DIRAC OPERATORS

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Abstract.

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Introduction

These are my notes taken at, or in the case of my own lectures made before, the 18.199 lectures in Spring 2006.

Please complain to me if the notes do not correspond to the talks or you notice other evils!

1. 9 February, 2006

Speaker:- Moi

Topic: An overview of Dirac operators.

(1) Dirac's idea. Clifford algebras and modules (see Taylor's book, [2]).
(2) First four talks:
    Ricardo Andrade:- on Clifford algebra
    Yakov Shapiro:- Clifford modules
    William Lopes:- Examples, signature, $\overline{\partial} + \overline{\partial}^*$, Gauss-Bonnet.
    Zuoqin Wang:- Lichnerowicz formula.
(3) Analytic properties
    Local elliptic regularity
    Spectrum and resolvent

Fredholm condition
Unique continuation
Heat kernel

(4) Parity – homologocal even/odd-ness; parity operator in even dimensions
(5) Numerical index theorem – McKean-Singer
(6) Getzler's rescaling, local index formula.
(7) Families index, determinant bundle
(8) Determinants, analytic torsion, holomorphic torsion.
(9) Odd families index, eta invariant
(10) Positive scalar curvature
(11) Non-compact manifolds – commutative, Callias' theorem and extensions
(12) Witten's proof of positivity of mass
(13) Non-compact manifolds – non-commutative. Atiyah-Patodi-Singer theorem.

Other things I added as a result of questions (and I will add some references later).

(1) Twisted Dirac operators
(2) Higher index, higher torsion invariants (twisting by the fundamental group). Novikov conjecture.
(3) Projective twisting (twisting by a gerbe).
(4) Families APS and K-theory.
(5) K-theory
(6) Connection with product-type pseudodifferential operators (from last semester).
(7) Equivarant index.
(8) Quantization commutes with reduction.
(9) Algebraic index theorem.
(10) Foliations.

Other comments and questions (as interpreted by me).

- Max suggests following Berline-Vergne proof of AS theorem via principal bundle (and equivariant index).
- Zuoqin asks – Can we prove APS without heat kernel? There is the proof by Piazza using the complex powers. It would be interesting to translate the McKean-Singer type proof into explicit operations on the resolvent.
- Silvia asked something interesting too, but I cannot remember what it was!

2. 16 February, 2006

Future talks

- William Lopes:- Examples, signature, $\overline{\partial} + \overline{\partial}^*$,
- Zuoqin Wang:- Lichnerowicz formula.
- Ricardo Andrade:- finish up periodicity for Clifford algebra.
- Yakov Shapiro:- finish up $\mathbb{Z}_2$ grading in even-dimensional case.

2.1. **Ricardo Andrade:- Clifford algebras, Pin and Spin groups.**

(1) Clifford algebra on a vector space V with quadratic from q (over field $\mathbb{K}$ of characteristic $\neq 2$)

$$\mathrm{Cl}(V,q) = \mathcal{T}(V)/\mathcal{J}_q,\ \mathcal{J}_q = \langle v \otimes v + q(v)1 \rangle \tag{1}$$

where $\mathcal{T}$ is the tensor algebra.

(2) $V \hookrightarrow \mathrm{Cl}(V,q)$

(3) Universal property for an algebra over $\mathbb{K}$, $V \longrightarrow A$ for an algebra A such that $f(v)\cdot f(v) = -q(v)1$ extends to an algebra homomorphism $\mathrm{Cl}(V) \longrightarrow A$.

(4) $\mathbb{Z}_2$ grading comes from $-\mathrm{Id}$.

(5) $\mathrm{Cl}(V,0) = \Lambda^* V$.

(6) The grading of the tensor algebra descends to a filtration of $\mathrm{Cl}(V,q)$ and the associated graded algebra is $\Lambda^* V$.

(7) So, as vector spaces $\mathrm{Cl}(V,q) \equiv \Lambda^* V$.

(8) $(V,q) \simeq (V_1,q_1) \oplus (V_2,q_2)$ orthogonal then

$$\mathrm{Cl}(V,q) \simeq \mathrm{Cl}(V_1,q_1) \hat{\otimes} \mathrm{Cl}(V_2,q_2). \tag{2}$$

(9) Pin and Spin.

$$\mathrm{Cl}^{\times}(V,q) = \{a\, \mathrm{Cl}(V,q); \exists b \in \mathrm{Cl}(V,q),\ ab = ba = \mathrm{Id}\} \tag{3}$$

Action of this group on $\mathrm{Cl}(V,q)$

$$\mathrm{Ad} : \mathrm{Cl}^{\times}(V,q) : \mathrm{Cl}(V,q) \longrightarrow \mathrm{Aut}(\mathrm{Cl}(V,q)). \tag{4}$$

For $v \in V$

$$-\mathrm{Ad}_v(w) = w - 2\frac{q(v,w)}{q(v)} \tag{5}$$

is reflection in the plane with normal v. This is an orthogonal transformation. Then $\mathrm{Pin}(V,q)$ is the subgroup of $\mathrm{Cl}^{\times}(V,q)$ generated by $v \in V$ and $\mathrm{Spin}(V,q)$ is the intersection with the even part.

