp(-t\lambda_k') = H(R^{-2}t) \; .$$

This shows that the invariants have the transformation law

(2.5) $$b_k' = R^{1-2k}\, b_k$$

under the transformation (2.3). Assigning to a monomial x^β the weight

(2.6) $$W(\beta) = \sum_{j=0}^{p} (j+1)\beta_j$$

it follows that (2.1) continues to hold if q_k is replaced by the sum of its terms of weight $W(\beta) = 2k$.

From this it follows immediately that

(2.7) $$b_{k+1} = c_k \int_0^L \left(\frac{d^k\kappa}{ds^k}\right)^2 ds + \int_0^L q_k'(\kappa,\kappa_1,\ldots,\kappa_{k-1})\, ds$$

where the lower order terms are at least cubic, but at most quadratic in κ_{k-1} for $k \geq 3$,

(2.8) $$q_k'(\kappa,\kappa_1,\ldots,\kappa_{k-1}) = q_k^{*}(\kappa,\kappa_1,\ldots,\kappa_{k-2})\kappa_{k-1}^2 + q_k^{*\prime}(\kappa,\kappa_1,\ldots,\kappa_{k-2}),$$

with all terms of weight $2k+2$. The main result of this note is:

(2.9) PROPOSITION. In the representation (2.7) of b_{k+1} the coefficient c_k is not zero.

This is proved in section 4 below. As a simple consequence we obtain:

(2.10) THEOREM. Each set $\mathfrak{F}(\Omega)$ of C^∞ bounded domains with the same Dirichlet spectrum as a given domain Ω is compact in the C^∞ topology on the curvature.

<u>Proof</u>. First note that the length L of the boundary is constant on $\mathfrak{F}=\mathfrak{F}(\Omega)$,

as is the number of boundary components. Similarly, $b_1 = c_1 \int_0^L \kappa^2 ds$ so from the constancy of b_1 on $\mathcal{S}$ it follows that $\mathcal{S}$ is bounded in L^2, as a norm on the curvature. Next consider

$$(2.11) \qquad b_2 = c_2 \int_0^L \kappa_1^2 \, ds + c_2' \int_0^L \kappa^4 \, ds ,$$

using (2.2). The Sobolev embedding theorem shows that for any $\delta > 0$,

$$\int_0^L \kappa^4 \, ds \leqslant \delta \int_0^L \kappa_1^2 \, ds + c_\delta \left(\left(\int_0^L \kappa^2 \, ds \right)^3 + \int_0^L \kappa^2 \, ds \right).$$

Thus from (2.11) and the constancy of b_2 on $\mathcal{S}$ it follows that $\int \kappa_1^2 \, ds$ is uniformly bounded on $\mathcal{S}$. This in particular implies the boundedness of the supremum norm $\|\kappa\|_\infty$. Next consider

$$(2.12) \qquad b_3 = c_3 \int_0^L \kappa_2^2 \, ds + c_3' \int_0^L \kappa_1^3 \, ds + \int_0^L q_2^* (\kappa, \kappa_1) \, ds ,$$

using (2.2), with q_2^* at most quadratic in κ_1. Thus the third term in (2.12) has already been shown to be bounded, and a similar argument to that above shows that the second term is also bounded in terms of b_0, b_1, b_2; thus the unifrom boundedness of the L^2 norm $\|\kappa_2\|_2$ follows from the boundedness of b_3.

Proceeding inductively we can suppose that for each $p < k$, $k > 2$, the boundedness of $b_0, b_1, \ldots, b_{p+1}$ implies the boundedness of $\|\kappa_r\|_\infty$, $r \leqslant p-1$ and of $\|\kappa_p\|_2$. From (2.7) and (2.8) these estimates imply the boundedness of the second term in (2.7) and hence the boundedness of $\|\kappa_k\|_2$, and so of $\|\kappa_p\|_\infty$ for $p \leqslant k-1$, given the boundedness of b_r, $0 \leqslant r \leqslant k+1$. This completes the proof, by induction, that the set $\mathcal{S}(\Omega)$ is bounded in C^∞. Since it is certainly closed the theorem follows.

