11. Lecture 8: Bott element
Monday, 22 September, 2008

The first thing I want to do today is to use the three Pauli matrices to construct the Bott element. Let's not worry about what this element is, or even where it is, for the moment. The initial objective is to find something non-trivial on $\mathbb{R}^2$.

The Pauli matrices, all elements of $M(2,\mathbb{C})$, I will denote

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \gamma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{11.1}$$

Clearly they are linearly independent over $\mathbb{C}$ and, together with $\mathrm{Id}_{2\times 2}$ span $M(2,\mathbb{C})$ as a linear space. Their products satisfy the cyclic conditions

$$\begin{gathered} \gamma_1\gamma_2 = i\gamma_3,\ \gamma_2\gamma_3 = i\gamma_1,\ \gamma_3\gamma_1 = i\gamma_2, \\ \gamma_2\gamma_1 = -i\gamma_3,\ \gamma_3\gamma_1 = -i\gamma_1,\ \gamma_1\gamma_3 = -i\gamma_2 \\ \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = \mathrm{Id},\ \gamma_1\gamma_2\gamma_3 = i\,\mathrm{Id}. \end{gathered} \tag{11.2}$$

and hence the Clifford identities

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij},\ i,j = 1,2,3. \tag{11.3}$$

This shows (see) that they give a representation, to wit the spin representation, of the complex Clifford algebra for $\mathbb{R}^2$. I will not use this explicitly here, but it is one way of getting a better understanding of what is going on. Each of the self-adjoint *involutions* γ_i has determinant -1 and so has two one-dimensional eigenspaces with eigenvalues ± 1.

Let us now choose a handy smooth function on $\mathbb{R}$ satisfying

$$\Theta : \mathbb{R} \longrightarrow \mathbb{R},\ \Theta(t) = 0,\ t < -2,\ \Theta(t) = \pi,\ t > -1,\ \Theta'(t) \geq 0. \tag{11.4}$$

So, $e^{i\Theta(t)}$ is a smooth flat pointed loop into the circle from the line.

Now consider the map from $\mathbb{R}^2$ into $M(2,\mathbb{C})$ given in terms of polar coordinates

$$\begin{gathered} (t,\tau) = r(\cos\theta, \sin\theta), \\ b(t,\tau) = \cos(\Theta(-r))\gamma_1 + \sin(\Theta(-r))\cos(\theta)\gamma_2 - \sin(\Theta(-r))\sin(\theta)\gamma_3, \\ b(t,\tau) = \begin{pmatrix} \cos(\Theta(-r)) & \sin(\Theta(-r))e^{i\theta} \\ \sin(\Theta(-r))e^{-i\theta} & -\cos(\Theta(-r)) \end{pmatrix} \end{gathered} \tag{11.5}$$

Let's hope I have made a sensible choice of signs!

The first thing to observe is that $-r$ is running from $-\infty$ to 0 as we come inwards from infinity, so

$$b(t,\tau) = \begin{cases} \gamma_1 & |(t,\tau)| > 2 \\ -\gamma_1 & |(t,\tau)| < 1. \end{cases} \tag{11.6}$$

It follows that $b : \mathbb{R}^2 \longrightarrow M(2,\mathbb{C})$ is smooth and is in fact a compactly-supported perturbation of γ_1 :

$$b - \gamma_1 \in \mathcal{C}_c^\infty(\mathbb{R}^2, M(2,\mathbb{C})). \tag{11.7}$$

Secondly, the Clifford identities show that

$$\begin{aligned} b^2 = {} & \cos^2(\Theta(-r))\gamma_1^2 + \sin^2(\Theta(-r))\cos^2(\theta)\gamma_2^2 + \sin^2(\Theta(-r))\sin^2(\theta)\gamma_3^2 \\ & + A(\gamma_1\gamma_2 + \gamma_2\gamma_1) + B(\gamma_2\gamma_3 + \gamma_3\gamma_2) + C(\gamma_3\gamma_1 + \gamma_1\gamma_3) = \mathrm{Id}. \end{aligned} \tag{11.8}$$

Thus, b is in fact a family of self-adjoint involutions. Moroever, it follows (without computation) that $\det(b(t,\tau)) = -1$, so again it has one-dimensional eigenspaces with eigenvalues ± 1. Let me denote the spectral decomposition by

$$(11.9)\quad b(t,\tau) = b_+(t,\tau) - b_-(t,\tau),$$
$$b_+ - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}^\infty_c(\mathbb{R}^2; M(2,\mathbb{C})),\ b_- - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{C}^\infty_c(\mathbb{R}^2; M(2,\mathbb{C})).$$

Here, $b_\pm$ are the orthogonal projections onto the ± 1 eigenspaces; probably b_+ best deserves to be called the Bott element. Its range is a 1-dimensional (over $\mathbb{C}$ of course) subbundle of $\mathbb{C}^2$ over $\mathbb{R}^2$ which is trivial (just the first component) near infinity but not globally trivial, as we shall see, if one only allows trivializations which are constant near infinity.