(10) Then $\mathrm{Pin} \longrightarrow O(V,q)$ is surjective as is $\mathrm{Spin}(V,q) \longrightarrow \mathrm{SO}(V,q)$.

(11) Short exact sequences

$$1 \longrightarrow F \longrightarrow \mathrm{Spin}(V,q) \longrightarrow O(V,q) \longrightarrow 1 \tag{6}$$

where $F = \mathbb{Z}_2$ if $i = \sqrt{-1} \in \mathbb{K}$ and $F = \mathbb{Z}_4$ if $\sqrt{-1} \in \mathbb{K}$.

2.2. **Yakov Shapiro:- Clifford modules and connections.** Mostly based on the appropriate section of my book [1].

(1) As before, V with positive definite quadratic form $-g$, $\mathrm{Cl}(V)$. Action on $\Lambda^* V$,

$$\mathrm{cl}(v)w = v \wedge w + i_{v^*} w \tag{7}$$

extends uniquely to an action, $\mathrm{Cl}(V) \longrightarrow \hom(\Lambda^* V)$. Proof by passing to an orthonormal basis.

(2) Let X be a Riemannian manifold, then $T^*_x X$ is a Euclidean vector space and $\mathrm{Cl}_x(X) = \mathrm{Cl}(T^*_x X)$ is a smooth bundle of algebras over X. Smoothness follows by reference to local coordinates. The action above gives a smooth action of $\mathrm{Cl}(X)$ on $\Lambda^* X$.

(3) A Clifford module on X is a vector bundle $E \longrightarrow X$ together with a smooth aclion $\mathrm{cl} : \mathrm{Cl}(X) \longrightarrow \hom(E)$.

(4) If E has an Hermitian inner product then the Clifford action can be required to be Hermitian.

(5) Connection is a linear differential operator

$$\nabla : \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(E \otimes T^*X) \tag{8}$$

satisfying

$$\nabla_V(fw) = f\nabla_V w + V(f)w. \tag{9}$$

(6) The connection is Clifford if

$$\nabla_V(\mathrm{cl}(\xi)w) = \mathrm{cl}(\xi)\nabla_V w + \mathrm{cl}(\overline{\nabla}_V \xi)w \tag{10}$$

where $\overline{\nabla}_V$ is the Levi-Civita connection.

(7) ∇ is Hermitian (unitary) if

$$u\langle v, w\rangle = \langle \nabla_u v, w\rangle + \langle v, \nabla_u w\rangle. \tag{11}$$

The Dirac operator associated to a Clifford module with Clifford connection is

$$\eth_E : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; E),\ \eth_E = i\overline{\mathrm{cl}}\nabla \tag{12}$$

defined as the composite of the connection and the contraction map given by the Clifford action $\overline{\mathrm{cl}} : \mathcal{C}^\infty(X; T^*X \otimes E) \longrightarrow \mathcal{C}^\infty(X; E)$.

(8) In terms of any local orthonormal basis v_i of TX and dual basis α_i of T^*X,

$$\eth_E = \sum_i \mathrm{cl}(\alpha_j)\nabla i v_j \tag{13}$$

where the factor of i corresponds to the sign normalization in the Clifford algebra.

(9)

Theorem 1. *Any Hermitian Clifford module has a unitary Clifford connection.*

3. 23 February, 2006

3.1. William Lopes:- Examples.

(1) (M, g) a compact Riemann manifold, ∇ the Levi-Civita connection on TM and hence on associated bundles. Clifford action on Λ^k

$$\mathrm{cl}(\phi)\omega = \phi \wedge \omega - i_{\phi^*}\omega \tag{1}$$

is Clifford. The associated Dirac operator is $d + d^*$.

(2) $\sigma_1(\eth)(\xi) = i\,\mathrm{cl}(\xi)$.

(3) Grading $\Lambda^* = \Lambda^{\mathrm{even}} \oplus \Lambda^{\mathrm{odd}}$ and $d + d^* : \mathcal{C}^\infty(M; \Lambda^{\mathrm{even}}) \longrightarrow \mathcal{C}^\infty(M; \Lambda^{\mathrm{odd}})$.

$$\mathrm{ind}((d + d^*)_{\mathrm{even}}) = \chi(M) = \int e(M) \tag{2}$$

is the Gauss-Bonnet theorem.

(4) M oriented and even-dimensional, $\dim M = 2n$. Volume form is $e_1 \wedge \dots e_{2n}$ and

$$\tau = i^n\,\mathrm{cl}(e_1)\cdots\mathrm{cl}(e_{2n}),\ \tau^2 = \mathrm{Id}\,. \tag{3}$$

$d + d^*$ is odd with respect to this grading.

(5) If n is odd then $\tau^* = \tau$ and $*$ is an isomorphism exchanging signs os.

$$\mathrm{ind}((d + d^*)_\tau) = 0 \tag{4}$$

(6) In case n is even

$$\operatorname{ind}((d+d^*)_\tau) = \operatorname{sgn}(M) = \int_M L(M) \tag{5}$$

is the signature and Hirzerruch's formula.