3. Perturbation formulae. The heat kernel associated to the Laplace operator with Dirichlet boundary condition can be computed in terms of the Schwartz kernel, $r(\lambda,z,z')$, of the resolvent through

$$(3.1)\qquad h(t,z,z') = (2\pi i)^{-1} \int_\Gamma e^{-\lambda t}\, r(\lambda,z,z')\, d\lambda \qquad z,z' \in \Omega,\ t > 0$$

where Γ is a contour in $\mathbb{C}/\bar{\mathbb{R}}^+$ tending to infinity in the negative direction in $Re(\lambda) > 0$, $Im(\lambda) > 0$ and in the positive direction in $Re(\lambda) > 0$, $Im(\lambda) < 0$. The kernel r satisfies

$$(3.2)\qquad \begin{cases} (\Delta_z - \lambda)\, r(\lambda,z,z') = \delta(z-z') & \text{in } \Omega \times \Omega \\ r(\lambda,z,z') = 0 & z \in \partial\Omega,\ z' \in \Omega . \end{cases}$$

From the standard properties of the kernels of pseudodifferential operators, and Poisson operators (see Boutet de Monvel [1]), $R(\lambda)$, with kernel $r(\lambda,z,z')$, is not trace class but $\partial_\lambda R(\lambda)$, with kernel $\partial_\lambda r(\lambda,z,z')$, is trace class having continuous kernel. Thus,

$$(3.3)\qquad H(t) = (2\pi i t)^{-1} \int_\Gamma \int e^{-\lambda t}\, \partial_\lambda\, r(\lambda,z,z)\, dz\, d\lambda , \quad t > 0 .$$

Provided Γ lies outside a conic neighbourhood of $\bar{\mathbb{R}}^+$, the trace

$$(3.4)\qquad T(\lambda) = \int \partial_\lambda\, r(\lambda,z,z)\, dz$$

is a classical symbol as $|\lambda| \to \infty$, along Γ, and computing the fractional powers in the expansion of $H(t)$ can be accomplished by computing the appropriate terms in the expansion of $T(\lambda)$.

Now, suppose that the boundary $\partial\Omega$ of Ω undergoes a smooth deformation giving the one parameter family of domains Ω_ϵ. The theory of elliptic boundary problems shows that

$$T^{\epsilon}(\lambda) = \int_{\Omega_\epsilon} \partial_\lambda\, r_\epsilon(\lambda,z,z)\, dz \tag{3.5}$$

is C^∞ in ϵ. To compute the coefficent c_k in (2.7) it is enough to be able to find the expansion, to all orders, of $d^2T^{\epsilon}(\lambda)/d\epsilon^2$ at $\epsilon=0$, around any convenient domain in terms of the curvature variation.

To simplify this computation we shall choose as base domain the half-space $y > 0$, i.e. $Z = \mathbb{R}_x \times \mathbb{R}_y^+$. Of course, this is not compact but variations satisfy the same formula, as we shall see. Thus, consider the solution of

$$\begin{cases} (\Delta - \lambda)\, r_\epsilon(\lambda,z,z') = \delta(z - z') \quad \text{in } Z_\epsilon \times Z_\epsilon \\ r_\epsilon(\lambda,x,\epsilon\tau(x)z') = 0 \\ r_\epsilon(\lambda,z,z') \longrightarrow 0 \quad \text{as } |z| \longrightarrow \infty \end{cases} \tag{3.6}$$

where $Z_\epsilon = \{y > \epsilon\tau(x)\}$ and the boundary perturbation, $y = \epsilon\tau(x)$ has compact support. Now, the arclength differential on the boundary is

$$ds = (1 + \epsilon^2(\tau'(x))^2)^{1/2}\, dx \tag{3.7}$$

and the curvature is

$$\kappa^{\epsilon}(x) = \epsilon\tau''(x)(1 + \epsilon^2(\tau'(x))^2)^{-1/2}\, . \tag{3.8}$$

Thus, at $\epsilon = 0$, the second variation is

$$\partial_\epsilon^2 \int_0^\infty (\kappa_k^{\epsilon}(s))^2\, ds = 2\int_0^\infty \left(\frac{d^{k+2}}{dx^{k+2}}\tau(x)\right)^2 dx\, . \tag{3.9}$$

If r_ϵ is defined by (3.6) then set

$$r = r_0(\lambda,z,z'),\quad r' = \frac{d}{d\epsilon} r_0(\lambda,z,z'),\quad r'' = \frac{d^2}{d\epsilon^2} r_0(\lambda,z,z') \tag{3.10}$$

all kernels on $Z \times Z$, and consider the symbol, outside a cone around $\bar{\mathbb{R}}^+$ in $\mathbb{C}$

$$T_2(\lambda) = \int_{-\infty}^{\infty}\int_0^{\infty} \partial_\lambda r''(\lambda,x,y;x,y)\,dy\,dx$$

$$(3.11) \qquad - \int_{-\infty}^{\infty} 2\tau(x)\,\partial_\lambda r'(\lambda,x,0;x,0)\,dx$$

$$+ \int_{-\infty}^{\infty} \tau^2(x)(\partial_y + \partial_{y'})\partial_\lambda r(\lambda,x,0;x,0)\,dx .$$

