30. Lecture 27: Analytic index of Atiyah and Singer Wednesday, 5 November, 2008

Reminder. *(And partly correction and revision) Last time, despite getting myself pretty seriously in knots on the board – and even in the notes to some extent, I more or less succeeded in defining the* semiclassical push-forward maps in K-theory. *Let me recall how this goes – see Theorem 11 – or at least should have gone. In fact I will add a little bit of stabilization. For a fibration with compact fibres,* (29.21), *(the compactness of the base is not needed at all here) we can define the stabilized semiclassical algebra on the fibres. As a space of 'functions' this is, as defined in Digression 1,*

$$\text{(30.1)}\quad \big\{F \in \mathcal{C}^\infty([[0,1]\times M^2_\phi;\{0\}\times \mathrm{Diag}];\mathbb{S}(\mathbb{R}^2)\otimes \pi_R^*\Omega);\\ F\equiv 0 \text{ at the old boundary.}\big\}$$

where I am using illegal blow-up notation and M^2_ϕ is the fibre diagonal – the submanifold of M^2 consisting of the pairs of points in the same fibre. I have also dispensed with the vector bundle E over M, because it is supposed to be embedded in $\mathcal{S}(\mathbb{R})$ by a projection-valued map $\pi_E : M \longrightarrow \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R})$. This globalized definition means that the kernel of $A\in\mathcal{C}^\infty((0,1];\Psi^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R})$ is following form in local coordinates in a patch U in the base and $V\subset\phi^{-1}(U)$ in the fibre

$$\text{(30.2)}\quad A(\epsilon,y,z,z',y,y')=\\ \epsilon^{-d}F'(\epsilon,y,z,\frac{z-z'}{\epsilon},y,y')|dy'|,\ F\in\mathcal{C}^\infty([0,1]\times U\times V;\mathcal{S}(\mathbb{R}^d_Z\times\mathbb{R}^2))$$

Here d is the dimension of the fibre, Z, and Ω has been trivialized over the coordinate patch, which is the $|dy'|$ and F' is the local representative of the function in (30.1). *Under composition these form an algebra of operators*

$$\text{(30.3)}\quad \Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\ni A:\mathcal{C}^\infty([0,1]\times M;\mathcal{S}(\mathbb{R}))\longrightarrow\mathcal{C}^\infty([0,1]\times M;\mathcal{S}(\mathbb{R})).$$

Moreover they have a symbol map which captures the behaviour at $\epsilon=0$ obtained by restricting F to $\epsilon=0$ and taking the Fourier transform in the 'adiabatic variable' Z which pieces together globally:

$$\text{(30.4)}\quad \epsilon\Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\longrightarrow\Psi^{-\infty}_{\mathrm{sl}(),\mathrm{iso}}(M/Y\times\mathbb{R})\xrightarrow{\sigma_{\mathrm{sl}}}\Psi^{-\infty}_{\mathrm{sus}(T^*(M/Y),\mathrm{iso}}(\mathbb{R}).$$

The image of the symbol map is just a family of isotropic smoothing operators on $\mathbb{R}$, i.e. elements of $\mathcal{S}(\mathbb{R}^2)$ depending smoothly, and in a Schwartz manner, on parameters in $T^*(M/Y)=T^*M/\phi^*(T^*Y)$ the bundle of 'fibre differentials'.

As usual $\Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})$ is a Neumann-Fréchet algebra and we can define our usual group and space of involutives:-

(30.5)

$$\mathcal{G}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})=\big\{A\in\Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R});\\ \exists\ B\in\Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R}),\ (\mathrm{Id}+B)=(\mathrm{Id}+A)^{-1}\big\}$$

$$\mathcal{H}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})=\big\{A\in\Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\otimes M(2,\mathbb{C});(\gamma_1+A)^2=\mathrm{Id}\,\big\}.$$

Now, the 'usual symbolic construction and correction' shows that the adiabatic symbol maps

$$\begin{aligned}\mathcal{G}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R}) &\xrightarrow{\sigma_{\mathrm{ad}}} \mathcal{G}^{-\infty}_{\mathrm{sus}(T^*(M/Y),\mathrm{iso}}(\mathbb{R})\\ \mathcal{H}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R}) &\xrightarrow{\sigma_{\mathrm{ad}}} \mathcal{H}^{-\infty}_{\mathrm{sus}(T^*(M/Y),\mathrm{iso}}(\mathbb{R})\end{aligned}\tag{30.6}$$

are 'homotopy equivalences' in the sense that they induce isomorphisms

(30.7)

$$\begin{aligned}\Pi_0\left(\mathcal{G}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\right) &\xrightarrow{\sigma_{\mathrm{ad}}} \Pi_0\left(\mathcal{G}^{-\infty}_{\mathrm{sus}(T^*(M/Y),\mathrm{iso}}(\mathbb{R})\right) = \mathrm{K}^1_c(T^*(M/Y))\\ \Pi_0\left(\mathcal{H}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\right) &\xrightarrow{\sigma_{\mathrm{ad}}} \Pi_0\left(\mathcal{H}^{-\infty}_{\mathrm{sus}(T^*(M/Y),\mathrm{iso}}(\mathbb{R})\right) = \mathrm{K}^0_c(T^*(M/Y)).\end{aligned}$$