Exercise 9. Try to show directly that b_+ is not homotopically trivial, in the sense that there is no homotopy through families of projections to a constant projection where the constant projection has to be always constant at infinity (either fixed forever there, or just constant for each value of the parameter, it doesn't matter). I think the easiest way to do this is to find an homotopy invariant which shows that such a deformation is not possible. The obvious one is the total curvature of the line bundle. The curvature is a 2-form on $\mathbb{R}^2$ of compact support. I will compute it later, probably not today.

We can proceed with either the involution b or the projection b_+. Since it is more in the spirit of what we have done so far, consider the involution. It is definitely of the form

$$(11.10)\qquad b(t,\tau) = \gamma_1 + \delta(t,\tau),\ \ \delta \in \mathcal{S}(\mathbb{R}^2; M(N,\mathbb{C}))$$

so we can apply 'semiclassical quantization' to the perturbation δ. Remember what this means: It just is the statement that

$$(11.11)\qquad \exists\ D \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2)\ \text{s.t.}\ \ \sigma_{\mathrm{sl}}(B) = \delta.$$

Here we need to work component by component in the 2×2 matrices. This we already know, from the surjectivity of the semiclassical symbol map, but we can do quite a lot more.

Lemma 9. *The semiclassical family D in* (11.11) *can be chosen so that (as operators on $\mathcal{S}(\mathbb{R};\mathbb{C}^2)$)*

$$(11.12)\qquad (\gamma_1 + D_\epsilon)^2 = \mathrm{Id}\,.$$

That is, the quantization can also be chosen to be a family of involutions. Notice that we are 'quantizing' the constant matrix to the same constant matrix – really componentwise a multiple of the identity as an operator – which is not by any means a semiclassical family of smoothing operators. However, it is consistent with the way we (or rather you) showed that differential operators with polynomial coefficients compose with semiclassical families. This we are just demanding that the identity be quantized to the identity and this is consistent with the semiclassical symbol map, etc.

Exercise 10. Show (without doing any work) that in the same sense as in the Lemma, the projections $b_\pm$ can be quantized to commuting projections (also called idemptotents if we do not demand they be selfadjoint) $B_\pm$ with

$$(11.13)\qquad \begin{gathered} B_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + D_+,\ D_+ \in \Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2),\ \sigma_{\text{sl}}(D_+) = b_\pm \\ B_- = \operatorname{Id} - B_+,\ B = B_+ - B_-,\ B_\pm^2 = B_\pm. \end{gathered}$$

Exercise 11. Check, if only mentally, that it is consistent to extend the semiclassical symbol map to constant matrices, where the symbol is just the matrix itself, in the sense that this gives us a multiplicative map symbol

$$(11.14)\qquad \sigma_{\text{sl,iso}} : M(N;\mathbb{C}) + \Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^N) \longrightarrow M(N;\mathbb{C}) + \mathcal{S}(\mathbb{R}^2; M(N;\mathbb{C}))$$

for this algebra into the algebra $\mathcal{S}(\mathbb{R}^2; M(N;\mathbb{C}))$ with constant multiples of the identity appended. We will do this more systematically later.

Proof. To do this I will need to check another couple of important facts about semiclassical quantization, but let's proceed anyway. For the first step we don't have much choice. Using the surjectivity of the symbol map, choose a D as in (11.11), but denoted D_0. The choice of symbol, together with the multipicativity and the exactness of the symbol sequence shows that

$$(11.15)\qquad \begin{gathered} E_1 = (\gamma_1 + D_0)^2 - \operatorname{Id} = \gamma_1 D_0 + D_0\gamma_1 + D_0^2 \in \Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2) \text{ satisfies} \\ \sigma_{\text{sl}}(E_1) = 0 \Longrightarrow E_1 = \epsilon E_1' \in \epsilon\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2). \end{gathered}$$

Thus our first choice 'works to first order'. We wish to modify D_0, the initial choice, by choosing $D_1 = \epsilon D_1' \in \epsilon\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2))$ to get the desired identity (11.12) to second order. Clearly adding $\epsilon D_1'$ changes the computation to

$$(11.16)\qquad \begin{aligned} E_2 = (\gamma_1 + D_0 + \epsilon D_1')^2 - \operatorname{Id} &= E_1 + \epsilon(\gamma + D_0)D_1' + \epsilon D_1'(\gamma + D_0) + \epsilon^2 (D_1')^2 \\ &= \epsilon(E_1' + (\gamma + D_0)D_1' + D_1'(\gamma + D_0)) + \epsilon^2 (D_1')^2. \end{aligned}$$

