3. Lecture 3: K-groups and loop groups
Wednesday, 3 September, 2008

Reconstructed, since I did not really have notes – because I was concentrating too hard on the 3 lectures on blow-up at MSRI!

(1) Odd K-theory
(2) Loop group
(3) Even K-theory
(4) Delooping sequence

Having defined the group $G^{-\infty}(\mathbb{N})$ and shown that it is open (and dense) in $\Psi^{-\infty}(\mathbb{N})$ we can define the odd K-theory of a space simply as the set of smooth equivalence classes of smooth maps. For the moment let us just consider a compact smooth manifold X then a map

$$f : X \longrightarrow G^{-\infty}(\mathbb{N}) \hookrightarrow \Psi^{-\infty}(\mathbb{N}) \tag{3.1}$$

is smooth if it is differentiable to infinite order. For a map into a fixed topological vector space this is quite a simple condition. Namely forget for that X is compact, then we certainly know what a continous map is. Differentiability at a point $\bar{x} \in X$ is the existence of the derivative, which is to be a continuous linear map,

$$Df(\bar{x}) : T_{\bar{x}}X \longrightarrow \Psi^{-\infty}(\mathbb{N}) \tag{3.2}$$

such that in local coordinates near $\bar{x}$, for given $\delta > 0$ there exists $\epsilon > 0$ such that

$$\|f(x) - f(\bar{x}) - Df(\bar{x})(x-\bar{x})\|_{(N)} \le \delta|x-\bar{x}| \text{ in } |x-\bar{x}| < \epsilon. \tag{3.3}$$

Then we require that $Df(\bar{x})$ exists everywhere so defines a map

$$Df : TX \longrightarrow \Psi^{-\infty}(\mathbb{N}) \tag{3.4}$$

which we then require to be continuous and differentiable. Proceeding inductively we can require the existence of higher derivatives by the same procedure, where differentiablilty in the linear variables is trivially true. Thus the kth derivative is required to be a map

$$D^kF(\bar{x}) : T_{\bar{x}}X \times T_{\bar{x}}X \cdots \times T_{\bar{x}}X \longrightarrow \Psi^{-\infty}(\mathbb{N}) \tag{3.5}$$

for each point $\bar{x} \in X$ which is the derivative of the $k-1$st derivative and which is continuous in all variables.

Examples are immediately provided by smooth maps $X \longrightarrow \mathrm{GL}(N,\mathbb{C})$ in the usual finite-dimensional sense, for any N because of the smooth inclusion

$$\mathrm{GL}(N,\mathbb{C}) \longrightarrow G^{-\infty}(\mathbb{N}). \tag{3.6}$$

So, having defined smoothness on compact manifold – including a compact manifold with boundary, we then define a smooth homotopy between two such maps. If $f_0,\ f_1 : X \longrightarrow G^{-\infty}(\mathbb{N})$ are smooth maps then they are said to be *smoothly homotopic* if there exists a smooth map

$$F : [0,1]_t \times X \longrightarrow G^{-\infty}(\mathbb{N}) \tag{3.7}$$

such that

$$F(0,x) = f_0(x),\ F(1,x) = f_1(x)\ \forall\ x \in X. \tag{3.8}$$

Definition 2. The odd K-theory of a compact manifold X is defined to be the set of equivalence classes under smooth homotopy of smooth maps $f : X \longrightarrow G^{-\infty}(\mathbb{N})$:

$$K^{-1}(X) = \{f : X \longrightarrow G^{-\infty}(\mathbb{N})\}/\sim . \tag{3.9}$$

The same definition applies to compact manifolds with boundaries, or with corners. For non-compact manifolds I will require the smooth maps to have 'compact support', meaning they reduce to the identity outside some compact set. The maps F in the homotopies are then also require to be equal to the identity outside a compact set, although of course the set is not itself fixed. I will use the slightly non-standard notation

$$K_{\mathrm{c}}^{-1}(X) = \{f : X \longrightarrow G^{-\infty}(\mathbb{N});\ \exists\, K \Subset X,\ f(x) = \mathrm{Id}\ \forall\, x \in X \setminus K\}/\sim \tag{3.10}$$

in this case.

