7. Topic 1: Determinant and eta
In place of lecture for Friday, 12 September, 2008

A natural question to ask yourself is:- Was it really necessary to construct the Fredholm determinant by hand? In fact Boris did suggest something pretty close to that! Indeed the answer is no, it was not really necessary, but how else would you appreciate the following?

Recall the delooping sequence

$$G^{-\infty}_{\text{sus}} \longrightarrow \tilde{G}^{-\infty}_{\text{sus}} \xrightarrow{R} G^{-\infty}. \tag{7.1}$$

Now, I showed that the first odd Chern form, $\operatorname{Tr}(g^{-1}dg)$ can be 'lifted' to the central group – not by using R don't be confused on this – by using the evaluation map and push-forward. This defines, in general, the eta forms or in this case the eta invariant on the central group:-

$$\eta : \tilde{G}^{-\infty}_{\text{sus}} \longrightarrow \mathbb{C},\ \eta(\tilde{g}) = \int_{\mathbb{R}} \operatorname{tr}\left(\tilde{g}^{-1}(t)\frac{d\tilde{g}(t)}{dt}\right) dt. \tag{7.2}$$

Is this really an eta invariant you ask? Well, in this context it is, it is not exactly *THE* eta invariant – that is essentially the same thing for the quantization sequence as we shall see later.

Exercise 1. Show that the eta invariant as defined in (7.2) is log-multiplicative:-

$$\eta(gh) = \eta(g) + \eta(h). \tag{7.3}$$

So, how does this construct the determinant? Well, I showed before that

$$d\eta = R^* \operatorname{Ch}_1 . \tag{7.4}$$

In particular, combined with (7.3) this just means

$$\eta\big|_{G^{-\infty}_{\text{sus}}} \text{ is locally constant.} \tag{7.5}$$

Exercise 2. Show by finite dimensional approximation (remember that $\mathcal{S}(\mathbb{R})$ behaves like $\dot{\mathcal{C}}^\infty[0,1]$)) that

$$\eta : G^{-\infty}_{\text{sus}} \longrightarrow 2\pi i\mathbb{Z}. \tag{7.6}$$

So we can define an integer-valued 'index' map (really it is the winding number of the Fredholm determinant)

$$\operatorname{ind} = \frac{\eta}{2\pi i} : G^{-\infty}_{\text{sus}} \longrightarrow \mathbb{Z}. \tag{7.7}$$

This can be added to the exact sequence above to get a commutative diagram:

$$\begin{array}{ccccc} G^{-\infty}_{\text{sus}} & \longrightarrow & \tilde{G}^{-\infty}_{\text{sus}} & \xrightarrow{R} & G^{-\infty} \\ \downarrow{\scriptstyle \operatorname{ind}} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \det} \\ \mathbb{Z} & \xrightarrow{2\pi i\times} & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^*. \end{array} \tag{7.8}$$

Now, what about

$$\begin{gathered} SG^{-\infty} = \{g \in G^{-\infty}; \det(g) = 1\} \hookrightarrow G^{-\infty}, \\ \tilde{G}^{-\infty}_{\text{sus},\eta=1} = \{\tilde{g} \in \tilde{G}^{-\infty}_{\text{sus}}; \eta(\tilde{g}) = 0\}, \\ G^{-\infty}_{\text{sus},\text{ind}=0} = \{g \in G^{-\infty}_{\text{sus}}; \eta(g) = 0\}. \end{gathered} \tag{7.9}$$

Exercise 3. Show that $\tilde{G}^{-\infty}_{\mathrm{sus},\eta=0}$ is contractible and that the combined diagram

$$
\begin{array}{ccccccccc}
 & & \{\mathrm{Id}\} & & \{\mathrm{Id}\} & & \{\mathrm{Id}\} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\{\mathrm{Id}\} & \longrightarrow & G^{-\infty}_{\mathrm{sus},\mathrm{ind}=0} & \longrightarrow & \tilde{G}^{-\infty}_{\mathrm{sus},\eta=0} & \xrightarrow{R} & SG^{-\infty} & \longrightarrow & \{\mathrm{Id}\} \\
 & & \downarrow{\scriptstyle \mathrm{ind}} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \det} & & \\
\{\mathrm{Id}\} & \longrightarrow & G^{-\infty}_{\mathrm{sus}} & \longrightarrow & \tilde{G}^{-\infty}_{\mathrm{sus}} & \xrightarrow{R} & G^{-\infty} & \longrightarrow & \{\mathrm{Id}\} \\
 & & \downarrow{\scriptstyle \mathrm{ind}} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \det} & & \\
\{0\} & \longrightarrow & \mathbb{Z} & \xrightarrow{2\pi i\times} & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* & \longrightarrow & \{1\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \{0\} & & \{0\} & & \{1\} & &
\end{array} \tag{7.10}
$$

is commutative and has exact rows and columns.

The first row is a reduced classifying sequence for K-theory.

Exercise 4. (Needs a bit more analysis) Define the unitary subgroup of $G^{-\infty}$, show that $G^{-\infty}$ retracts to it and construct a diagram as in (7.10) based on the unitary group and its loop groups and their subgroups.