31. Lecture 28: Relative and compactly-supported K-theory Friday, 7 November, 2008

Reminder. *There are still some gaps in the definition of the analytic index of Atiyah and Singer which I wish to fill today – and try to give a little more background as well.*

Let me start by considering a compact manifold with boundary, X. In the main case of initial interest here it is the radially fibre compactified cotangent bundle of a fibration, $X = \overline{T^*(M/Y)}$. Given our basic odd and even classifying spaces $G^{-\infty}$ and $\mathcal{H}^{-\infty}$ there are six 'obvious' K-groups although with several possible, but equivalent definitions:

$$\begin{aligned}\mathrm{K}^0_{\mathrm{c}}(X \setminus \partial X) =& [X \setminus \partial X; \mathcal{H}^{-\infty}]_{\mathrm{c}} \xrightarrow{=} \\ & \Pi_0\left(\{f : X \longrightarrow \mathcal{H}^{-\infty}; f\big|_{\partial X} = \gamma_1\}\right) = \mathrm{K}^0_{\mathrm{c}}(X, \partial X)\end{aligned} \tag{31.1}$$

$$\mathrm{K}^0(X) = [X; \mathcal{H}^{-\infty}] \tag{31.2}$$

$$\mathrm{K}^0(\partial X) = [\partial X; \mathcal{H}^{-\infty}] \tag{31.3}$$

$$\begin{aligned}\mathrm{K}^1_{\mathrm{c}}(X \setminus \partial X) =& [X \setminus \partial X; \mathcal{G}^{-\infty}]_{\mathrm{c}} \xrightarrow{=} \\ & \Pi_0\left(\{f : X \longrightarrow \mathcal{G}^{-\infty}; f\big|_{\partial X} = \mathrm{Id}\}\right) = \mathrm{K}^1_{\mathrm{c}}(X, \partial X)\end{aligned} \tag{31.4}$$

$$\mathrm{K}^1(X) = [X; \mathcal{G}^{-\infty}] \tag{31.5}$$

$$\mathrm{K}^1(\partial X) = [\partial X; \mathcal{G}^{-\infty}]. \tag{31.6}$$

Note the natural equality between the K-spaces 'with compact supports in the interior' and the K-spaces 'relative to the boundary'. The maps are induced by inclusions and are isomorphism because we can make small perturbations.

Proposition 47. *For any compact manifold with boundary there is a* 6*-term exact sequence involving the K-spaces above:*

$$\begin{array}{ccccc} \mathrm{K}^0_c(X, \partial X) & \longrightarrow & \mathrm{K}^0(X) & \longrightarrow & \mathrm{K}^0(\partial X) \\ \uparrow{\scriptstyle \mathrm{cl_{oe}}} & & & & \downarrow{\scriptstyle \mathrm{cl_{eo}}} \\ \mathrm{K}^1(\partial X) & \longleftarrow & \mathrm{K}^1(X) & \longleftarrow & \mathrm{K}^1_c(X, \partial X) \end{array} \tag{31.7}$$

in which the horizontal arrows are induced by inclusions or pull-backs and the vertical, connecting, maps involve identification of a collar neighbourhood of the boundary with $\overline{\mathbb{R}} \times \partial X$.

Exercise 30. I will probably not have time to go through the proof in class – of course this is a standard topolical argument, it is *just the details* that require checking! The spaces have been defined, so the definitions of the six maps need to be checked, and then the 12 statements corresponding to exactness at each space need to be checked. The horizontal maps are clear enough – inclusion of maps which are trivial near the boundary and restriction to the boundary respectively and these project to the homotopy classes. Make sure to check that the vertical maps are well-defined – really they involve retraction to finite rank, followed by 'suspension' from odd to even or even to odd by adding a real parameter and then this 'suspended' object can be converted into a compactly-supported map which is trivial outside a little collar neighbourhood $(0, 1) \times \partial X$ of the boundary.

Then to exactness. In the middle this is clear enough – a map trivial on the boundary comes from one which is compactly supported in the interior. Exactness at the relative spaces can be seen by observing that a class that is mapped to zero in the absolute, central, spaces generates a homotopy on the boundary from which it comes by the map from the boundary space. The exactness at the boundary spaces is really Bott periodicity, at least in the sense that it corresponds to the fact that $\mathrm{cl}_{\mathrm{eo}}$ and $\mathrm{cl}_{\mathrm{oe}}$ induce isomorphism – it is therefore perhaps the trickiest. Roughly said, a class on the boundary, say in $\mathrm{K}^1(\partial X)$, is represented by a map into $g \in \mathcal{C}^\infty(\partial X; \mathcal{G}^{-\infty})$. This is mapped into $\mathrm{cl}_{\mathrm{oe}}(g)$ which is a family of projections with an additional parameter, interpreted as the normal variable near the boundary (sse the sketch). If this is mapped to zero in the interior then this generates an homotopy. Twisting the neck of the boundary around – again see sketch – and using the essential surjectivity of $\mathrm{cl}_{\mathrm{oe}}$ this can be used to construct an absolute class on the manifold which restricts to the original class on the boundary.