3.2. **Zuoqin Wang:- (notes as well as talk) Lichnerowicz Formula.**

▶ **Notation/Review**

• (X, g) is Riemannian manifold.
• Clifford algebra $\mathcal{C}l(V,q) \triangleq T(V)/\mathcal{I}_q = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}/\{v\otimes w + w\otimes v + 2q(v,w) \mid v,w\in V\}$.
• Clifford bundle $\mathcal{C}l(X) =$ bundle with fiber $\mathcal{C}l_x(X) = \mathcal{C}l(T^*_xX, g^*_x)$.
• Clifford Module $\mathcal{E}$ = bundle with a C^∞ action $\mathcal{C}l : C^\infty(\mathcal{C}l(X)) \times C^\infty(E) \to C^\infty(E)$.
• Clifford connection $\mathbb{A}$ =linear connection with $[\mathbb{A}_v, \mathcal{C}l(\xi)] = \mathcal{C}l(\nabla_v \xi)$.
• $\Delta^{\mathbb{A}} = -\sum_{i,j} g^{ij}[\mathbb{A}_i\mathbb{A}_j - \sum_k \Gamma^k_{ij}\mathbb{A}_k]$ is the Laplacian with respect to $\mathbb{A}$.
• $r_X = \sum_{l,m} R_{lmlm}$ is the scalar curvature of X.
• $R^{\mathcal{E}}$ is the action of Riemannian curvature R on $\mathscr{E}$, by $R^{\mathcal{E}}(e_i,e_j) = \sum_{k,l} R_{lkij}\mathcal{C}l(e^k)\mathcal{C}l(e^l)$.
• $F^{\mathcal{E}/S} = \mathbb{A}^2 - R^{\mathcal{E}}$ is the twisting curvature.
• Dirac operator $\mathcal{D}_{\mathbb{A}} = \mathcal{C}l \circ \mathbb{A} = \sum_j \mathcal{C}l(\xi_j)\mathbb{A}_j$.
• $c(F^{\mathcal{E}/S}) = \sum_{i<j} F^{\mathcal{E}/S}(e_i,e_j)\mathcal{C}l(e^i)\mathcal{C}l(e^j)$.

▶ **Lichnerowicz Formula**

Lemma 1. *(1)* $R_{ijkl} = -R_{ijlk}$, *(2)* $R_{ijkl} = -R_{jikl}$, *(3)* $R_{ijkl} + R_{iklj} + R_{iljk} = 0$, *(4)* $R_{ijkl} = R_{klij}$.

Lemma 2. *Under the decomposition* $End(\mathscr{E}) \cong \mathcal{C}l(X)\otimes End_{\mathcal{C}l(X)}(\mathscr{E})$*, the curvature* $\mathbb{A}^2$ *decomposes as* $\mathbb{A}^2 = R^{\mathscr{E}} + F^{\mathscr{E}/S}$.

Proof: Let $a \in \Gamma(X, T^*X)$. Since $\nabla^2 a = Ra$ and $[R^{\mathscr{E}}, c(a)] = c(Ra)$, we get

$$[\mathbb{A}^2, c(a)] = [\mathbb{A}, [\mathbb{A}, c(a)]] = [\mathbb{A}, c(\nabla a)] = c(\nabla^2 a) = c(Ra) = [R^{\mathscr{E}}, c(a)].$$

So $F^{\mathscr{E}/S}$ commutes with the operators $c(a)$, i.e. $F^{\mathscr{E}/S}$ is a differential form with value in $\operatorname{End}_{\mathcal{C}l(X)}(\mathscr{E})$. **Q.E.D.**

Lemma 3. $\sum_{i<j}\mathcal{C}l(e^i)\mathcal{C}l(e^j)\mathbb{A}^2(e^i,e^j) = -\frac{1}{8}\sum_{ijkl} R_{klij}\mathcal{C}l(e^i)\mathcal{C}l(e^j)\mathcal{C}l(e^k)\mathcal{C}l(e^l) + c(F^{\mathscr{E}/S})$.

Lemma 4.

$$\mathcal{C}l(e^i)\mathcal{C}l(e^j)\mathcal{C}l(e^k) = \frac{1}{6}\sum_{\sigma\in S_3} sgn(\sigma)\mathcal{C}l(e^{\sigma(i)})\mathcal{C}l(e^{\sigma(j)})\mathcal{C}l(e^{\sigma(k)}) - \delta^{ij}\mathcal{C}l(e^k) - \delta^{jk}\mathcal{C}l(e^i) + \delta^{ki}\mathcal{C}l(e^j).$$

Theorem 2 (Lichnerowicz Formula)**.**

$$\mathcal{D}^2_{\mathbb{A}} = \Delta^{\mathbb{A}} + c(F^{\mathcal{E}/S}) + \frac{r_X}{4}$$

