6. Lecture 6: Higher loop groups and determinant Wednesday, 10 September, 2008

Before picking up where I left off last time, let me outline where I will head after the coming week-long break. What we need to justify the definition of the odd K-theory of a space as the homotopy classes of maps into $G^{-\infty}(\mathbb{N})$ is to prove *Bott periodicity*:

$$\pi_j(G^{-\infty}(\mathbb{N})) = \begin{cases} \{0\} & j \text{ even} \\ \mathbb{Z} & j \text{ odd.} \end{cases} \tag{6.1}$$

We already know that $G^{-\infty}(\mathbb{N})$ is connected and using the Fredholm determinant it is reasonably easy to show the result for π_1. The general case will following from this by constructing a weak homotopy equivalence

$$G^{-\infty}_{\mathrm{sus}(2)}(\mathbb{N}) \longrightarrow G^{-\infty}(\mathbb{N}). \tag{6.2}$$

I have not defined the group on the left yet, but will do so today. It is just an iterated loop group; see (6.8).

At the end last time I talked about turning flat loops into Schwartz functions. The basic statement is simple enough. Consider the diffeomorphism of the open interval to the line

$$T : (0, 2\pi) \ni \theta \longmapsto \tan\left(\frac{\theta - \pi}{2}\right). \tag{6.3}$$

The derivative is $\frac{1}{2}\sec^2(\frac{\theta-\pi}{2}) > 0$ on $(0, 2\pi)$. As $\theta \downarrow 0$, $\theta T(\theta) \to -2$ and similarly at the other end, $(2\pi - \theta)T(\theta) \to 2$.

Lemma 7. *Pull-back under T in* (6.3) *gives a topological isomorphism*

$$T^* : \mathcal{S}(\mathbb{R}) \longrightarrow \dot{\mathcal{C}}^\infty([0, 2\pi]) = \left\{u \in \mathcal{C}^\infty([0, 2\pi]);\ \frac{d^k u}{d\theta^k}(e) = 0,\ e = 0, 2\pi,\ \forall\ k \geq 0\right\}. \tag{6.4}$$

Proof. Clearly under T^* an element $v \in \mathcal{S}(\mathbb{R})$ pulls back to be smooth in the interior. The derivatives of T all grow at most polynomially at the end points from which it follows easily that $T^* v \in \dot{\mathcal{C}}^\infty([0, 2\pi])$ and the converse is similar. □

Of course this proof could do with a bit of expansion!

Anyway, it follows from this Lemma, or a rather a generalization of it, that the suspended group can be moved to the real line.

$$\begin{gathered} b \in G^{-\infty}_{\mathrm{sus}}(\mathbb{N}) = \{b \in \dot{\mathcal{C}}^\infty([0, 2\pi]; \Psi^{-\infty}(\mathbb{N}); (\mathrm{Id} + b(\theta)) \in G^{-\infty}(\mathbb{N})\} \\ \Longleftrightarrow \\ b = T^* b',\ b' \in \mathcal{S}(\mathbb{R}; \Psi^{-\infty}(\mathbb{N})) \text{ and } \mathrm{Id} + b'(t) \in G^{-\infty}(\mathbb{N})\ \forall\ t \in \mathbb{R}. \end{gathered} \tag{6.5}$$

The point here, as usual, is that having values in $\Psi^{-\infty}(\mathbb{N})$ is really no different from usual complex-valued functions.

It is also important to note that the even Chern forms, defined on $G^{-\infty}_{\mathrm{sus}}(\mathbb{N})$ are independent of such a change of parameterization, even though it is from a compact to a non-compact space. Namely they are all given by push-forward of forms on this manifold, and this is independent of choice. At a more prosaic level the forms look like

$$\int_0^{2\pi} \operatorname{tr}(\dots \frac{dg}{d\theta} \dots) d\theta = \int_{\mathbb{R}} \operatorname{tr}(\dots \frac{dg}{dt} \dots) dt \tag{6.6}$$

with the Jacobians cancelling.