(3.12) PROPOSITION. $T_2(\lambda)$ has a classical expansion as $|\lambda| \to \infty$ outside any cone containing $\bar{\mathbb{R}}^+$, of the form

$$(3.13) \qquad T_2(\lambda) \sim \sum_{p \geq 1} e_p(\lambda) \int_{-\infty}^{\infty} \left(\frac{d^p\tau}{dx^p}\right)^2 dx + f(\lambda)\int_{-\infty}^{\infty} \tau(x)\,dx$$

where the order of e_p tends to $-\infty$ as $p \to \infty$. The coefficients $e_p(\lambda)$ are, modulo $S^{-\infty}$, the same as would be obtained by expansion of $d^2T^\epsilon(\lambda)/d\epsilon^2$, at $\epsilon=0$, for the same perturbation $y = \epsilon\tau(x)$ of the boundary $y=0$ of the cylindrical domain $\mathbb{R}_x/\rho\mathbb{Z} \times [0,1]$ for ρ so large that $\mathrm{supp}(\tau) \subset (-\frac{1}{2}\rho, \frac{1}{2}\rho)$.

__Proof__. Consider the compact manifold, with flat metric,

$$X = \mathbb{R}_x/\rho\mathbb{Z} \times [0,1]$$

under perturbation to X_ϵ, with the boundary $y=0$ modified to $y = \epsilon\tau(x)$, $-\frac{1}{2}\rho < x < \frac{1}{2}\rho$ being considered as a fundamental domain for $\mathbb{R}/\rho\mathbb{Z}$. The existence of an expansion for $T^\epsilon(\lambda)$, given by (3.5) is standard, with the coefficients again being the invariants a_k, b_k of (1.1). The existence of an expansion (3.13) for $d^2T^0(\lambda)/d\epsilon^2$ therefore follows from formulae (3.7), (3.8) but now for perturbatiion X_ϵ of X. Only terms linear or quadratic in τ can occur, so the expansion must be of the form (3.13). Moreover, if $\bar{r}^\epsilon(\lambda,z,z')$ is the corresponding kernel, satisfying

$$(3.14)\qquad \begin{cases} (\Delta - \lambda)\bar{r}^{\epsilon}(\lambda;z;z') = \delta(z - z') & X_{\epsilon} \times X_{\epsilon} \\ \bar{r}^{\epsilon}(\lambda,z,z') = 0 & y = \epsilon\tau(x) \text{ on } y = 1 \end{cases}$$

then from (3.5),

$$(3.15)\qquad T(\lambda) = \int_{-\frac{1}{2}p}^{\frac{1}{2}p} \int_{\epsilon\tau(x)}^{1} \partial_{\lambda}\bar{r}^{\epsilon}(\lambda,x,y;x,y)\, dy\, dx .$$

Direct differentiation gives:

$$(3.16)\qquad d^2T^0/d\epsilon^2 = \int_X \partial_{\lambda}\bar{r}''(\lambda,z,z)\, dz - 2\int_{-\frac{1}{2}p}^{\frac{1}{2}p} \tau(x)\, \partial_{\lambda}\bar{r}'(\lambda,x,0;x,0)\, dx$$

$$- \int_{-\frac{1}{2}p}^{\frac{1}{2}p} \tau^2(x)(\partial_y+\partial_{y'})\partial_{\lambda}\bar{r}(\lambda,x,0;x,0)\, dx$$

where $\bar{r}, \bar{r}', \bar{r}''$ are defined as in (3.10) but from $\bar{r}^{\epsilon}$.

Finally, to prove (3.13) it is enough to show that the difference between each term in (3.11) and the corresponding term in (3.16) is rapidly decreasing as $|\lambda| \to \infty$, outside a cone containing $\bar{R}^+$. Consider first the passage from the kernel r^{ϵ} satisfying (3.6) to the corresponding kernel $\tilde{r}^{\epsilon}$ where R_x is replaced by $R_x/p\mathbb{Z}$, assuming always that the perturbation has support in $(-\frac{1}{2}p, \frac{1}{2}p)$. In fact,

$$\tilde{r}^{\epsilon}(\lambda,z,z') = \sum_{k\in\mathbb{Z}} r^{\epsilon}(\lambda,z+kp,z') .$$

Standard elliptic estimates show that on X_ϵ the difference

$$\partial_\epsilon \tilde{r}^\epsilon - \partial_\epsilon r^\epsilon = O(|\lambda|^{-N}) \quad \forall N . \tag{3.17}$$

Similarly, in passing from $\tilde{r}^\epsilon$ to $\bar{r}^\epsilon$ a correction term must be added to give the Dirichlet condition at $y = 1$, rather than decrease at infinity. Since the boundary condition is independent of ϵ , and at a finite distance from the perturbed boundary, the difference between each term in (3.11) and the corresponding term in (3.16) is rapidly decreasing as $|\lambda| \to \infty$, along Γ . This completes the proof of Proposition 3.12.