Now, in fact $K^{-1}(X)$ is not just a set, but is an abelian group. That it is a group is relatively clear. The commutativity of the group structure follows from the approximation properties.

Proposition 5. *Group composition in $G^{-\infty}(\mathbb{N})$ induces the structure of an abelian group on $K^{-1}(X)$.*

Proof. Given two smooth maps $f_i : X \longrightarrow G^{-\infty}(\mathbb{N})$, $i = 1, 2$, the product $f_1f_2(x) = f_1(x)f_2(x)$ is smooth in view of the smoothness of the product map on $G^{-\infty}(\mathbb{N})$. To see that $K^{-1}(X)$ inherits a group structure from this, we need to check that it is consistent with homotopy, i.e. is independent of the choice of representative. However, that is obvious enough since if $f_i' : X \longrightarrow G^{-\infty}(\mathbb{N})$, $i = 1, 2$ are two other representatives of the same K-classes then, by definition, there homotopies $F_i : [0, 1] \times X \longrightarrow G^{-\infty}(\mathbb{N})$, $i = 1, 2$ which are smooth and such that

$$F_i(0, x) = f_i(x),\ F_i(1, x) = f_i'(x). \tag{3.11}$$

Then, F_1F_2 is a smooth homotopy between the products, so the class $[f_1f_2] \in K^{-1}(X)$ only depends on the classes $[f_1]$, $[f_2] \in K^{-1}(X)$. This product makes $K^{-1}(X)$ into a group since the inverse of $[f]$ is clearly $[f^{-1}]$ and the other group conditions follow from $G^{-\infty}(\mathbb{N})$.

So, the remaing thing to show is that the product is commutative. To do so, we show that each element $[f] \in K^{-1}(X)$ can be represented by a smooth map $f' : X \longrightarrow \mathrm{GL}(N, \mathbb{C})$ for some N (and hence for any larger N by stabilization). This follows from the approximation result proved early. Namely, the image of f is certainly compact (since X is assumed to be so) and thus

$$\Pi_N f(x) \longrightarrow f(x) \text{ uniformly for } x \in X. \tag{3.12}$$

It follows from the openness of $G^{-\infty}(\mathbb{N})$ that for N large enough the smooth homotopy $F(t, x) = (1 - t)f(x) + t\Pi_N f(x)$ takes values in $G^{-\infty}(\mathbb{N})$ and so $\Pi_N f$ also represents $[f] \in K^{-1}(X)$.

Now, having take two classes, represented by f and g. For N large enough, these classes are represented by $\Pi_N f$ and $\Pi_N g$ which take values in $\mathrm{GL}(N, \mathbb{C})$. We can also embed $\mathrm{GL}(N, \mathbb{C})$ in $\mathrm{GL}(2N, \mathbb{C})$ by stabilization and see that each of these classe is represented by a map taking values in matrices like this

$$\begin{pmatrix} * & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix} \tag{3.13}$$

which are block $N \times N$ matrices. Now consider the following homotopy which is really just 'rotation by a 2×2 matrix' for say f :
(3.14)
$$F(t,x) = \begin{pmatrix} \cos(\frac{1}{2}\pi t) & \sin(\frac{1}{2}\pi t) \\ -\sin(\frac{1}{2}\pi t) & \cos(\frac{1}{2}\pi t) \end{pmatrix} \begin{pmatrix} f(x) & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix} \begin{pmatrix} \cos(\frac{1}{2}\pi t) & -\sin(\frac{1}{2}\pi t) \\ \sin(\frac{1}{2}\pi t) & \cos(\frac{1}{2}\pi t) \end{pmatrix}.$$
The outer matrices are inverses of each other (of course there are hidden Id_N's in the 2×2 matrices). At $t = 0$, F reduces to the suspended f but at $t = 1$ it is