Now, one reason why this change of point of view, which is all I have really done here, is that combined with the shift last time from the sequential to isotropic forms of $G^{-\infty}$ it makes things look much more uniform. Thus we can write

$$\begin{aligned} G^{-\infty}(\mathbb{R}) &= \{a \in \mathcal{S}(\mathbb{R}^2); (\operatorname{Id} + a)^{-1} = \operatorname{Id} + a',\ a' \in \mathcal{S}(\mathbb{R}^2)\} \\ G^{-\infty}_{\text{sus}}(\mathbb{R}) &= \{b \in \mathcal{S}(\mathbb{R}^3); (\operatorname{Id} + b(t, \bullet)) \in G^{-\infty}(\mathbb{R})\ \forall\ t \in \mathbb{R}\} \\ \tilde G^{-\infty}_{\text{sus}}(\mathbb{R}) &= \big\{\tilde b \in \mathcal{C}^\infty(\mathbb{R}^3); (\operatorname{Id} + \tilde b(t, \bullet)) \in G^{-\infty}(\mathbb{R})\ \forall\ t \in \mathbb{R} \text{ and} \\ &\qquad \exists\ \tilde a_\pm \in \mathcal{S}(\mathbb{R}^3),\ a_\infty \in \Psi^{-\infty}(\mathbb{R}) \text{ such that} \\ &\qquad \tilde b(t) = a_-(t) \text{ in } t < 0,\ \tilde b = a_\infty + a_+(t) \text{ in } t > 0\big\}. \end{aligned} \tag{6.7}$$

In the second case, t is the first variable, which is just a parameter, and the other two are 'non-commutative' in the sense of the operator product. Of course the assertion here is that these two groups are, as has already been shown for the first two, isomorphic to $G^{-\infty}(\mathbb{N})$, $G^{-\infty}_{\text{sus}}(\mathbb{N})$ and $\tilde G^{-\infty}_{\text{sus}}(\mathbb{N})$, respectively, using the Hermite expansion and in the second two cases the compactification of $\mathbb{R}$ corresponding to T^{-1}. I leave the last case to you.

Now, since the suspended group is just the invertible Schwartz perturbations of the identity, it is reasonable to define higher loop groups in the same way:

$$G^{-\infty}_{\text{sus}(p)}(\mathbb{R}) = \{b \in \mathcal{S}(\mathbb{R}^{p+2}); (\operatorname{Id} + b(\tau, \bullet)) \in G^{-\infty}(\mathbb{R})\ \forall\ \tau \in \mathbb{R}^p\},\ p \in \mathbb{N}. \tag{6.8}$$

Having earlier shown that the spaces on $\mathbb{N}$ and $\mathbb{R}$ are the same I will simplify the notation and now write $\Psi^{-\infty}$, $G^{-\infty}$ and $G^{-\infty}_{\text{sus}(p)}$ for either case – and some more which will appear later.

Next I meant to discuss the Fredholm determinant rather briefly – actually I spent the rest of the lecture doing so!

Theorem 1. *There is a unique $\mathcal{C}^\infty$ functions, which is in fact entire analytic,* $\det(a) = \det_{Fr}(\operatorname{Id} + a)$

$$\det : \Psi^{-\infty} \longrightarrow \mathbb{C} \tag{6.9}$$

satisfying the multiplicativity condition

$$\det_{Fr}((\operatorname{Id} + a)(\operatorname{Id} + b)) = \det_{Fr}(\operatorname{Id} + a)\det_{Fr}(\operatorname{Id} + b)\ \forall\ a, b \in \Psi^{-\infty} \tag{6.10}$$

and the normalization

$$\frac{d}{dt}\det_{Fr}(\operatorname{Id} + ta)\big|_{t=0} = \operatorname{tr}(a). \tag{6.11}$$

The last condition prevents one from replacing det by a power.