We shall now proceed to derive the expansion (3.13) explicitly. The point of the proposition is that, by comparison with the expansion of $d^2T^0/d\epsilon^2$, the coefficients can be related to the c_k in (2.7).

4. Computation of coefficients. Starting from the formula (3.11) we shall compute the form of the expansion (3.13). Thus, it is first necessary to find r, r', r'' and associated kernels. If $(\xi^2 - \lambda)^{1/2}$ is for $\xi \in \mathbb{R}$, $\lambda \notin \bar{\mathbb{R}}^+$ the branch with positive real part, then

$$(4.1)\quad F(\lambda,z,z') = \begin{cases} (4\pi)^{-1} \int e^{i(x-x')\xi-(\xi^2-\lambda)^{1/2}(y-y')}(\xi^2-\lambda)^{-1/2}d\xi, & y > y' \\ (4\pi)^{-1} \int e^{i(x-x')\xi-(\xi^2-\lambda)^{1/2}(y'-y)}(\xi^2-\lambda)^{-1/2}d\xi, & y' > y \end{cases}$$

is the preferred, i.e. temperate, fundamental solution of $\Delta - \lambda$ on $\mathbb{R}^2$. In terms of this kernel,

$$(4.2)\qquad r(\lambda,z,z') = F(\lambda,z,z') - G(\lambda,z,z')$$

$$(4.3)\qquad G(\lambda,z,z') = (4\pi)^{-1} \int e^{i(x-x')\xi-(\xi^2-\lambda)^{1/2}(y+y')}(\xi^2-\lambda)^{-1/2}\,d\xi .$$

Thus, by continuity

$$(4.4)\qquad (\partial_y + \partial_{y'})\,\partial_\lambda r(\lambda,x,0;x,0) = -2\partial_y\partial_\lambda G(\lambda,x,0;x,0) = 0 ,$$

so the third term in (3.11) is identically zero.

The corresponding Poisson kernel, solving

$$(4.5)\qquad \begin{cases} (\Delta - \lambda)P(\lambda,z;x') = 0 \\ P(\lambda,x,0;x') = \delta(x-x') \end{cases}$$

and vanishing at infinity is

$$P(\lambda,z;x') = (2\pi)^{-1}\int e^{i(x-x')\xi-(\xi^2-\lambda)^{1/2}y}\,d\xi \,. \tag{4.6}$$

Differentiating (3.6) gives the boundary problem for r':

$$(\Delta-\lambda)r'(\lambda,z;z')=0\ ,\quad r'(\lambda,x,0;z') = -\tau(x)\,\partial_y r(\lambda,x,0;z')\ , \tag{4.7}$$

with r' vanishing at infinity. Thus, from (4.6) and (4.2),

$$r'(\lambda,z;z') = \lim_{\chi\to 1} -(2\pi)^{-2}\int e^{i(x-x')\xi+i(x''-x)\xi'-(\xi^2-\lambda)^{1/2}y-((\xi')^2-\lambda)^{1/2}y'} \cdot \tau(x'')\,\chi(\xi)\,\chi(\xi')\,d\xi'\,d\xi\,dx'' \,. \tag{4.8}$$

Where the compactifying cutoffs χ have been inserted to ensure absolute convergence, and the limit is taken over $\chi \longrightarrow 1$ in the symbol topology S^{δ}, $\delta > 0$. Differentiating (3.6) twice gives the corresponding boundary problem for r'':

$$\begin{cases} (\Delta-\lambda)r''(\lambda,z;z') = 0 \\ r''(\lambda,x,0;z') = -2\tau(x)\partial_y r'(\lambda,x,0;z') - \tau^2(x)\partial_y^2 r(\lambda,x,0;z'). \end{cases} \tag{4.9}$$