Thus to ensure that $E_2 = \epsilon^2 E_2' \in \epsilon^2\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2)$ we wish to choose D_1' so that

$$(11.17)\qquad \begin{gathered} \sigma_{\text{sl}}(E_1' + (\gamma + D_0)D_1' + D_1'(\gamma + D_0)) = 0 \Longleftrightarrow \\ b\sigma_{\text{sl}}(D_1') + \sigma_{\text{sl}}(D_1')b = -\sigma_{\text{sl}}(E_1'). \end{gathered}$$

Here I have used the original choice of $\sigma_{\text{sl}}(D_0) = \delta$. The symbols are still non-commutative, but only because they take values in 2×2 matrices; this is just matrix algebra. So why is such a choice possible? The matrix on the left in the last identity is not arbitrary. Indeed, recalling that $b = b_+ - b_-$, $\operatorname{Id} = b_+ + b_-$ it is necessarily diagonal with respect to this decomposition, since it is just

$$(11.18)\qquad 2b_+\sigma_{\text{sl}}(D_1')b_+ - 2b_-\sigma_{\text{sl}}(D_1')b_-.$$

Thus, (11.17) can only be solved if $\sigma_{\text{sl}}(E_1')$ is also diagonal. Fortunately it is, because of the associativity of the (operator) product which shows that

$$(11.19)\qquad \begin{gathered} B_0'E_1 = B_0'((B_0')^2 - \operatorname{Id}) = ((B_0')^2 - \operatorname{Id})B_0' = E_1 B_0' \Longrightarrow \\ b\sigma_{\text{sl}}(E_1) = \sigma_{\text{sl}}(E_1)b \Longrightarrow \sigma_{\text{sl}}(E_1) = b_+\sigma_{\text{sl}}(E_1)b_+ + b_-\sigma_{\text{sl}}(E_1)b_-. \end{gathered}$$

Thus indeed we can choose D_1' to satisfy (11.17), for instance just require

$$(11.20)\qquad \sigma_{\text{sl}}(D_1') = -\frac{1}{2}b_+\sigma_{\text{sl}}(E_1)b_+ + \frac{1}{2}b_-\sigma_{\text{sl}}(E_1)b_-.$$

So, to complete the 'formal' part of the construction we just repeat this argument inducively. Suppose we have shown the existence of $D'_j \in \Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2)$ for $0 \le j \le k$ so that (11.12) holds to order k (or is it $k+1$?)

$$(\gamma_1 + \sum_{0\le j\le k} \epsilon^j D'_j)^2 - \mathrm{Id} = E_{k+1} \in \epsilon^{k+1}\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2). \tag{11.21}$$

Then we want to choose D'_{k+1} to get to the next step. Adding $e^{k+1}D'_{k+1}$ changes the left side by

$$\epsilon^{k+1}\left((\gamma_1 + D_0)D'_{k+1} + D'_{k+1}(\gamma_1 + D_0)\right) \quad \mathrm{mod}\ \epsilon^{k+1}\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2). \tag{11.22}$$

The choice (11.20), with 1 replaced by $k+1$ throughout works for the same reason. Thus, we can find a full formal solution.

Now, to proceed further we need first to pass from the formal series

$$\sum_{j=0}^{\infty} e^j D'_j \tag{11.23}$$

to an actual element of $\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2)$. This is Borel'e lemma.

Lemma 10. *[É. Borel] Given any sequence $D'_j \in \Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^N))$ there exists an element $D' \in \Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^N))$ such that*

$$D' - \sum_{j=0}^{k} \epsilon^j D'_j \in \epsilon^{k+1}\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^N). \tag{11.24}$$

Proof. I will not do this, since it is just Borel's lemma when applied to the functions $F'_j(\epsilon;t,s)$ representing the opertors. Namely if $F'_j \in \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2))$ is such a sequence, with no constraints, then there exists $F \in \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2))$ such that

$$F - \sum_{0\le j\le k} \epsilon^j F'_j \in \epsilon^{k+1}\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2))\ \forall\ k. \tag{11.25}$$

□

With this D' for our sequence, or series, as constructed above we conclude that

$$(\gamma + D')^2 - \mathrm{Id} \in \epsilon^\infty\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^N)) = \bigcap_{j=0}^{\infty} \epsilon^j\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2). \tag{11.26}$$

That is, this one element works to all orders.

Lemma 11. *The residual space*

$$\epsilon^\infty\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^N)) = \bigcap_{j=0}^{\infty} \epsilon^j\Psi^{-\infty}_{\text{sl,iso}}(\mathbb{R};\mathbb{C}^2) = \bigcap_{j=0}^{\infty} \epsilon^j\mathcal{C}^\infty([0,1];\Psi^{-\infty}_{\text{iso}}(\mathbb{R};\mathbb{C}^2)) \tag{11.27}$$

is just the space of smooth families, in the ordinary sense, of Schwartz smoothing operators on $\mathbb{R}$ vanishing to infinite order at $\epsilon = 0$.