Proof: By the local formula of $\mathcal{D}_{\mathbb{A}}$, we get

$$\begin{aligned}
\mathcal{D}_{\mathbb{A}}^2 &= \sum_{i,j} \mathcal{C}l(\xi_i)\ \mathbb{A}_i\ \mathcal{C}l(\xi_j)\ \mathbb{A}_j \\
&= \sum_{i,j} \frac{1}{2}[\mathcal{C}l(\xi_i)\mathcal{C}l(\xi_j) - \mathcal{C}l(\xi_j)\mathcal{C}l(\xi_i)]\mathbb{A}_i\mathbb{A}_j + \sum_{i,j}[\mathcal{C}l(\xi_i)\mathbb{A}_i\mathcal{C}l(\xi_j)\mathbb{A}_j - \mathcal{C}l(\xi_i)\mathcal{C}l(\xi_j)\mathbb{A}_i\mathbb{A}_j] \\
&\quad + \sum_{i,j} \frac{1}{2}\mathcal{C}l(\xi_i)\mathcal{C}l(\xi_j)[\mathbb{A}_i\mathbb{A}_j + \mathbb{A}_j\mathbb{A}_i] \\
&= \sum_{i,j}(-g^*(\xi_i,\xi_j)\mathbb{A}_i\mathbb{A}_j) + \sum_{i,j}\mathcal{C}l(\xi_i)\mathcal{C}l(\nabla_i\xi_j)\mathbb{A}_j + \sum_{i<j}\mathcal{C}l(\xi_i)\mathcal{C}l(\xi_j)[\mathbb{A}_i,\mathbb{A}_j] \\
&= -\sum_{i,j} g^{ij}\mathbb{A}_i\mathbb{A}_j + \sum_{ij}\sum_k \Gamma_{ik}^j\mathcal{C}l(\xi_i)\mathcal{C}l(\xi_k)\mathbb{A}_j + \sum_{i<j}\mathcal{C}l(\xi_i)\mathcal{C}l(\xi_j)[\mathbb{A}_i,\mathbb{A}_j] \\
&= -\sum_{i,j} g^{ij}[\mathbb{A}_i\mathbb{A}_j - \sum_k \Gamma_{ij}^k\mathbb{A}_k] + \sum_{i<j}\mathcal{C}l(\xi_i)\mathcal{C}l(\xi_j)[\mathbb{A}_i,\mathbb{A}_j] \\
&= \Delta^{\mathbb{A}} + \sum_{i<j}\mathcal{C}l(e^i)\mathcal{C}l(e^j)\mathbb{A}^2(e^i,e^j) \\
&= \Delta^{\mathbb{A}} - \frac{1}{8}\sum_{ijkl} R_{klij}\mathcal{C}l(e^i)\mathcal{C}l(e^j)\mathcal{C}l(e^k)\mathcal{C}l(e^l) + c(F^{\mathscr{E}/S}) \\
&= \Delta^{\mathbb{A}} - \frac{1}{8}\sum_{ijl} R_{jlij}\mathcal{C}l(e^i)\mathcal{C}l(e^l) + \frac{1}{8}\sum_{ijl} R_{ilij}\mathcal{C}l(e^j)\mathcal{C}l(e^l) + c(F^{\mathscr{E}/S}) \\
&= \Delta^{\mathbb{A}} + \frac{1}{8}\sum_{ijl} R_{ilij}[Cl(e^j)\mathcal{C}l(e^l) + Cl(e^l)\mathcal{C}l(e^j)] + c(F^{\mathscr{E}/S}) \\
&= \Delta^{\mathbb{A}} + \frac{r_X}{4} + c(F^{\mathscr{E}/S}).
\end{aligned}$$

▶ **Applications**

★ Spin manifold

Let $\mathscr{C}$ be the spinor bundle over spin manifold X, $\mathscr{W}\otimes\mathscr{C}$ be a twisted spinor bundle, $D_{\mathscr{W}\otimes\mathscr{C}}$ be the Dirac operator on $\mathscr{C}$ associated to the Clifford connection $\nabla^{\mathscr{W}\otimes\mathscr{C}} = \nabla^{\mathscr{W}}\otimes 1 + 1\otimes\nabla^{\mathscr{C}}$. Then the twisting curvature $F^{\mathcal{E}/S}$ equals $F^{\mathscr{W}}$, the curvature of $\nabla^{\mathscr{W}}$. So we get

Lichnerowicz formula for twisted spinor bundle

$$D_{\mathscr{W}\otimes\mathscr{C}}^2 = \Delta^{\mathscr{W}\otimes\mathscr{C}} + c(F^{\mathscr{W}}) + \frac{r_X}{4}.$$

In particular, for the spinor bundle $\mathscr{C}$ itself, we have

Lichnerowicz formula for spinor bundle

$$D^2 = \Delta^{\mathscr{C}} + \frac{r_X}{4},$$

where D is **the** Dirac operator on $\mathscr{C}$ w.r.t the Levi-Civita connection $\nabla^{\mathscr{C}}$.

