16. Lecture 13: Involutions and K^0 Friday, 3 October

Last time I introduced the space of smooth involutions $\mathcal{H}^{-\infty}(\mathbb{R})$, let me immediately note some properties of it.

Proposition 14. *There is a surjective index, or relative dimension, map*

$$\operatorname{ind} : \mathcal{H}^{-\infty}(\mathbb{R}^k) \longrightarrow \mathbb{Z},\ \operatorname{ind}(I_\infty + a) = \frac{1}{2}\operatorname{tr}(a) \tag{16.1}$$

which labels the components, $\mathcal{H}^{-\infty}_k(\mathbb{R}^k)$, of $\mathcal{H}^{-\infty}(\mathbb{R}^k)$. The base component, where the index vanishes, is a homogeneous space

$$\mathcal{H}^{-\infty}_0(\mathbb{R}^k) = G^{-\infty}_{\text{iso}}(\mathbb{R}^k;\mathbb{C}^2)/\left(G^{-\infty}_{\text{iso}}(\mathbb{R}^k) \oplus G^{-\infty}_{\text{iso}}(\mathbb{R}^k)\right) \tag{16.2}$$

through conjugation and the other components are isomorphic to the base component – but not naturally so.

Proof. We use finite rank approximation to prove this. In the construction of the quantized Bott element I used the idea which lies behind:

Lemma 18. *For each $I \in \mathcal{H}^{-\infty}(\mathbb{R}^k)$ there is a neighbourhood*

$$0 \in B \subset \Psi^{-\infty}(\mathbb{R}^k;\mathbb{C}^2)$$

such that if $b \in B$ then the complex integral

$$J(b) = -\operatorname{Id} - \frac{\pi i}{\int}_{|z-1|=\frac{1}{2}} \left(\frac{1}{2}(I+b) - (z-\frac{1}{2})\operatorname{Id}\right)^{-1} dz \tag{16.3}$$

is an element of $\mathcal{H}^{-\infty}(\mathbb{R}^k)$.

Proof. As Boris said: Just use the functional calculus!

If $b = 0$ in (16.3), then the inverse of $\frac{1}{2}I - (z-\frac{1}{2})\operatorname{Id} = (1-z)B_+ - zB_-$ is $(1-z)^{-1}B_+ - z^{-1}B_-$ where $I = B_+ - B_-$ is the decomposition into projections. The inverse is uniformly bounded on $|z-1| = \frac{1}{2}$ so remains invertible there if perturbed by $b/2$ in a small ball around the origin. Thus the integrand in (16.3) does exist and is of the form

$$\left(\frac{1}{2}(I+b) - (z-\frac{1}{2})\operatorname{Id}\right)^{-1} = (1-z)^{-1}B_+ - z^{-1}B_- + \gamma(z;b) \tag{16.4}$$

where $\gamma(z;b)$ is holomorphic near $|z-1| = \frac{1}{2}$ and valued in smoothing operators. The integral of the first term on the right in (16.4) is $-B_+$ so $J(b) = I + b'$ with $b' \in \Psi^{-\infty}(\mathbb{R}^k;\mathbb{C}^2)$. Moreover, b' is small with b and depends continuously on it. It remains to check that $J(b)$ is an involution. The square can be written

$$\begin{aligned} J(b)^2 = \operatorname{Id} + 2\frac{1}{\pi i}\int_{|z-1|=\frac{1}{2}} \left(\frac{1}{2}(I+b) - (z-\frac{1}{2})\operatorname{Id}\right)^{-1} dz \\ + \frac{1}{(\pi i)^2}\int_{|z-1|=\frac{1}{2}}\int_{|t-1|=\frac{1}{2}+\delta} \left(\frac{1}{2}(I+b) - (z-\frac{1}{2})\operatorname{Id}\right)^{-1} \\ \left(\frac{1}{2}(I+b) - (t-\frac{1}{2})\operatorname{Id}\right)^{-1} dzdt \end{aligned} \tag{16.5}$$

