18.100B, TEST 1, SOLUTIONS WITH EXPLANATIONS AND DISCUSSION

In-class test, 18.100B, March 19, 1996

Try each of the questions; they will be given equal value. You may use theorems from class, or the book, provided you can recall them correctly! I remind you only that a set in a metric space is compact if every open cover of it has a finite subcover. Don't bother the proctor with questions, he or she is not supposed to know anything about the test.

Problem 1

Consider the set S defined as follows. The elements of S are sequences, $\{s_n\}_{n=1}^{\infty}$ with all entries either 1 or 2 and with the additional property that every 2 is followed by a 1. Said more precisely, for every n, $s_n = 1$ or $s_n = 2$ and if $s_n = 2$ then $s_{n+1} = 1$. Say why precisely one of the following is true

(a) S is finite
(b) S is countably infinite
(c) S is uncountably infinite

and then decide which one is true and *prove* it.

Problem 2

Consider the metric space $M = [0,1] = \{x \in \mathbb{R}; 0 \leq x \leq 1\}$ with the usual metric, $d(x,y) = |x-y|$. Is the set $A = [0, \frac{1}{2}) = \{x \in \mathbb{R}; 0 \leq x < \frac{1}{2}\}$ open as a subset of M? What is the closure of A as a subset of M? Is A compact? Is the closure of A compact? In each case justify your answer.

Problem 3

Let M be a *compact* metric space. Suppose $A \subset M$ is *not* compact. Show, directly from the definition or using a theorem proved in class, that A is *not* closed.

Problem 4

Recall that a set S in a metric space M is connected if any separated decomposition of it, $S = A \cup B$ where $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, is 'trivial' in the sense that either A or B is empty. Show that the whole metric space M is connected if and only if the only subsets $A \subset M$ of it which are *both open and closed* are the 'trivial' cases $A = \emptyset$ and $A = M$.

Results

Total marks varied between 24 and 41 (well I gave one additional mark to anyone who pointed out the stupidity of question 3 as stated on the test. The median was 31 or thereabouts. There was quite a cluster around the median – which reflects the fact that many people did not make much out of the last question. I would not want to base a letter grade on this test – my basic position is that I will give as many A's as there are people who achieve a clear understanding of the material, B's for people with a good understanding and C for those (very few if any) who achieve only a shaky understanding.

Solutions and chat

Problem 1

By definition a set S is finite if it is either empty of else is in 1-1 correspondence with the set $\{1,\dots,n\}$ for some n. It is countably infinite if it is in 1-1 correspondence with $\mathbb{N}$ and it is uncountably infinite if it is neither finite nor countably infinite. Only one of these can hold (since we know that $\mathbb{N}$ cannot be in 1-1 correspondence with $\{1,\dots,n\}$ for any finite n.

The set S is uncountably infinite. Here is a proof that reduces it to the case we looked at in class. Namely, we know that S' which consists of the set of sequences with values 0 or 1 is uncountably infinite. We show that S and S' are in 1-1 correspondence (and hence have the same cardinality by definition). Take a sequence in S' and replace evey occurence of 0 by two terms, 2 followed by 1. This gives a sequence in S. Moreover no two sequences in S' are mapped to the same sequence in S. Thus the map is injective. We can construct an inverse the same way, replace every pair $2,1$ by one element 0. Thus S is indeed uncountably infinite.

Remarks. It is also fairly straightforward to use the diagonalization procedure, but not completely trivial since you have to make sure that the new sequence is in S and different from the others.

Marks: $4+6$ – Why did a lot of people not do the first part? Bravado I suppose, since those marks were clearly too easy to get. There was still some confusion here – in counting the elements of S each element counts as 1! Maybe it is an Australian joke (and a very weak one at that) to say that the way to count the number of sheep in a field is to count the legs and divide by 4.

Problem 2

Everything is relative to the metric space $M=[0,1]$.

1. As a subset of M, A is indeed open. If $x\in A$ then $B(x,\epsilon)\subset A$ if $\epsilon=\frac12-x$, since $|y-x|<\epsilon$, $y\in[0,1]$ implies $y<\frac12$ and hence $y\in A$.
2. Clearly $\frac12$ is a limit point of A so the closure $\bar A\supset[0,\frac12]$. By the same argument as above $(\frac12,1]$ is open in $[0,1]$ so this set is closed and hence $\bar A=[0,\frac12]$.
3. Since A is not closed it cannot be compact.
4. By the Heine-Borel theorem $\bar A=[0,\frac12]$ is compact since is closed and bounded.

0.1. **Remarks.** Usual errors here were to say that A is not open, thinking of it as a subset of $\mathbb{R}$ this is certainly true but it is open as a subset of M. Similarly in the third part it does not follow directly from the fact that A is open that it is not

compact! It does follow from the fact that it is not closed, but in a finite metric space (which this is not) there are open compact sets.

Marks 2+3+2+3.

Problem 3

Disaster and dismal failure of proofreading here. The word 'compact' was supposed to appear before the first 'metric space' and then the question becomes very easy. I decided to give 10 marks to everyone and subtract a few marks for really unwise statements – in particular I took of 1 mark for anyone who said 'we have a theorem to the effect that every closed subset of a metric space is compact'. Not so and I should really have taken off at least 8 marks, but .... Only one person got 11 marks by pointing out the stupidty of the question.

Theorem The earth is flat.

Proof If it wasn't flat then professors would eventually roll away from MIT. They seem to hang around for years so it is flat.

Marks 10 ± 1.

Problem 4

Suppose first that M is connected. Let A be a subset of M which is both open and closed. Then $B = M \setminus A$ is also both open and closed and $M = A \cup B$. Since A and B are separated ($A \cap B = \emptyset$ and $\bar{A} = A$, $\bar{B} = B$) it follows, from the assumption that M is connected, that one of them is empty, so $A = \emptyset$ or $A = M$ are the only sets which are both open and closed.

Conversely, suppose that the only subsets of M which are both open and closed are M and $\emptyset$. Then let A and B be separated sets in M such that $M = A \cup B$. This means that $B = M \setminus A$ is the complement of A in M. The conditions that A and B be separated imply that $\bar{A} \cap B = \emptyset$, so $\bar{A} \subset M \setminus B = A$ hence A must be closed. Similarly B must be closed and hence A must be both open and closed. Thus one of A or B must be empty and hence, by definition, M is connected.

Remarks. The main problem here seems to be logical confusion. Some people proved one direction (typically that if there is a set which is neither $\emptyset$ nor M and is both open and closed then M is not connected) and then did it again. Some people gave long proofs that $\emptyset$ and M *are* both open and closed. This is a 'real' question, it is important that you understand both the logic and the details of the (or a) proof.

Marks 5+5 for the two directions.