The following result is the first application, due to Lichnerowicz:

Corollary 1. *If X is a compact spin manifold with nonnegative scalar curvature which is strictly positive at some point. Then the kernel of* **the** *Dirac operator vanishes. In particular, its index is 0.*

Proof:

$$\int_X (D^2 s, s)\ dx = \int_X |\nabla s|^2\ dx + \frac{1}{4}\int_X r_X |s|^2\ dx.$$

So $Ds = 0 \Longrightarrow \nabla s = 0$ and $r_X|s|^2 = 0 \Longrightarrow s$ is constant$\Longrightarrow s = 0$. **Q.E.D.**

The above result is always referred as "no harmonic spinor". From the above proof, we see that in the case $r_X \equiv 0$, $|\nabla s| = 0$ for any harmonic spinor s, i.e.

Corollary 2. *If $r_X \equiv 0$, then the harmonic spinors are globally parallel.*

Since the Laplacian is positive operator, we get immediately

Corollary 3. *The eigenvalues of the Dirac operator satisfy $|\lambda|^2 > \dfrac{\min_{x\in X} r_X(x)}{4}$.*

★ Riemann manifold

The Dirac operator associated to $\wedge T^*X$ and its Levi-Civita connection is the operator $d + d^*$. In this case, the Lichnerowicz formula becomes the famous Weitzenbock Formula:

$$(d + d^*)^2 = \Delta^{\wedge T^*X} - \sum_{ijkl} R_{ijkl}\varepsilon^k \iota^l \varepsilon^j \iota^j.$$

Since

$$\sum_{ijkl} R_{ijkl}\varepsilon^i \iota^j \varepsilon^k \iota^l = -\sum_{ijkl} R_{ijkl}\varepsilon^i \varepsilon^k \iota^j \iota^l - \sum_{ij} Ric_{ij}\varepsilon^i \iota^j,$$

we get

$$(d + d^*)^2 = \Delta^{\wedge T^*X} + \mathrm{Ric} \quad \text{on 1-forms.}$$

Corollary 4. *If X is compact Riemannian manifold with positive Ricci tensor, then the deRham cohomology group $H^1_{dR}(X, \mathbb{R}) = 0$.*

Proof: By Hodge theorem, we only need to prove $\mathcal{H}^1(X) = 0$. In fact,

$$u \in \mathcal{H}^1(X) \Longleftrightarrow du = 0, d^*u = 0 \Longrightarrow (d + d^*)u = 0.$$

So by Weitzenbock Formula,

$$0 = |(d + d^*)u|^2_{L^2} = (Ric(u), u) + |\nabla u|^2_{L^2}.$$

Thus u is constant and the constant must be 0. **Q.E.D.**

★ Kähler manifold

Suppose X is Kähler manifold, then $\bar\partial + \bar\partial^*$ is Dirac operator. Let $\mathscr{W}$ be an Hermitian holomorphic bundle over X. In this case, the Lichnerowicz formula is Bochner-Kodaira Formula

$$\bar\partial\bar\partial^* + \bar\partial^*\bar\partial = \Delta^{0,\bullet} + \sum_{ij} \varepsilon(d\bar z^i)\iota(dz^j) F^{\mathscr{W}\otimes\wedge^n T^{1,0}X}(\partial_{z^j}, \partial_{\bar z^i}).$$

Corollary 5 (Kodaira Vanishing Theorem.)**.** *If $\mathscr{L}$ is a Hermitian holomorphic line bundle over a compact Kahler manifold X such that the line bundle $\mathscr{L} \otimes \wedge^n T^{1,0}X$ is positive, then*

$$H^i(X, \mathcal{O}(\mathscr{L})) = 0 \qquad \text{for } i > 0.$$

References:
[1]N.Berlne, E.Getzler and M.Vergne, Heat Kernels and Dirac Operators. §3.5, §3.6.
[2]R.Melrose, The Atiyah-Patodi-Singer index Theorem. §8.8
[3]M.Taylor, Partial Differential Equations II. §10.4.
[4]B.Lawson and M.Michelsohn, Spin Geometry. §2.8.

4. 28 February, 2006

Future talks

- Maksim Lipyanskiy:- Spin and spin-$\mathbb{C}$ structures
- Yakov Shapiro:- Finish $\mathbb{Z}_2$ grading
- Ricardo Andrade:- Unique continuation
- William Lopes:- K-theory
- Zuoqin Wang:- Heat kernel

4.1. **Ricardo Andrade:- Periodicity.** Periodicity – completing earlier talk.