Finally, I 'explained' but did not prove that the corresponding restriction to $\epsilon = 1$

$$\begin{aligned}\mathcal{G}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R}) &\xrightarrow{R_{\epsilon=1}} \mathcal{G}^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R})\\ \mathcal{H}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R}) &\xrightarrow{R_{\epsilon=1}} \mathcal{H}^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R})\end{aligned}\tag{30.8}$$

lead to

$$\begin{aligned}\Pi_0\left(\mathcal{G}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\right) &\xrightarrow{R_{\epsilon=1}} \Pi_0\left(\mathcal{G}^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R})\right) = \mathrm{K}^1_c(Y)\\ \Pi_0\left(\mathcal{H}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\right) &\xrightarrow{R_{\epsilon=1}} \Pi_0\left(\mathcal{H}^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R})\right) = \mathrm{K}^0_c(Y)\end{aligned}\tag{30.9}$$

where it is these last identifications which are not obvious, because of the twisting in the bundles.

Combining these two maps, the 'invertible' one for the symbol and the second one for the restriction to $\epsilon = 1$ *gives the (semiclassical) push-forward, or index, maps*

$$\begin{aligned}(\phi\pi)_! &: \mathrm{K}^1_c(T^*(M/Y)) \longrightarrow \mathrm{K}^1_c(Y)\\ (\phi\pi)_! &: \mathrm{K}^1_c(T^*(M/Y)) \longrightarrow \mathrm{K}^1_c(Y)\end{aligned}\tag{30.10}$$

where $\pi : T^*(M/Y)$ *is the projection onto* M *and* $\phi : M \longmapsto Y$ *so the composite is* $\phi\pi : T^*(M/Y)$.

This is a 'generalization' (it isn't really nbecause of the non-compactness of the fibres) of the Thom isomorphism(s) – where for a real vector bundle, $V \longrightarrow Y$, $T^*(V/Y) \equiv V \times_Y V^* = V \oplus V^* = W$ *is naturally symplectic, and ultimately the Bott periodicity maps, where "*$M = Y \times \mathbb{R}^k$*" and "*$T^*(M/Y) = Y \times \mathbb{R}^{2k}$*". In contrast to these cases the maps* (30.10) *need not be isomorphisms.*

So, either I will go throught that or I will explain the content of the Atiyah-Singer index theorem in K-theory.

In the same setting of the fibration (29.21) we can consider *differential operators* or *pseudodifferential operators* on the fibres of M. These are just operators 'between sections of bundles on Z' but twisted by the diffeomorphisms involved in the transition maps for ϕ. In fact they can be defined perfectly directly. I will not go through the definition here, unless there is a desire for me to do so. Let me just 'remind you' further.

Pseudodifferential operators of order K on a fixed compact manifold Z can be defined either globally through their kernels or locally through coordinate patches – naturally I prefer the first definition but I will briefly describe the second one. First on $\mathbb{R}^n$ we define operators which *are not the isotropic pseudodifferential operators* *discussed above.* They are very closely related but they are not the same. To see the difference in terms of kernels, recall that the kernel of an isotropic pseudodifferential operator of order k (recall that there is an issue of a $\frac{1}{2}$ with the orders) is a distribution on $\mathbb{R}^{2n}$ which can be written in Weyl form as

$$(30.11)\quad A(\frac{z+z}{2}, z-z') = (2\pi)^{-n}\int_{\mathbb{R}^n} a(\frac{z+z}{2},\tau)e^{i(z-z')\cdot\tau}d\tau,\ a\in\rho_q^{-k/2}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2n}}) \Longleftrightarrow A\in\Psi^k_{\text{qiso}}(\mathbb{R}^n).$$

The integral here is not convergent if $k\ge -2n$ but is then to be interpreted as the inverse Fourier transform on tempered distibutions. The 'ordinary' as opposed to 'isotropic' pseudodifferential operators are given by the same formula but with a different class of amplitudes:-

$$(30.12)\quad A(\frac{z+z}{2}, z-z') = (2\pi)^{-n}\int_{\mathbb{R}^n} a(\frac{z+z}{2},\tau)e^{i(z-z')\cdot\tau}d\tau,\ a\in\mathcal{S}(\mathbb{R}^n;\mathcal{C}^\infty(\overline{\mathbb{R}^2})\Longleftrightarrow A\in\Psi^k_{\mathcal{S}}(\mathbb{R}^n).$$

This is not quite standard notation, but the $\mathcal{S}$ denotes the rapid vanishing of the coefficients. This is again an algebra of operators on $\mathcal{S}(\mathbb{R}^n)$, but neither of these algebras is contained in the other,

$$(30.13)\quad \begin{gathered}\Psi^k_{\mathcal{S}}(\mathbb{R}^n)\circ\Psi^l_{\mathcal{S}}(\mathbb{R}^n)\subset\Psi^{k+l}_{\mathcal{S}}(\mathbb{R}^n),\\ \Psi^{k-1}_{\mathcal{S}}(\mathbb{R}^n)\longrightarrow\Psi^k_{\mathcal{S}}(\mathbb{R}^n)\xrightarrow{\sigma_k}\mathcal{S}(\mathbb{R}^n;\mathcal{C}^\infty(\mathbb{R}^{n-1};N_k)).\end{gathered}$$

Here N_k is the line bundle over the sphere at infinity generated by the $-k$th power of the defining function – it just hides homogeneity. The symbol sequence here is exact and multiplicative.