Exercise 31. Since we do have Bott periodicity at our disposal there is a rather clearer way to look at the maps in (31.7). Namely we can work with the suspended classifying spaces $\mathcal{G}^{-\infty}_{\mathrm{sus}(p)}$ and consider the spiral of groups

(31.8)

$$\begin{array}{ccccc} [(X,\partial X),(G^{-\infty}_{\mathrm{sus}(p)},\{\mathrm{Id}\})] & \longrightarrow & [X, G^{-\infty}_{\mathrm{sus}(p)}] & \longrightarrow & [\partial X, G^{-\infty}_{\mathrm{sus}(p)}] \\ \uparrow{\scriptstyle \text{to level } p-2} & & & & \downarrow \\ [\partial X, G^{-\infty}_{\mathrm{sus}(p-1)}] & \longleftarrow & [X, G^{-\infty}_{\mathrm{sus}(p-1)}] & \longleftarrow & [(X,\partial X),(G^{-\infty}_{\mathrm{sus}(p-1)},\{\mathrm{Id}\})] \end{array}$$

In the top left and bottom right the maps and homotopies are required to preserve the pairs, i.. the boundary is mapped to the identity. The horizontal maps are the same as before but the vertical maps involve 'using up' one of the suspension variables and turning it into the normal variable in the collar. See what it takes to show that (31.8) is a long exact sequence as a semi-infinite spiral – starting at $p = \infty$. Show that the maps commute with Bott periodicity and hence it collapses to a 6-term sequence and this is the same as in (31.7).

I have included this boundary sequence, both because it is important (and I plan/planned to include some discussion of analysis for operators on manifolds with boundary) and because it motivates the 'mixed' characterization of the K-theory with compact supports in the interior – however the discussion above is not actually needed for this.

Lemma 36. *For a compact manifold with boundary there is a natural identification*

$$\mathrm{K}^0(X,\partial X) = \Pi_0\left(\mathcal{R}^{-\infty}(X,\partial X)\right) \tag{31.9}$$

$$\mathcal{R}^{-\infty}(X,\partial X) =$$
$$\{(\gamma,g) \in \mathcal{C}^\infty(X;\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R})) \times \mathcal{C}^\infty(\partial X;\tilde{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^2);(Rg)^{-1}\gamma\big|_{\partial X}(Rg) = \gamma_1\}.$$

Recall that $\tilde{G}$ is the 'half-open loop group'. Meaning that the elements are smooth maps $G : \mathbb{R} \longrightarrow G^{-\infty}$ which approach the identity rapidly at $-\infty$ and approach some element Rg rapidly at $+\infty$.

Proof. To get the map from the space on the right into $\mathrm{K}^0(X,\partial X)$ first observe that there is alway a diffeomorphism from X onto X with an extra boundary strip

$[-1,1]\times\partial X$ glued on and this diffeomorphism well defined up to homotopy through such. Then just 'glue' the curve $g(t)^{-\infty}\gamma\big|_{\partial X}g(t)$ onto the end of γ – which can be assumed to be flat to its limit at the boundary, by identifying $\overline{\mathbb{R}}$ with $[-1,1]$. The map the other way can be taken to be inclusion where γ is flat to γ_1 at the boundary and so can be mapped to (γ,Id). Of course we need to check that these maps induce isomorphism at the level of homotopy but that is clear enough if one recalls Lemma 30 to use in a strip near the boundary. □

In this is the way we can see not only that the symbol data $(a,\mathbb{E})$ of an elliptic operator generates a class in $\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y))$ but also where the identification of the Atiyah-Singer index and the semiclassical push-forward map comes from.

First the K-class of the symbol. Here we use a function $\Theta:\mathbb{R}\longrightarrow\mathbb{R}$ which is flat to 0 at $-\infty$ and to $\pi/2$ at $+\infty$.