With $V = \mathbb{R}^{r+s}$, $q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}$

$$\mathrm{Cl}_{r,s} = \mathrm{Cl}(V, q),\ v^2 = -q(v). \tag{1}$$

In terms of generators $\{e_i\}_{i=1}^{r+s}$

$$e_i e_j + e_j e_i = \begin{cases} -\delta_{ij} & i \le r \\ \delta_{ij} & i, j > r. \end{cases} \tag{2}$$

The $\Pi_{a \in S} e_a$ for $S \subset \{1, \dots, r+s\}$. Then in terms of $\mathbb{Z}_2$ graded tensor product

$$\mathrm{Cl}_{r,s} \simeq \mathrm{Cl}_{1,0}^{\hat{\otimes}^r} \hat{\otimes} \mathrm{Cl}_{0,1}^{\hat{\otimes}^s}. \tag{3}$$

Main statements are ungraded

$$\begin{gathered} \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2} = \mathrm{Cl}_{0,n+2}, \\ \mathrm{Cl}_{0,n} \otimes \mathrm{Cl}_{0,2} = \mathrm{Cl}_{n+2,0}\, \mathrm{Cl}_{r,s} \otimes \mathrm{Cl}_{1,1} = \mathrm{Cl}_{r+1,s+1}. \end{gathered} \tag{4}$$

Proved by using the universal property. Explicitly compute $\mathrm{Cl}_{1,0} = \mathbb{C}$, $\mathrm{Cl}_{2,0} = \mathbb{H}$, $\mathrm{Cl}_{0,1} = \mathbb{R} \otimes \mathbb{R}$, $\mathrm{Cl}_{0,2} = \mathbb{R}(2)$, $\mathrm{Cl}_{1,1} = \mathbb{R}(2)$ etc.

Complex case.

4.2. **Fangyun Yang:- Elliptic regularity.** M closed Riemannian, i.e. compact without boundary, E Hermitian bundle over M, ∇ a connection on E. Sobolev norms

$$\|\phi\}_{L_k^2} = \int_M |\phi|^2 + |\nabla\phi|^2 + \dots |\nabla \circ \dots \nabla\phi|^2 \tag{5}$$

complete to Sobolev space $L_k^2(M, E)$. Hilbert space independent of choices up to equivalence of norms.

Alternatively use local coordinates. On $\mathcal{S}(\mathbb{R}^n)$ Sobolev s-norm for

$$\|u\|_s^2 = \int (1 + |\xi|^2)^2 s |\hat{u}(\xi)|^2 d\xi. \tag{6}$$

Localize and use partition of unity to get the same spaces and, using coordinate-invariance get general real order Sobolev spaces.

Elliptic differential operator. A differential operator is a linear map between sections of vector bundles

$$P : \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M; F) \tag{7}$$

and in local coordinates $P = \sum_\alpha A^\alpha(x)\partial_x^\alpha$. The principal symbol is a well-defined map

$$\sigma(P) : \pi^* E \longrightarrow \pi^* F \tag{8}$$

$\pi : T^*M \longrightarrow M$, $\xi \in T^*_x M$

$$\sigma(P)(\xi) = i^m \sum_{|\alpha|=m} A^\alpha(x)\xi^\alpha. \tag{9}$$

If $\sigma(P)(\xi)$ is an isomorphism for each $\xi \neq 0$ then P is said to be elliptic.

Coordinate-free definition $\phi \in \mathcal{C}^\infty(M)$, $\phi(x) = 0$ $d\phi(x) \neq 0$ then for $u \in \mathcal{C}^\infty(M; E)$

$$\sigma(P)(d\phi(x)u(x)) = P((i\phi)^m u)(x). \tag{10}$$

Compute the symbol of a Dirac operator

$$\sigma(D)(\xi)u = i\,\mathrm{cl}(\xi)u. \tag{11}$$

So Dirac operators are elliptic.

If P is a differential operator of order m then P^* is the formal adjoint of P

$$\int_M \langle Pu, v\rangle = \int_M \langle u, P^* v\rangle. \tag{12}$$

Since $\sigma(P^*)$ is $\sigma(P)^*$ if P is elliptic then so is P^*.

Garding inequality.

Proposition 1. *If $P : \mathcal{C}^\infty(M, E) \longrightarrow \mathcal{C}^\infty(M; F)$ is elliptic of order m then there exists a constant C_k then*

$$\|\phi\|_{L^{k+m}} \leq C_k(\|P\phi\|_{L^k} + \|\phi\|_{L^k}). \tag{13}$$

Proof. Assume same result for domains in $\mathbb{R}^n$ with trivial bundles. Use Fourier transform in constant coefficient case. Freezing coefficients to pass to general case. □

4.3. **Maksim Lipyanskiy:- Spin representation.** In the even dimensional, complexified, case the Clifford algebra has a unique irreducible representation. Algebraically this is because $\mathbb{C}\mathrm{l}(\mathbb{R}^{2n})$ is a matrix algebra. To see the spin representation directly. In terms of orthonormal bundle $c_i = ie_{2i}e_{2i+1}$ all have square Id and $[c_i, c_j] = 0$. If $\mathbb{C}\mathrm{l}(\mathbb{R}^{2n})$ acts on W then W splits into the summands on which each c_i acts with a a fixed sign. Let W_0 be the subspace on which they are all Id. So in fact

$$W = W_0 \otimes \mathbb{C}^{2^n} \tag{14}$$

decomposes into the irreducible.

Example:- Almost complex case.