Exercise 12. I will almost certainly not have time to do this but it is straightforward. Think in terms of the kernel of the semiclassical family written out as

$$a(\epsilon, z, z') = \epsilon^{-1}F(\epsilon, \frac{\epsilon(z+z')}{2}, \frac{z-z'}{\epsilon}). \tag{11.28}$$

The function on the left is unque, it is the kernel of the operator and is certainly Schwartz for $\epsilon > 0$. The assumption is that the one function on the left can be written in the form

$$\epsilon^k \epsilon^{-1} F_k(\epsilon, \frac{\epsilon(z+z')}{2}, \frac{z-z'}{\epsilon}) \tag{11.29}$$

for each k where $F_k \in \mathcal{C}^\infty([0,1]; \mathcal{S}(\mathbb{R}^2))$. What we want are the estimates on a

$$\sup \epsilon^{-N} |D_\epsilon^p z^j (z')^k D_z^l D_{z'}^m a| < \infty \tag{11.30}$$

for all indices. Just check that any finite set of them follows from (11.29) by taking k large enough. The converse is easy.

So, now we know that our summed-up choice of quantization, D' satisfies

$$(\gamma_1 + D')^2 - \mathrm{Id} = E' \in \epsilon^\infty \mathcal{C}^\infty([0,1]; \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2)) \tag{11.31}$$

where the, perhaps improper, notation is too suggestive not to use. We still want to actually solve the problem, to get rid of the error term on the right. Just to keep you oriented, remember that $\sigma_{\mathrm{sl}}(D') = \delta$ and we are way beyond changing the leading term.

So finally the claim is that we can add an element of $\epsilon^\infty \mathcal{C}^\infty([0,1]; \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2))$ to D' to get rid of E'. This is now 'genuinely non-linear' where up to this point we have been linearizing.

So, first notice that if $d > 0$ and $z \in \mathbb{C}$ is such that $|z - \pm 1| \geq d$ then

$$\begin{gathered}(B' - z)^{-1} = \left((1-z)^{-1}\frac{1}{2}(B' + \mathrm{Id}) - (1+z)^{-1}\frac{1}{2}(B' - \mathrm{Id})\right)(\mathrm{Id} + F), \\ F \in \epsilon^\infty \mathcal{C}^\infty([0,1]; \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2)),\ 0 < \epsilon < \epsilon_0(R) > 0.\end{gathered} \tag{11.32}$$

Here the operator on the left 'should have' eigenvalues $1 - z$ and $-(1+z)$ on the positive and negative pieces of B' if this were an involution, so the first term on the right side is formally the inverse of this. To prove (11.32) compute the product

$$\begin{aligned}(B' - z)&\left((1-z)^{-1}\frac{1}{2}(B' + \mathrm{Id}) - (1+z)^{-1}\frac{1}{2}(\mathrm{Id} - B')\right) \\ &= (B' - z)\left(\frac{1}{1-z^2}B' + \frac{z}{1-z^2}\mathrm{Id}\right) \\ &= \frac{1}{1-z^2}(B')^2 - \frac{z^1}{1-z^2}\mathrm{Id} = \mathrm{Id} + \frac{1}{1-z^2}((B')^2 - \mathrm{Id}).\end{aligned} \tag{11.33}$$

As a result of our work so far, $(B')^2 - \mathrm{Id} = E' \in \epsilon^\infty \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2)$. Thus, by using Neumann series, the operator on the right in (11.33) is invertible, with inverse of the same form – at least for $0 \leq \epsilon \leq \epsilon_0$ for some $\epsilon_0 > 0$ depending on d and other constants. Thus (11.32) follows, with $F \in \epsilon^\infty \mathcal{C}^\infty([0,\epsilon_0]; \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2))$ and holomorphic in $|z - \pm 1| > d$.

Finally then we can use this to construct the quantization D, or $B = \gamma_1 + D$ we want. Just set

$$B_+ = \frac{i}{2\pi}\int_{|z-1|=\frac{1}{2}} (B' - z)^{-1} dz,\ B = 2B_+ - \mathrm{Id}. \tag{11.34}$$

I probably will not even get to this point today, but a few things remain. Namely we need to show that B_+ makes sense and is a projection, that

$$B - B_+ \in \epsilon^\infty \mathcal{C}^\infty([0,\epsilon_0]; \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2))$$

and that B satisfies all our requirements. I leave this as an exercise in contour shifting – notice that the fact that $\epsilon_0 < 1$ is only a technical inconvenience, we can simply rescale the parameter (starting at $\frac{1}{2}\epsilon_0$) to make B exist out to $\epsilon = 1$. Mostly the semiclassical families are only of interest near $\epsilon = 0$. $\square$