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & f(x) \end{pmatrix}. \tag{3.15}$$

Thus, f is homotopic to a map which commutes with g. The product is therefore commutative. □

For the moment, I will not go into any detail, but these abelian groups (which are written additively, so that the class [Id] = 0) behave like (and indeed form) a cohomology theory. Thus, under a smooth map between compact manifold, $h : X \longrightarrow Y$, these odd k-groups pull back:

$$h^* : H^{-1}(Y) \longrightarrow K^{-1}(X),\ h^*[f] = [f \circ h]. \tag{3.16}$$

Check for yourself that this is well-defined.

As well as this bald definition of odd K-theory, which is only really justified by subsequent properties, I want to introduct even K-theory and also the delooping sequence today. Even K-theory here will be defined in terms of the appropriate loop group as a classifying space. Loops in general are just maps from the circle. In the case of a group, for us $G^{-\infty}(\mathbb{N})$ one can restrict to smooth pointed loops, which take the value Id at the base point, $1 \in \mathbb{S}$. In fact, for analytic reasons that will appear later, it is best here to take an even smaller group the flat-pointed loop (smooth) loop group:

$$G^{-\infty}_{\mathrm{sus}}(\mathbb{N}) = \{b : \mathbb{S} \longrightarrow G^{-\infty}(\mathbb{N}); \mathcal{C}^\infty,\ b(1) = \mathrm{Id},\ \frac{d^k b}{d\theta^k}(1) = 0\ \forall\ k \geq 1\}. \tag{3.17}$$

Thus these loops not only take the value Id at the point $1 \in \mathbb{S}$ but all derivatives vanish there as well, making the loop 'flat'. I use the abbreviation 'sus' for this group to indicate that it is obtained by 'suspension' from $G^{-\infty}(\mathbb{N})$ in a way that will be clarified below.

Now, I did not do the following in the lecture, because I did not have my notes!

Lemma 2. *The suspended group $G^{-\infty}_{\mathrm{sus}}(\mathbb{N})$ is open and dense in the Fréchet algebra*
(3.18)
$$\dot{\mathcal{C}}^\infty([0,2\pi]; \Psi^{-\infty}(\mathbb{N})) = \{b : [0,2\pi]_\theta \longrightarrow \Psi^{-\infty}(\mathbb{N}); \frac{d^k b}{d\theta^k} b(\theta) = 0,\ \theta = 0,\ 2\pi\ \forall\ k \geq 0\}.$$

Proof. So, to do this properly I need to show that

(1) The space (3.18) is a Fréchet algebra
(2) There is a natural map from the group in (3.17) into it.
(3) The range is open (and in fact it is dense).

So, this is just like the relationship between $\Psi^{-\infty}(\mathbb{N})$ and $G^{-\infty}(\mathbb{N})$. □

Having defined this suspended group, we can set by direct analogy with the odd case above

$$K^{-2}(X) = \{f : X \longrightarrow G^{-\infty}_{\mathrm{sus}}(\mathbb{N}); \mathcal{C}^\infty\}/\sim \tag{3.19}$$

with the equivalence relation being smooth homotopy in the same sense. Thus f_0 and f_1 are homotopic if there exists

$$F : [0,1] \times X \longrightarrow G^{-\infty}_{\text{sus}}(\mathbb{N}); F(0,x) = f_0(x),\ F(1,x) = f_1(x)\ \forall\ x \in X\}. \tag{3.20}$$

I need to expand a bit on the things I said in the later part of the lecture. Namely there is a natural injection

$$K^{-2}(x) \longrightarrow K^{-1}(\mathbb{S} \times X) \tag{3.21}$$

which corresponds to the fact that an element of $G^{-\infty}_{\text{sus}}(\mathbb{N})$ is already a smooth map of $\mathbb{S}$ into $G^{-\infty}(\mathbb{N})$ and hence a map from X into $G^{-\infty}_{\text{sus}}(\mathbb{N})$ can be regarded as a map from $\mathbb{S} \times X$ into $G^{-\infty}(\mathbb{N})$. In the lecture I did not prove injectivity but I did say:

