18.100B PRACTICE TEST 1, FOR ACTUAL TEST ON 18 MARCH 2004

The test on Thursday will be open book – just the book, nothing else is permitted (and no notes in your book!) Note that where $\mathbb{R}^k$ is mentioned below the standard metric is assumed.

(1) Let $C \subset \mathbb{R}^n$ be closed. Show that there is a point $p \in C$ such that $|p| = \inf\{|x|; x \in X\}$.

(2) Give a counterexample to each of the following statements:
   (a) Subsets of $\mathbb{R}$ are either open or closed
   (b) A closed and bounded subset of a metric space is compact.
   (c) In any metric space the complement of a connected set is connected.
   (d) Given a sequence in a metric space, if every subsequence of that sequence itself has a convergent subsequence then the original sequence converges.

(3) Suppose A and B are connected subsets of a metric space X and that $A \cap B \neq \emptyset$, show that $A \cup B$ is connected.

(4) Let K_i, $i = 1, \dots, N$, be a finite number of compact sets in a metric space X. Show that $\bigcup_{i=1}^N K_i$ is compact.

(5) Let $G_i \subset X$, $i \in \mathbb{N}$ be a countable collection of open subsets of a complete metric space, X. Suppose that for each $N \in \mathbb{N}$, $\bigcap_{i=1}^N G_i \neq X$ and that for each n, $x, y \in X \setminus G_n \Longrightarrow d(x, y) < 1/n$. Show that $\bigcup_{i \in \mathbb{N}} G_i \neq X$.

(6) Let x_n be a bounded sequence in $\mathbb{R}$. Show that there exists $x \in \mathbb{R}$ and a subsequence $x_{n(k)}$ such that $\sum\limits_{k=1}^{\infty} x_{n(k)} - x$ converges absolutely.