If the 'full symbol' a in (30.12) is supported in a set of the form $K\times\overline{\mathbb{R}^n}$ where K is compact (this is essentially impossible for isotropic operators) then the kernel $A(z,z')$ is smooth in $\Omega\times\Omega$ for any open set $\Omega\subset\mathbb{R}^n$ with $\Omega\cap K=\emptyset$. In particular there are plenty of pseudodifferential operators with kernels having compact support in sets of the form $U\times U$ where $U\subset\mathbb{R}^n$ is open. Furthermore, taking such an operator and 'changing coordinates' by making a diffeomorphism to U' again gives a kernel of the same form, with a different amplitude. This coordinate invariance allows pseudodifferential operators to be defined by coordinate covering on any compact manifold Z and acting between sections of any two vector bundles E_+ and E_- over Z. Thus

$$(30.14)\quad \Psi^k(Z;\mathbb{E})\ni A:\mathcal{C}^\infty(Z;E_+)\longrightarrow\mathcal{C}^\infty(Z;E_-)$$

can now be taken to be well-defined. There is a multiplicative symbol map which is invariantly defined and gives

$$(30.15)\quad \Psi^{k-1}(Z;\mathbb{E})\longrightarrow\Psi^k(Z;\mathbb{E})\xrightarrow{\sigma_k}\mathcal{C}^\infty(S^*Z;\pi^*\hom(\mathbb{E})\otimes N_k).$$

Here $S^*Z=\partial\overline{T^*Z}$ is the boundary of the radial compactification of the cotangent bundle, $\pi:S^*Z\longrightarrow Z$ (and I usuall drop the π^* from the notation) and N_k is the same bundle as before. Multiplicativity means

$$(30.16)\quad \Psi^k(Z;F,E_-)\circ\Psi^l(Z;E_+,F)\subset(=)\Psi^{k+l}(Z;E_+,E_-),\ \sigma_{k+l}(AB)=\sigma_k(A)\sigma_l(B).$$

Of course there is a lot more to be said, but I am assuming this is all 'well-known'. Now for the fibration (29.21) we can define the 'pseudodifferential operators acting on the fibres' because of coordinate invariance and we get a similar space of operators 'depending on Y as parameters':

(30.17)
$$\Psi^k(M/Y;\mathbb{E}) \ni A : \mathcal{C}^\infty(M;E_+) \longrightarrow \mathcal{C}^\infty(M;E_-),\ \psi^{-\infty} = \bigcap_k \Psi^k(M/Y;\mathbb{E}).$$

Finally then we arrive at the Atiyah-Singer setting where we have elliptic operators, meaning the symbol has an inverse.

$$\begin{aligned}\operatorname{Ell}^k(M/Y;\mathbb{E}) &= \\ \{A \in \Psi^k(M/Y;\mathbb{E});\ &\exists\, b \in \mathcal{C}^\infty(S^*(M/Y);\mathbb{E}^- \otimes N_{-k},\ \sigma_k(A)b = b\sigma_k(A) = \operatorname{Id}\}.\end{aligned} \tag{30.18}$$

Then we can construct parametrices and set

$$\begin{aligned}\mathcal{P}^k(M/Y;\mathbb{E}) = \{(A,B) &\in \Psi^k(M/Y;\mathbb{E}) \oplus \Psi^{-k}(M/Y;\mathbb{E}); \\ R_R = \operatorname{Id} - A\circ B \in \Psi^{-\infty}(M/Y;E_-),&\ R_L = \operatorname{Id} - BA \in \Psi^{-\infty}(M/Y;E_+)\}.\end{aligned} \tag{30.19}$$

Theorem 12. *For any fibration* (29.21) *with compact total space there are natural maps inducing the* analytic index map

(30.20)
$$\begin{array}{ccccc}
\mathcal{P}^k(M/Y;\mathbb{E}) & \longrightarrow & \operatorname{Ell}^k(M/Y;\mathbb{E}) & & \\
\Big\downarrow & & \Big\downarrow & & \\
 & & \{a \in \mathcal{C}^\infty(S^*(M/Y);\hom(\mathbb{E})\otimes N_k)\ \textit{invertible}\} & \longrightarrow & \mathrm{K}^0_c(T^*(M/Y)) \\
 & & & & \Big\downarrow \operatorname{ind}_a \\
\mathcal{H}^{-\infty}(M/Y;\mathbb{E}) & \longrightarrow & \mathcal{H}^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R}) & \longrightarrow & \mathrm{K}^0(Y)
\end{array}$$

which is equal to the semiclassical push-forward map.