5. 2 March, 2006

5.1. **Maksim Lipyanskiy:- Spin and Spin-$\mathbb{C}$ structures.** Source:- Hatcher, Bott and Tu.oups

Principal bundle P with structure group G on a space X, conventionally a right action. Example – frame bundle of a vector bundle. Conversely a representation of G gives a vector bundle or a general action of G gives a fibre bundle with structure group G. For a homomorphism of groups $G \longrightarrow H_1 \longrightarrow H_2$

$$(P \times_G H_1) \times_{H_1} H2 = P \times_G H_2. \tag{1}$$

For a group G BG is a classifying space for G, quotient of a contractible space by a free G-action. Equivalence class of principal bundles over X are classified by $[X, BG]$. To construct, consider

$$EG \times_G P \tag{2}$$

which is a bundle over X with contractible fibre so has a section which gives a map into BG.

In real case $H^*(GR_n(\mathbb{R}), \mathbb{Z}_2) = \mathbb{Z}_2[x_1, \dots, x_n,\ |x_i| = i$. In the complex case $H^*(GR_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_n],\ |c_i| = 2i$.

Given a homomorphism of groups $G \longrightarrow H$, there is an equivariant map $\phi : EG \longrightarrow EH$.

Here we are interested in $\mathrm{SO} \longrightarrow \mathrm{O}$ and $\mathrm{Spin} \longrightarrow \mathrm{SO}$. The first corresponds to orientation of a real bundle. This corresponds to asking for a lift of $f : X \longrightarrow BSO$ to $\tilde{f} : X \longrightarrow B\,\mathrm{Spin}$. This is the case if and only if $f^* x_2 = 0$. So,

Theorem 3. *For an oriented manifold manifold M the* SO *bundle given by oriented orthonormal frames has a spin structure, i.e. the manifold has a* Spin *bundle, if and only if $w_2(TM) = 0$.*

Uniqueness:- All other structures arise from non-trivial $H^1(X) \longrightarrow H^1(P) \longrightarrow H^1(\mathrm{SO})$. Thus spin structures form an affine space by $H^1(X, \mathbb{Z}_2)$.

The group $\mathrm{Spin}-\mathbb{C}$ is the image of $\mathrm{Spin}(n) \times \mathrm{U}(1)$ in $\mathbb{C}\mathrm{l}$. This is a 2-1 cover of $\mathrm{Spin}-\mathbb{C}$. The map

$$\mathrm{Spin}-\mathbb{C} \longrightarrow \mathrm{SO}(n) \times \mathrm{SO}(2) \tag{3}$$

The existence of a $\mathrm{Spin}-\mathbb{C}$ structure is equivalent to the existence of a Spin structure on the tangent bundle on $TM \oplus \det(M)$, the determinant bundle. Then

$$w_2(P_{\mathrm{SO}(n)\times\mathrm{SO}(2)} = w_2(P_{\mathrm{SO}(n)} + w_2(P_{\mathrm{SO}(2)}. \tag{4}$$

So a Spin-$\mathbb{C}$ structure exists if and only if w_2 has an integral lift.

5.2. **William Lopes.** On almost complex manifolds there is a natural $\mathrm{Spin}-\mathbb{C}$ structure. The complexified form bundle decomposes into types

$$\Omega^r = \sum_{p+q=r} \Omega^{p,q}. \tag{5}$$

The Clifford algebra acts on $\Omega^{0,*}$ by

$$v \cdot w = \sqrt{2}(v^{0,1} \wedge w - \iota v^{0,1} w. \tag{6}$$

Hence there is a Dirac operator

$$\eth \psi = \sum_{i=1}^{2n} e_i \nabla_{e_i} \psi. \tag{7}$$

In the Kähler case (iff) there is a holomorphic coordinate system at each point in terms of which $g_{ij} = \delta_{ij} + O(|z|^2)$.

5.3. **Yakov Shapiro:- $\mathbb{Z}_2$-grading.** Suppose $E \longrightarrow X$ is a Clifford module over an even-dimensional oriented manifold. Then if $\xi_1, \dots, \xi_{2n}$ is an oriented orthonormal basis,

$$Z = i^{n(2n-1)} \operatorname{cl}(\xi_1) \cdots \operatorname{cl}(\xi)2n), \tag{8}$$

is independent of the choice of basis. Z anticommutes with $\operatorname{cl}(\alpha)$ for and 1-form α and $Z^2 = \operatorname{Id}$. Has eigenspaces ± 1 when acting on any Clifford module which are of equal dimensions, $E = E_+ \oplus E_-$. Then the Dirac operator is odd with respect to this grading.