From (4.2), $\partial_y^2 r = 0$ at $y = 0,\ z' \in Z$ so the second term in the boundary condition for r'' is zero. Applying (4.6) again allows one to construct r'', we only need

$$\partial_\lambda r''(\lambda,z,z) = \lim_{\chi\to 1}\partial_\lambda - 2(2\pi)^{-3}\int e^{i(x-x')\xi+i(x'-x'')\xi'+i(x''-x)\xi''} \cdot e^{-(\xi^2-\lambda)^{1/2}y-((\xi'')^2-\lambda)^{1/2}y((\xi')^2-\lambda)^{1/2}} \cdot \tau(x')\tau(x'')\chi(\xi)\chi(\xi')\chi(\xi'')\,d\xi''\,d\xi'\,d\xi\,dx''\,dx' \,. \tag{4.10}$$

Again from (4.8), the second term in (3.11) is identically zero, so

$$T_2(\lambda) = \lim_{\chi\to 1} \partial_\lambda -2(2\pi)^{-3} \int e^{i(x-x')\xi+i(x'-x'')\xi'+i(x''-x)\xi''}$$

$$(4.11) \qquad \cdot\ \tau(x')\tau(x'')\chi(\xi)\chi(\xi')\chi(\xi'')((\xi')^2-\lambda)^{1/2}((\xi^2-\lambda)^{1/2}+((\xi'')^2-\lambda)^{1/2})^{-1}$$

$$\cdot\ d\xi\ d\xi'\ d\xi''\ dx'\ dx''\ dx\ .$$

In (4.11) the integral over x , ξ'' can be evaluated directly, since the integrand is independent of x , giving:

$$T_2(\lambda) = \lim_{\lambda\to 1} \partial_\lambda\ -(2\pi)^{-2} \int e^{i(x'-x)(\xi'-\xi)}\tau(x')\tau(x)$$

(4.12)

$$\cdot\ \chi(\xi)\chi(\xi')((\xi')^2-\lambda)^{1/2}(\xi^2-\lambda)^{-1/2}\ d\xi\ d\xi'\ dx'\ dx\ .$$

The integrals over x' and ξ' can now be evaluated by the lemma of stationary phase:

$$T_2(\lambda) \sim \sum_{k\geq 1} e_k(\lambda) \int (d^k\tau(x)/dx^k)^2\ dx$$

(4.13)

$$e_k(\lambda) = -\ ((2k)!(2\pi))^{-1}\ \partial_\lambda \int (\xi^2-\lambda)^{-1/2}\ \partial_\xi^{2k}(\xi^2-\lambda)^{1/2}\ d\xi$$

the odd order terms being automatically zero.

The coefficients $e_k(\lambda)$ is actually homogeneous of degree $-k-\frac{1}{2}$, so from its holomorphy properties

$$(4.14) \qquad e_k(\lambda) = \gamma_{k-2}(-\lambda)^{-k-\frac{1}{2}}\ ,$$

with

$$\gamma_{k-2} = -(k-\tfrac{1}{2})((2k)!(2\pi))^{-1} \int (s^2+1)^{-1/2} d^{2k}(s^2+1)^{1/2}/ds^{2k}\, ds$$

(4.15)

$$= (-1)^k (k-\tfrac{1}{2})^2 ((2k)!\ 2\pi)^{-1} \int (d^{k-1}(s^2+1)^{-1/2}/ds^{k-1})^2\, ds \ .$$

This shows that all the coefficients γ_{k-2} , $k \geq 2$, are non-zero. Tracing backwards through (3.13) it follows easily that all the c_k , $k \geq 0$, are non-zero, providing Proposition 2.9.

More explicitly a short computation gives the formula

$$(4.16) \qquad c_k = \tfrac{1}{2}\, \gamma_k\, \frac{1}{2\pi i} \int_\Gamma e^{-\lambda}(-\lambda)^{-k-5/2}\, d\lambda \qquad k > 0 \ ,$$

using (4.15) this becomes:

$$(4.17) \quad c_k = -(4\pi^2 \cdot (2k+4)!)^{-1} \Gamma(-k-3/2)(k+3/2)^2 \int (d^{k+1}(1+s^2)^{-1/2}/ds^{k+1})\, ds.$$

References

1. L. Boutet de Monvel, Boundary problems for pseudo-differential operators. Acta. Math. 126 (1971) 11-51.

2. M. Kac, Can one hear the shape of a drum? Amer. Math. Monthly 73 (1966), 1-23.

3. S. Marvizi and R.B. Melrose, Spectral invariants of convex planar regions. J. Diff. Geom. 17 (1982) 475-502.

4. S. Marvizi and R.B. Melrose, Wave invariants and spectral determinacy. In preparation.

5. L. Smith, The asymptotics of the heat equation for a boundary value problem. Invent. Math. 63 (1981), 467-493.