Proof. The multiplicativity means that at any point $(\operatorname{Id} + b)$ the derivative can be computed:-

$$\begin{aligned} &\frac{d}{dt}\det_{\mathrm{Fr}}(\operatorname{Id} + b + ta)\big|_{t=0} = \\ &\quad \det_{\mathrm{Fr}}(\operatorname{Id} + b)\frac{d}{dt}\det_{\mathrm{Fr}}(\operatorname{Id} + t(\operatorname{Id} + b)^{-1}a)\big|_{t=0} = \det_{\mathrm{Fr}}(\operatorname{Id} + b)\operatorname{tr}\left((\operatorname{Id} + b)^{-1}a\right). \end{aligned} \tag{6.12}$$

This is just the first odd Chern form discussed earlier. Namely, the total derivative must satisfy

$$d\det_{\mathrm{Fr}}(g) = \det_{\mathrm{Fr}}(g)\,\mathrm{tr}\left(g^{-1}dg\right). \tag{6.13}$$

Of course, (6.13) is just the standard formula for the logarithmic derivative of the determinant for $N \times N$ matrices.

So, to define the function $\det_{\mathrm{Fr}}$ we can use the connectedness of $G^{-\infty}$ to choose a smooth curve

$$\chi : [0,1] \longrightarrow G^{-\infty},\ \chi(0) = \mathrm{Id},\ \chi(1) = g \tag{6.14}$$

and then set

$$\det_{\mathrm{Fr}}(g) = \exp\left(\int_\chi \mathrm{tr}\left(g^{-1}dg\right)\right) = \exp\left(\int_0^1 \mathrm{tr}\left(\chi^{-1}(t)\frac{d\chi(t)}{dt}\right)dt\right). \tag{6.15}$$

Since the integrand is smooth the integral on the right certainly exists. However, we need to show that the result does not depend on the choice of path. We will do this in two stages, first showing homotopy invariance.

Thus, suppose that $\chi(t,s)$, $t,s \in [0,1]$ is a smooth homotopy in $G^{-\infty}$ with $\chi(0,s) = \mathrm{Id}$, $\chi(1,s) = g$ fixed. The homotopy invariance follows from the fact that Ch_1 is closed, but let me prove it directly for reassurance. Thus we just compute the derivative

$$\begin{aligned}
&\frac{d}{ds}\int_0^1 \mathrm{tr}\left(\chi^{-1}(t)\frac{d\chi(t)}{dt}\right)dt \\
&= \int_0^1 \mathrm{tr}\left(-\chi^{-1}(t,s)\frac{d\chi(t,s)}{ds}\chi^{-1}(t,s)\frac{d\chi(t,s)}{dt} + \chi^{-1}(t,s)\frac{d^2\chi(t,s)}{dtds}\right)dt \\
&= \int_0^1 \mathrm{tr}\left(-\chi^{-1}(t,s)\frac{d\chi(t,s)}{dt}\chi^{-1}(t,s)\frac{d\chi(t,s)}{ds} + \chi^{-1}(t,s)\frac{d^2\chi(t,s)}{dtds}\right)dt \\
&= \int_0^1 \frac{d}{dt}\,\mathrm{tr}\left(\chi^{-1}(t,s)\frac{d\chi(t)}{ds}\right)dt = 0.
\end{aligned} \tag{6.16}$$

Here the trace identity has been used and of course the constancy at the ends.

It also follows directly that the result is independent of the parameterization of the curve. We can use this, as discussed above, to reparameterize the curve so that it is flat at both ends. Alternatively, this could have been required in the original definition. This flatness allows us to 'add' to curves and keep smoothness. Thus if

$$\chi_i : [0,1] \longrightarrow G^{-\infty} \text{ are smooth with } \frac{d^k}{dt^k}\chi_i(t') = 0\ \forall\ k \geq 1,\ t' = 0,1, \quad \chi_i(0) = \mathrm{Id},\ \chi_i(1) = g_i,\ i = 1,2 \tag{6.17}$$

then we can simply define

$$\chi : [0,1] = \begin{cases} \chi_1(2t) & 0 \leq t \leq \frac{1}{2} \\ a_1\chi_1(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{6.18}$$

and conclude (using left invariance of Ch_1) that

$$\det_{\mathrm{Fr}}(g_1g_2) = \det_{\mathrm{Fr}}(g_1)\det_{\mathrm{Fr}}(g_2) \tag{6.19}$$

once we have shown independence of the choice of homotopy class of path.

