9. Topic 3: Clifford algebras
In place of lecture for Wednesday, 17 September

Clifford algebra. If you know all about Clifford algebras, go to item 17 below.

(1) Let V be a real vector space, of dimension n, equipped with a Euclidean struture – that is a positive-definite symmetric bilinear form

$$h : V \times V \longrightarrow \mathbb{R}. \tag{9.1}$$

We will associate with V two algebras, the real and complex Clifford algebras. The latter can also be defined for a complex vector space with a positive definite hermitian bilinear form. In fact only non-degeneracy of the form is really needed but the Euclidean case is the one we want.

(2) Recall the full tensor algebra on V. This consists of the formal direct sum of the tensor products

$$\mathcal{T} = \sum_{k=0}^{\infty} (V^*)^{\otimes k}. \tag{9.2}$$

Thus an element of $\mathcal{T}$ is a sequence with kth term an element of $(V^*)^{\otimes k}$ and with all but finitely many terms zero. This is an algebra in the obvious way, with component-wise addition and tensor product. Of course the sequences should be thought of as finite sums terminating at some point.

(3) Note that I have not fixed the coefficients here. Thus (9.2) corresponds to real coefficients, with $(V^*)^0 = \mathbb{R}$. We are actually more interested in the complex version

$$\mathcal{T}_{\mathbb{C}} = \mathbb{C} \otimes \mathcal{T} \tag{9.3}$$

in which V^* is replaced throughout by its complexification.

(4) The tensor product of two copies $V^* \otimes V^*$ is the space of bilinear forms on V and so decomposes into the symmetric and antisymmetric parts, S^2V^* and Λ^2V^*. The former has dimension $\frac{1}{2}n(n+1)$ and the latter dimension $\frac{1}{2}n(n-1)$, with S^2V^* spanned by elements of the form

$$w_1 \otimes w_2 + w_2 \otimes w_1,\ w_1,\ w_2 \in V^*. \tag{9.4}$$

(5) If h^* is the dual metric, induces on V^* by h as a metric on V consider

$$J = \{w_1 \otimes w_2 + w_2 \otimes w_1 - 2h^*(w_1, w_2) \in \mathcal{T}. \tag{9.5}$$

This is a linear space, with complexification $J_{\mathbb{C}}$ given by the same terms with complex coefficients. Note that it is more conventional to replace the $-$ sign in (9.5) by a $+$. If that is the way you like it, bad luck.

(6) In $\mathcal{T}$ consider the ideal generated by J

$$\mathcal{J} = \mathcal{T} \otimes J \otimes \mathcal{T} \subset \mathcal{T},\ \mathcal{J}_{\mathbb{C}} = \mathbb{C} \otimes J \subset \mathcal{T}_{\mathbb{C}}. \tag{9.6}$$

(7) Finally then we have the Clifford algebras, real and complex:

$$\mathrm{Cl}(V) = \mathcal{T}/\mathcal{J},\ \mathbb{C}\mathrm{l}(V) = \mathcal{T}_{\mathbb{C}}/\mathcal{J}_{\mathbb{C}} = \mathbb{C} \otimes \mathrm{Cl}(V). \tag{9.7}$$

(8) Show by an inductive argument (or otherwise) that *as linear spaces* the Clifford algebras are isomorphic to the corresponding (real and complex)

exterior vector spaces

$$\Lambda^* V^* = \sum_{k=0}^{n} \Lambda^k V^* \tag{9.8}$$

of sums of totally antisymmetric k-linear forms on V.

(9) Show that the odd an even parts of the tensor product descend to the quotient so that the Cliffor algebras are $\mathbb{Z}_2$-graded

$$\mathrm{Cl}(V) = \mathrm{Cl}_{\mathrm{even}}(V) \oplus \mathrm{Cl}_{\mathrm{odd}}(V) \tag{9.9}$$

with the product graded in the sense that the product of two even, or two odd, elements is even and the product of an odd and an even element is odd.

(10) Show that the Clifford algebras are filtered by degree where an element is of degree k or less if it can be written as a sum of products each consisting of at most k elements of V^* :

$$\mathrm{Cl}^{(k)}(V) = \{u \in \mathrm{Cl}(V); u = \sum_{l \le k} c_\bullet w_1 \dots w_l, \tag{9.10}$$

$$\mathrm{Cl}^{(k)}(V)\,\mathrm{Cl}^{(l)}(V) \subset \mathrm{Cl}^{(k+l)}(V),\ \mathrm{Cl}(V) = \sum_{j=0}^{n} \mathrm{Cl}^{(n)}(V).$$

(11) Check that

$$\mathrm{Cl}^{(0)} = \mathbb{R},\ \mathrm{Cl}^{(1)} = V^* \oplus \mathbb{R} \Longrightarrow V^* \hookrightarrow \mathrm{Cl}(V). \tag{9.11}$$

(12) Show that an element of V^*, injected into $\mathrm{Cl}(V)$ has an inverse if and only if it is non-zero.

(13) Show that the associated graded algebra is canonically the exterior algebra

$$\sum_j \mathrm{Cl}^{(j)} \,/\, \mathrm{Cl}^{(j-1)} = \Lambda^* V \text{ as algebras.} \tag{9.12}$$

(14) Show that if V is given an orientation, so (using the metric as well) $\Lambda^n V^* = \mathbb{R}$ (or $\mathbb{C}$) this map defines the supertrace

$$\mathrm{str} : \mathrm{Ch}(V) \longrightarrow \mathrm{Cl}(V)/\,\mathrm{Cl}^{(n-1)}(V) = \Lambda^n V^* = \mathbb{R},\ \mathrm{str}(ab - (-1)^{\pm} ba) = 0 \tag{9.13}$$

where a and b are either even or odd and the sign is $+$ unless they are of opposite persuasions.

(15) Proceeding inductively (or otherwise) construct the fundamental spin representations, which are to say algebra isomorphisms to the matrix algebras

$$\begin{gathered} \mathbb{C}\mathrm{l}(\mathbb{R}) = \mathbb{C} \oplus \mathbb{C},\ \mathbb{C}\mathrm{l}(\mathbb{R}^2) = M(2,\mathbb{C}),\ \mathbb{C}\mathrm{l}(\mathbb{R}^3) = M(2,\mathbb{C}) \oplus M(2,\mathbb{C}), \\ \mathbb{C}\mathrm{l}(\mathbb{R}^{2k}) = M(2^k,\mathbb{C}),\ \mathbb{C}\mathrm{l}(\mathbb{R}^{2k+1}) = M(2^k,\mathbb{C}) \oplus M(2^k,\mathbb{C}). \end{gathered} \tag{9.14}$$

(16) Work out, if you have the time and energy, the 8-fold periodicity analogous to (9.14) for the real Clifford algebras!

(17) There is plenty more about Clifford algebras, but why are they useful here? In the even dimensional case, so $V = \mathbb{R}^{2p}$, the injection of V^* into $\mathrm{Cl}^{(1)}(V)$ leads to an embedding of the unit sphere

$$\mathbb{S}^{2k-1} = \{w \in \mathbb{R}^{2k}; |w|_{h^*} = 1\} \longrightarrow \mathbb{C}\mathrm{l}(\mathbb{R}^{2k}) = \mathrm{GL}(2^k,\mathbb{C}) \longrightarrow G^{-\infty}. \tag{9.15}$$

(18) We will show that (9.15) generates all the homotopy groups of $G^{-\infty}$.