where the t contour has been moved slightly where $\delta > 0$. Applying the resolvent identity

$$\left(\frac{1}{2}(I+b)-(z-\frac{1}{2})\operatorname{Id}\right)^{-1}\left(\frac{1}{2}(I+b)-(t-\frac{1}{2})\operatorname{Id}\right)^{-1} = \\ (z-t)^{-1}\left(\frac{1}{2}(I+b)-(z-\frac{1}{2})\operatorname{Id}\right)^{-1} \\ -(z-t)^{-1}\left(\frac{1}{2}(I+b)-(t-\frac{1}{2})\operatorname{Id}\right)^{-1}$$

and inserting this into the the last term allows it to be evaluated by residues as

$$\text{(16.6)}\quad \frac{1}{(\pi i)^2}\int_{|z-1|=\frac{1}{2}}\int_{|t-1|=\frac{1}{2}+\delta}\left(\frac{1}{2}(I+b)-(z-\frac{1}{2})\operatorname{Id}\right)^{-1} \\ \times\left(\frac{1}{2}(I+b)-(t-\frac{1}{2})\operatorname{Id}\right)^{-1}dzdt \\ =-2\frac{1}{\pi i}\int_{|z-1|=\frac{1}{2}}\left(\frac{1}{2}(I+b)-(z-\frac{1}{2})\operatorname{Id}\right)^{-1}dz.$$

Thus indeed, $J(b)^2 = \operatorname{Id}$. □

This 'retraction onto $\mathcal{H}^{-\infty}(\mathbb{R}^k)$' allows any element $I_\infty + a$ to be connected to a finite rank perturbation of I_∞. Namely, if k is large enough, depending on a, then

$$\text{(16.7)}\qquad I_\infty + (1-t)a + t\Pi_k a \Pi_k$$

is sufficiently close to $I_\infty + a$, for $t \in [0,1]$, for the Lemma to apply. Moreover it follows directly from the formula for $J(b)$ that

$$\text{(16.8)}\qquad J(\Pi_k a\Pi_k) = I + \Pi_k a' \Pi_k$$

is indeed a finite rank perturbation. Thus, as an involution it is equal to

$$\text{(16.9)}\qquad I_\infty(\operatorname{Id}-\Pi_k) + \Pi_k A \Pi_k$$

where the second term is an involution in $M(2,\mathbb{C})\otimes M(k,\mathbb{C})$, the latter being matrices acting on the range of Π_k in $\mathcal{S}(\mathbb{R}^k)$.

For finite rank involutions the first statements in the Proposition become obvious. In a given vector space they correspond to a decomposition as a direct sum, of the 1 and -1 eigenspaces, of dimensions d_+ and d_-, $d_+ + d_- = N$ being the dimension of the space on which the involution acts. Moreover, for fixed N any two such decompositions are linearly equivalent if and only the positive eigenspaces have the same dimension, d_+. The trace of the involution, $d_+ - d_- = -2N + 2d_-$, is an even integer which determines the involution up to linear equivalence. It follows that for the decomposition (16.9), in which Π_k acts as a multiple of the identity on the $\mathbb{C}^2$ factor,

$$\text{(16.10)}\qquad \operatorname{tr}(J(\Pi_k a\Pi_k) - I_\infty) = \operatorname{tr}(\Pi_k A\Pi_k) - \operatorname{tr}(I_\infty \Pi_k) = 2p \in 2\mathbb{Z}$$

determines the linear equivalence class.

So, it remains to show that $\frac{1}{2}\operatorname{Tr}(a)$ is locally constant. However differentiating the identity $I_t^2 = \mathrm{Id}$ shows that

$$\begin{gathered} I_t I_t' + I_t' I_t = 0 \Longrightarrow \operatorname{tr}(I_t') = 0, \\ \text{hence } \frac{d}{dt}\operatorname{tr}(I_t - I) = 0. \end{gathered} \tag{16.11}$$

since I_t' is off-diagonal with respect to I_t.