To show this independence it suffices, using (6.18) to see that if χ is a closed smooth curve into $G^{-\infty}$ starting and ending at the identity then

$$\int_0^1 \operatorname{tr}\left(\chi^{-1}(t)\frac{d\chi(t)}{dt}\right) dt \in 2\pi i\mathbb{Z}. \tag{6.20}$$

Since we have already shown homotopy invariance it suffices to show this where $\chi(t) = \operatorname{Id} + a(t)$ is replaced by $\operatorname{Id} + \Pi_N a \Pi_N$. This reduces to the case of the determinant on $\operatorname{GL}(N, \mathbb{C})$ where I assume it is well-known and I leave it to you. □

Below is what I meant to cover today, instead of spending so long discussing the Fredholm determinant.

So, that takes care of what was left over from last time. What I wanted to do today was consider the effect of the delooping sequence on the Chern forms. Written in terms of this isotropic version of the groups the delooping sequence is

$$G^{-\infty}_{\mathrm{sus}}(\mathbb{R}) \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{R}) \xrightarrow{R} G^{-\infty}(\mathbb{R}). \tag{6.21}$$

So far I have defined Chern forms on the first and last groups. The even Chern forms were defined by pull-back and integration and the same idea works for the middle group. However, I will call the resulting form an 'eta form'.

Thus, for each $k \geq 0$ there is a form on $\tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{R})$ given explicitly by

$$\eta_{2k} = -\int_{\mathbb{R}} \operatorname{tr}\left(g^{-1}\frac{\partial g}{\partial t}(g^{-1}dg)^{2k}\right) dt. \tag{6.22}$$

The main thing to observe here is that the integral is absolutely convergent. This follows as before from the fact that the term $\partial g/\partial t$ is Schwartz in t, since both the identity term and the constant term in the expansion as $t \to \infty$ are killed by differentiation. Thus the whole integrand, evaluated on $2k$ tangent elements is itself an element of $\mathcal{S}(\mathbb{R}^3)$.

Now, the main difference between the eta form and the Chern forms is that the latter were closed. The eta form is a 'transgression form'.

Proposition 10. *The eta forms in* (6.22) *restrict to the subgroup* $G^{-\infty}_{\mathrm{sus}}(\mathbb{R})$ *to the even Chern forms and have 'basic' derivatives in the sense that*

$$d\eta_{2k} = R^* \operatorname{Ch}_{2k+1}. \tag{6.23}$$

Proof. The first statement is immediate, since we are using the same formula, (6.22) to define both even Chern and even eta forms. To prove the second part we need to compute the derivative. Let me remind you of the proof that the even Chern forms are closed – obviously we just follow that and see what happens.

We know that the odd Chern form is closed. When pulled back under the evaluation map from $G^{-\infty}$ to $\mathbb{R} \times \tilde{G}^{-\infty}_{\mathrm{sus}}$ this becomes the condition

$$D \operatorname{tr}((g^{-1}Dg)^{2k+1}) = 0,\ D = dt\frac{\partial}{\partial t} \wedge + d. \tag{6.24}$$

Expanding out the form according in terms of the product we find

$$\operatorname{ev}^* \operatorname{Ch}_{2k+1} = dt \wedge A(t) + B(t) \Longrightarrow \frac{\partial B}{\partial t} = dA(t) \text{ and } dB(t) = 0. \tag{6.25}$$

Integrating over t we find by definition

$$\eta_{2k} = \int \operatorname{tr}(A(t))dt \Longrightarrow d\eta_{2k} = \int \operatorname{tr}(dA(t))dt = \int \operatorname{tr}(\frac{dB(t)}{dt})dt = B(\infty). \tag{6.26}$$

In the case of the Chern forms A and B are Schwartz. Here A is Schwartz, since there is a t-derivative somewhere. However, $B(t)$ involves no t derivative, so it has a possibly non-zero limit as $t \to \infty$. Clearly in fact

$$B(\infty) = \operatorname{tr}((g_\infty^{-1} dg_\infty)^{2k+1}) = R^* \operatorname{Ch}_{2k+1}\,. \tag{6.27}$$

□