Lemma 37. *The symbol data* $a\in\mathcal{C}^\infty(S^*(M/Y);\pi^*\hom(\mathbb{E}))$ *of an elliptic family* $A\in\mathrm{Ell}^0(M/Y;\mathbb{E})$ *(so a is invertible) generates a K-class through* (31.9), *namely if* E_+ *and* $E_=$ *are identified with the ranges of projections* $\pi_\pm\in\mathcal{C}^\infty(M;\mathbb{C}^N)$ *for some* N *then*

$$\mathrm{Ell}^0(M/Y;\mathbb{E})\longmapsto[(\gamma_E,g)]\in\mathrm{K}_c(\overline{T^*(M/Y)},S^*(M/Y))=\mathrm{K}^0_c(T^*(M/Y),$$

$$\gamma_E=\begin{pmatrix}\mathrm{Id}_N-\pi_- & 0 & 0 & 0\\ 0 & -\pi_- & 0 & 0\\ 0 & 0 & \pi_+ & 0\\ 0 & 0 & 0 & -(\mathrm{Id}_N-\pi_+)\end{pmatrix},$$

(31.10)

$$g(t)=\begin{pmatrix}\mathrm{Id}_N-\pi_- & 0 & 0 & 0\\ 0 & \cos(\Theta(t))\pi_- & \sin(\Theta(t))a & 0\\ 0 & \sin(\Theta(t))a^{-1} & \cos(\Theta(t))\pi_+ & 0\\ 0 & 0 & 0 & \mathrm{Id}_N-\pi_+\end{pmatrix}.$$

Here both γ_E and g should be stabilized; γ_E is defined on the whole of M but should be lifted to $\overline{T^*(M/Y)}$ by the projection but g is only defined over the boundary $S^*(M/Y)$, but depends on a parameter and has the property that the value at $+\infty$

$$Rg=\begin{pmatrix}\mathrm{Id}_N-\pi_- & 0 & 0 & 0\\ 0 & 0 & a & 0\\ 0 & a^{-1} & 0 & 0\\ 0 & 0 & 0 & \mathrm{Id}_N-\pi_+\end{pmatrix}\tag{31.11}$$

conjugates γ_E on the boundary to γ_1. There is an issue of orientations here which, as usual, I have not checked.

Proof. It is only necessary to show that this does what it is supposed to in the sense that it defines an element of the space in (31.9). □

So, the point here is that the symbol of the elliptic operator allows us to identify the two bundles E_+ and E_- over the boundary of $\overline{T^*(M/Y)}$ and hence to deform them back into a family of involutions which has compact support in the interior.

Easy part of Theorem 12. The step I have not discussed in the diagram (30.20) is the surjectivity of the map on the right in the middle row, onto $\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y))$ – which is what we have just been discussing. In fact we know from the discussion above that every compactly supported class on $T^*(M/Y)$ can be represented by a pair (γ,g) in the space on the right in (31.9) and that the class is invariant under homotopies in this space. Now, $T^*(M/Y)$ is a real vector bundle and hence $\overline{T^*(M/Y)}$ is a bundle of balls. If γ is restricted to the zero section of $T^*(M/Y)$ it defines a family $\tilde{\gamma} \in \mathcal{C}^\infty(M; \mathcal{H}^{-\infty})$. Moreover the ball bundle can be 'retracted' onto the zero section. This means that γ is homotopic in $\mathcal{C}^\infty(\overline{T^*(M/Y)}; \mathcal{H}^{-\infty})$ to the pull-back of $\tilde{\gamma}$. One the other hand we know that $\tilde{\gamma}$ is homotopic to a family of the form γ_E in (31.9) (in principle the ranks might be different but we already know that they are equal since the projection is homotopic to γ_1 at the boudnary. Thus γ is homotopic to a γ_E in $\mathcal{C}^\infty(\overline{T^*(M/Y)}; \mathcal{H}^{-\infty})$. Under this homotopy, 'information' is streaming out across the boundary, in particular there is an homotpy $\gamma' \in \mathcal{C}^\infty([0,1] \times S^*(M/Y); \mathcal{H}^{-\infty})$ starting at γ_1 and finishing at γ_E lifted under π. In fact we know from Proposition ? that such an homotopy can be realized as a curve under cojugation, that is there exists

$$\tilde{g} \in \mathcal{C}^\infty(S^*(M/Y); \tilde{G}^{-\infty}_{\text{iso}}(\mathbb{R}; \mathbb{C}^2)) \text{ s.t. } \gamma'(s(t)) = \tilde{g}^{-1}(t)\gamma_1\tilde{g}(t),$$
$$s : [0,1] \longrightarrow \overline{\mathbb{R}},\ s(0) = \infty,\ s(1) = -\infty. \tag{31.12}$$

It follows from this that the map $R(g)$ conjugates γ_E to γ_1 and I claim that it is, after stabilization, homotopic to the image of an elliptic symbol.

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Hence the map to $\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y))$ is surjective. □