18.100B TEST 1, 18 MARCH 2004
11:05AM – 12:25PM, WITH SOLUTIONS

Total Marks possible: $10 \times 6 = 60$
Average Mark: 43
Median: 40

You are permitted to bring the book 'Rudin: Principles of Mathematical Analysis' with you – just the book, nothing else is permitted (and no notes in your book!) You may use theorems, lemmas and propositions from the book. Note that where $\mathbb{R}^k$ is mentioned below the standard metric is assumed.

(1) Suppose that $\{p_n\}$ is a sequence in a metric space, X, and $p \in X$. Assuming that every subsequence of $\{p_n\}$ itself has a subsequence which converges to p show that $p_n \to p$.

Solution:- To say that $\{p_n\}$ converges to p is to say that for every $\epsilon > 0$ the set $\{n \in \mathbb{N}; d(p, p_n) > \epsilon\}$ is finite. Thus if $\{p_n\}$ were not to converge to p then for some $\epsilon > 0$ this set would be infinite. Then we can take the subsequence $\{p_{n_k}\}$ where $\{n_k\}$ is the unique increasing sequence with range $\{n \in \mathbb{N}; d(p, p_n) > \epsilon\}$. Thus sequence cannot have any subsequence converge to p since for any susbsequence (of the subsequence) $\{p_{n_{k_l}}\}$ all points lie outside $B(p, \epsilon)$. This proves the result by contradiction.

(2) Let x_n, $n = 1, 2, \ldots$, be a sequence of real numbers with $x_n \to 0$. Show that there is a subsequence $x_{n(k)}$, $k = 1, 2, \ldots$, such that $\sum\limits_k |x_{n(k)}| < \infty$.

Solution:- By definition of convergence, give for every k there exists N such that $n \geq N$ implies $|x_n| < 2^{-k}$. Thus we can choose a subsequence $\{x_{n_k}\}$ with $|x_{n_k}| < 2^{-k}$ for all k. Then $\sum\limits_{k=1}^{N} |x_{n_k}| < \sum\limits_{k=1}^{N} 2^{-k} < 1$ so the sequence of partial sums is increasing and bounded above, hence convergent.

(3) Give examples of:
- (a) A countable subset of $\mathbb{R}^2$ which is infinite and closed.
- (b) A subset of the real interval $[-2, 2]$ which contains $[0, 1]$ but is not compact.
- (c) A metric space in which all subsets are compact.
- (d) A cover of $[0, 1]$ as a subset of $\mathbb{R}$ which has no finite subcover.

Solution:- Note that I did not ask you to prove this, but they do need to be right. Not just discrete in the third case, etc.
- (a) The subset $\{x = (1/n, 0); n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}^2$ is closed since the only limit point is 0. It is certainly infinite and countable.
- (b) For example $(-2, 1]$ contains $[0, 1]$ but is not compact since it is not closed.
- (c) Any finite metric space has the property that all subsets are finite, hence compact.
- (d) Taking the collection of single-point subsets $V_x = \{x\}$, $x \in [0, 1]$ is a cover of $[0, 1]$ but has no finite subcover. (I did not say *open*).

(4) Let K be a compact set in a metric space X and suppose $p \in X \setminus K$. Show that there exists a point $q \in K$ such that

$$d(p, q) = \inf\{d(p, x); x \in K\}.$$

Solution:-

Method A By definition of the infimum, there exists a sequence q_n in K such that $d(p, q_n) \to I = \inf\{d(p, x); x \in K\}$. Since K us compact, this has a convergent subsequence. Replacing the original sequence by this subsequence we may assume that $q_n \to q$ in K. Now, by the triangle inquality

$$|d(p, q) - d(p, q_n)| \leq d(q, q_n) \to 0$$

by the definition of convergence. Thus $d(p, q)$ must be the limit of the sequence $d(p, q_n)$ in $\mathbb{R}$, so $d(p, q) = I$ as desired.

Method B Let $I = \inf\{d(p, x); x \in K\}$. If there is no point $q \in K$ with $d(p, q) = I$ then the open sets $E_\epsilon = \{x \in X; d(p, x) > I + \epsilon\}$, $\epsilon > 0$ cover K. By compactness there is a finite subcover, so $K \subset E_\epsilon$ for some $\epsilon > 0$ which contradicts the definition of I.

Method C (really uses later stuff) Since $f : K \ni x \longmapsto d(p, x)$ is continuous (prove using triangle inequality) and K is compact, $f(K)$ is compact in $\mathbb{R}$, so contains its infimum. Thus there exists $q \in K$ with $d(p, q) = \inf\{d(p, x); x \in K\}$.

(5) (a) Let X be a (non-empty) metric space with metric d and let $q \notin X$ be an external point. Show that there is a unique metric d_Y one $Y = X \cup \{q\}$ satisfying

$$d_Y(x, x') = d(x, x'),\ \forall\ x, x' \in X,\ d_Y(q, x) = 1,\ \forall x \in X.$$

(b) Show that with this metric Y is not connected.

Solution:- Unfortunately I got carried away here and this is not true! I should have said that X is a metric space with $d(x, x') \leq 2$ for all $x, x' \in X$; then it works fine. I hope I did not confuse anyone too much by this. I gave everyone full marks for the whole question.

(6) Let X be a metric space and $A \subset X$. Let A° be the union of all those open sets in X which are subsets of A. Show that the complement of A° is the closure of the complement of A.

Solution:- A set contained in A is exactly one with complement containing the complement of A. Thus, from the definition, the complement of A° is the intersection of all closed sets which contain the complement of A. This we know to be its closure.

Or:- $A^\circ = A \setminus (A^{\complement})'$ – since a point in A is either an interior point (lies in an open subset of A or else is a limit point of the complement). Thus $(A^\circ)^{\complement} = A^{\complement} \cup (A^{\complement})'$ which is the closure of the complement.