This proves (16.1) and that the 'index' map is constant on the components of $\mathcal{H}^{-\infty}(\mathbb{R}^k)$.

In the case that $\operatorname{ind}(I_\infty + a) = 0$ it follows from the discussion above that $I + a$ is connected by a smooth path $I_\infty + a(t)$, $t \in [0,1]$, in $\mathcal{H}^{-\infty}(\mathbb{R}^k)$ to I_∞ itself, so $a(1) = a$, $a(0) = 0$. For each $I \in \mathcal{H}^{-\infty}(\mathbb{R}^k)$, if b is small enough and $I + b \in \mathcal{H}^{-\infty}(\mathbb{R}^k)$ then

$$T = (I+b)_+ I_+ + (I+b)_- I_- \in G^{-\infty}(\mathbb{R}^k; \mathbb{C}^2) \tag{16.12}$$

where $I + b = (I_b)_+ - (I+b)_-$ is the decomposition into projections. Moreover,

$$TI = (I+b)T \Longrightarrow (I+b) = T^{-1} I T.$$

Thus, nearby involutions are conjugate under the action of $G^{-\infty}(\mathbb{R}^k; \mathbb{C}^2)$.

Apply this at each point $t \in [0,1]$ it follows that there is a finite decomposition of the interval such that $I + a(t)$ at each lower end-point is so conjugate to the upper end-point. Composing the action shows that $I + a$ is conjugate to I_∞.

Thus we see that the action by conjugation of $G^{-\infty}(\mathbb{R}^k; \mathbb{C}^2)$, is transitive on $\mathcal{H}_0^{-\infty}(\mathbb{R}^k)$. It is clear that the subgroup fixing I_∞ is the diagonal group $G^{-\infty}(\mathbb{R}^k) \oplus G^{-\infty}(\mathbb{R}^k)$ which is (16.2).

In each $\mathcal{H}_k^{-\infty}(\mathbb{R}^k)$ there is a 'base point'

$$\begin{cases} I_\infty + (\mathrm{Id} - I_\infty)\Pi_k & \in \mathcal{H}_k^{-\infty} \\ I_\infty - (\mathrm{Id} + I_\infty)\Pi_k & \in \mathcal{H}_{-k}^{-\infty} \end{cases}, k > 0 \tag{16.13}$$

Thus it suffices to show that these are conjugate to I_∞. This can be done by renumbering the bases – of course these conjugating operators are not in $G^{-\infty}(\mathbb{R}^k; \mathbb{C}^2)$. □

This result has quite a few consequences for our defintion of $\mathrm{K}_{\mathrm{c}}^0(X)$. However, the first thing I need to do – to finish the proof of Bott periodicity – is to go back and look at the quantized Bott involution constructed in Lemma 9. What we want to do is to compute $\frac{1}{2}\operatorname{tr}(D_\epsilon)$, which we now know to be constant as a function of $\epsilon > 0$. Of course we must somehow compute it in terms of the semiclassical limit as $\epsilon \downarrow 0$. By construction D_ϵ comes from a semiclassical family, with kernels

$$D_\epsilon = \epsilon^{-1} D(\epsilon, \frac{\epsilon(t+t')}{2}, \frac{t-t'}{\epsilon}) \tag{16.14}$$

valued in 2×2 matrices. So, for $\epsilon > 0$

(16.15)

$$\operatorname{tr}(D_\epsilon) = \epsilon^{-1}\int_{\mathbb{R}} \operatorname{tr} D(\epsilon, \epsilon t, 0)dt = \epsilon^{-2}\int_{\mathbb{R}} D(\epsilon, T, 0)dT = \frac{1}{2\pi\epsilon^2}\int_{\mathbb{R}^2} \operatorname{tr} \hat{D}(\epsilon, t, \tau)dtd\tau.$$

So, what we know is $\hat{D}(0, t, \tau) = \delta(t, \tau)$ and what we need to compute is the (integral of the trace of) the coefficient of ϵ^2 in the Taylor series expansion of $\hat{D}(\epsilon, \dots)$. Fortunately, the ϵ^2 term is the next after the leading term.