Lemma 3. *For any compact manifold there is a natural short exact sequence*

$$K^{-2}(X) \longrightarrow K^{-1}(\mathbb{S} \times X) \longrightarrow K^{-1}(X). \tag{3.22}$$

Proof. □

The relationship between the circle $\mathbb{S}$ and the interval $[0, 2\pi]$, which already appears (at the moment implicitly) above gives rise to the delooping sequence. This comes above by cutting the circle at 1. So, consider in place of (3.17) the group

$$\tilde{G}^{-\infty}_{\text{sus}}(\mathbb{N}) = \\ \{\tilde{b} : [0,2\pi]_\theta \longrightarrow G^{-\infty}(\mathbb{N}); \mathcal{C}^\infty,\ \tilde{b}(0) = \text{Id},\ \frac{d^k\tilde{b}}{dt^k}(t) = 0,\ t = 0, 2\pi\ \forall\ k \geq 1\}. \tag{3.23}$$

Thus, these are smooth maps from the interval, hence 'open loops', which take the value Id at 0 and which are flat at both ends. However the value at the far end, $t = 2\pi$, is not specified. The topology on this group is of the same type as is (not yet) discussed above.

Proposition 6. *The natural maps, given by the identification $\mathbb{S} = \mathbb{R}/2\pi\mathbb{N}$ with fundamental domain $[0, 2\pi]$ and by restriction to 2π, give a short exact sequence of groups*

$$\{1\} \longrightarrow G^{-\infty}_{\text{sus}}(\mathbb{N}) \longrightarrow \tilde{G}^{-\infty}_{\text{sus}}(\mathbb{N}) \xrightarrow{|_{2\pi}} G^{-\infty}(\mathbb{N}) \longrightarrow \{1\}. \tag{3.24}$$

Proof. Identifying $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$ gives a smooth map $[0, 2\pi] \longrightarrow \mathbb{S}$, explicitly $\theta \longmapsto e^{i\theta}$, under which 0 and 2π are both identified with $1 \in \mathbb{S}$. Thus, elements of $G^{-\infty}_{\text{sus}}(\mathbb{N})$, being maps on $\mathbb{S}$, pull back to $[0, 2\pi]$. In fact, comparing (3.17) and (3.23) this map is injective into $\tilde{G}^{-\infty}_{\text{sus}}(\mathbb{N})$ and has range the subset on which $\tilde{b}(2\pi) = \text{Id}$. The restricition map in (3.24) is just evaluation at $\theta = 2\pi$ so exactness at $\tilde{G}^{-\infty}_{\text{sus}}(\mathbb{N})$ also follows. The surjectivity of this map follows from the connectedness of $G^{-\infty}(\mathbb{N})$. In fact the argument above, by approximation gave a piecewise smooth curve from any given point of $G^{-\infty}(\mathbb{N})$ to Id. To prove surjectivity we need to show that this curve, which can be assumed to be from $[0, 2\pi]$ can be chosen smooth and flat at the ends. If it is smooth, reparameterization makes it flat. Namely consider a map $\psi : [0, 2\pi] \longrightarrow [0, 2\pi]$ with is smooth and constant near the ends with $\psi(0) = 0$, $\psi(2\pi) = 2\pi$. Then if $b' : [0, 2\pi] \longrightarrow G^{-\infty}(\mathbb{N})$ is smooth, $b' \circ \psi$ is smooth and flat at the ends. The same construction allows a piecewise smooth curve to be mad smooth, by making it flat at the special points. This completes the proof of the exactness of (3.24). □

Add some words about the contractibility of $\tilde{G}^{-\infty}_{\text{sus}}(\mathbb{N})$ and why this might be important.