In fact if you recall the construction of D what we did was start with D_0 which is a quantization of δ; we can take it not to depend explicitly on ϵ. Then we need to compute the semiclassical symbol of the error term

$$(I_\infty + D_0)^2 - \mathrm{Id} = \epsilon^2 E_1,\ \sigma_{\mathrm{sl}}(E_1) = \frac{1}{2i}\left(\partial_t\delta\partial_\tau\delta - \partial_\tau\delta\partial_t\delta\right). \tag{16.16}$$

Now, the *correction term* is $\epsilon^2 D_1$ where $\sigma_{\mathrm{sl}}(D_1)$ has to satisfy

$$b\sigma_{\mathrm{sl}}(D_1) + \sigma_{\mathrm{sl}}(D_1)b = \sigma_{\mathrm{sl}}(E_1) \tag{16.17}$$

which we did by noting that the right side satisfies

$$b\sigma_{\mathrm{sl}}(E_1) = \sigma_{\mathrm{sl}}(E_1)b \text{ so } \sigma_{\mathrm{sl}}(D_1) = \frac{1}{2}b\sigma_{\mathrm{sl}}(E_1)$$

works. Thus combining these formulæ we need to compute

$$-\frac{1}{8\pi}\int_{\mathbb{R}^2} \mathrm{tr}\left(b\left(\partial_t\delta\partial_\tau\delta - \partial_\tau\delta\partial_t\delta\right)\right) dtt\tau. \tag{16.18}$$

Since $\partial_t\delta$ and $\partial_\tau\delta$ are derivatives of $b = I_\infty + \delta$ we know that $b(\partial_t\delta) = -(\partial_t\delta)b$, etc, anticommute. So in fact the two terms in (16.18) are the same. Since δ is written in terms of polar coordinates, it is natural to change variable and use a similar rearrangement to reduce to the integral

$$-\frac{1}{4\pi}\int_0^\infty\int_0^{2\pi} \mathrm{tr}\left(b(\partial_r\delta)(\partial_\theta\delta)\right) drd\theta. \tag{16.19}$$

Now, recall what $b = I_\infty + \delta$ is! It was defined in terms of Pauli matrices

$$b(t,\tau) = \cos(\Theta(-r))\gamma_1 + \sin(\Theta(-r))\cos(\theta)\gamma_2 - \sin(\Theta(-r))\sin(\theta)\gamma_3. \tag{16.20}$$

There are three constant matrices in (16.20). Each of them has trace zero and the product of any two of them (which is $\pm i$ times the other one) has trace zero. The product of all three $\gamma_1\gamma_2\gamma_3 = -\mathrm{Id}_{2\times 2}$ has trace -2. Thus there are four terms which can contribute. Namely the product of $\Theta'(-r)$ and

$$\begin{aligned}\sin^3(\Theta)\sin^2\theta\gamma_3\gamma_1\gamma_2 - \sin(\Theta)\cos^2(\Theta)\sin^2\theta\gamma_1\gamma_3\gamma_2\\ -\sin^3(\Theta)\cos^2\theta\gamma_2\gamma_1\gamma_3 + \sin(\Theta)\cos^2(\Theta)\cos^2\theta\gamma_1\gamma_2\gamma_3\\ = -\sin(\Theta)\,\mathrm{Id},\end{aligned} \tag{16.21}$$

where $\Theta = \Theta(-r)$. The integral is therefore

$$-\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\sin(\Theta)d\theta d\Theta = 8\pi. \tag{16.22}$$

Combining all this we conlude that

$$\mathrm{ind}(B) = \frac{1}{2}\,\mathrm{tr}(D) = 1. \tag{16.23}$$

Phew, that proves Bott periodicity.