5.4. **Me.**

- Azumaya bundles – bundles of algebras over a manifold X which are locally trivial, with their algebra structures, and isomorphic to matrix algebras over $\mathbb{C}$. The case $\hom(E)$ for a vector bundle E is supposed to be the 'trival case'
- Equivalence – two such bundles A_1, A_2 are equivalent if there are vector bundles E_1, E_2 such that $A_1 \otimes \hom(E_1) \equiv A_2 \otimes \hom(E_2)$ (as bundles of algebras).
- Serre's theorem – the equivalence classes form an Abelian group under tensor product and this group (the small Brauer group) is the torsion part of $H^3(X; \mathbb{Z})$.
- For a vector bundle E denote by $\operatorname{Hom}(E)$ the bundle over X^2 which has fibre $E_x \otimes E'_y$ at (x, y), linear maps from E_y to E_x. This is important because, for instance, differential operators on E can be identified with certain distributional sections of $\operatorname{Hom}(E)$ over X^2. Even though these distributions are supported at the diagonal (and are 'smooth along it') they are *not* sections of $\hom(E) = \operatorname{Hom}(E)\big|_{\mathrm{Diag}}$.
- For an Azumaya algebra it is therefore natural to ask whether it can be extended from the diagonal in X^2 ($\mathrm{Diag} = X$) to a neighbourhood U as a bundle $\tilde{A}$ with a multiplication like $\operatorname{Hom}(E)$, namely

$$\tilde{A}_{(x,y)} \otimes \tilde{A}_{(y,z)} \longrightarrow \tilde{A}_{(x,z)},\ (x, z), (x, y), (y, z) \in U. \tag{9}$$

  satisfying the obvious associatitivity condition with three points.
- The answer is YES, it is always possible to find such an extension and hence to define an algebra of differential operators 'valued in A' (even though there is in general no bundle for this to act on).

6. 7 March, 2006

6.1. **Ricardo Andrade:- Unique continuation.** Reference, Aronszagn – strong unique continuation. For scalar operators, if

$$|Au(x)|^2 \le M \left(\sum_{i=1}^{n} |\frac{\partial u(x)}{\partial x_i}| + |u(x)|^2 \right) \tag{1}$$

where A is elliptic an everything is real, $u \in H^2$ and the inequality holds ae and if u vanishes to infinite order at a point in the sense that

$$\int_{B_r} |u| = O(r^{\alpha+n})\ \forall\ n, \tag{2}$$

then $u = 0$.

Instead deal with weak continuation. Consider a 'Dirac bundle', $E \longrightarrow M$, i.e. Clifford with compatibile connection. Then $D = \mathrm{cl}\circ\nabla$, $D = \sum_i \mathrm{cl}(e_i)\nabla_{e_i}$. The symbol

$$\sigma(D) = i\,\mathrm{cl}(\xi) \Longrightarrow D = \sum_i \mathrm{cl}(e_i)e_i + \text{ order } 0. \tag{3}$$

Booss-Bavnbek: Consider a closed hypsersurface, $\Sigma \subset M$, normal bundle trivial. Collar neighbourhood

$$\Sigma \times (-1,1) \hookrightarrow M,\ \Sigma = \Sigma_0,\ \Sigma_t \longrightarrow M. \tag{4}$$

If X_{dt}, is the unit normal X_k form a local orthonormal basis then

$$D = \mathrm{cl}(dt)X_{dt} + \sum_k \mathrm{cl}(X_k)X_k + \text{ order } 0 = \mathrm{cl}(X_{dt})X_{dt} + \mathrm{cl}(X_{dt})B_t \tag{5}$$

is the splitting into normal and tangential part. THen the adjoint is

$$B_t^* = \sum_k X_k^*(\mathrm{cl}(X_k))^*(\mathrm{cl}(X_{dt}))^* = B_t^*. \tag{6}$$

So the self-adjoint part $\frac{1}{2}(B_t + B_t^*)$ is self-adjoint and elliptic, the skew part is of order 0.

We take the collar to be derived from Riemannian normal coordiantes around a point with Σ the spheres.

Now, we want to show that if is M connected, $u \in \Gamma(E)$ and $u\big|_\Omega = 0$ for some open set, then $u = 0$. Take $\widetilde{\Omega} = \{u = 0\}$ so $\widetilde{\Omega}$ has non-empty interior. Choose a ball centred in the interior with T the radius at which the sphere hits the boundary. Show that in fact u vanishes in a larger ball. Take t to be the radius shifted by a constant so $t = 0$ is the ball on which u vanishes.

Take a cutoff function $\phi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \le \phi \le 1$, $\phi(t) = 1$ in $t < 8/10$, $\phi(t) = 0$ in $t \ge 9/10$. Consider $v(t,y) = \phi(t)u(t,y)$ and show that

$$R\int_0^T \int_{S_t} e^{R(T-t)^2}\|v(t,y\|^2 dydt \le \int_0^T \int_{S_t} e^{R(T-t)^2}\|Dv(t,y\|^2 dydt,\ \forall\ R >> R_0. \tag{7}$$

This implies the result. Indeed,

$$\begin{aligned} &e^{RT^2/4}\int_0^{T/2}\int_{S_t}\|v\|^2 dydt \le \int_0^{T/2}\int_{S_t} e^{R(T-t)^2}\|v\|^2 dydt \le \\ &\frac{C}{R}\int_0^{T/2}\int_{S_t} e^{R(T-t)^2}\|Dv\|^2 dydt \le \frac{C}{R}e^{\frac{RT^2}{25}}\int_0^{T/2}\int_{S_t}\|Dv\|^2 dydt \\ &\qquad \Longrightarrow e^{RT^2/4}\int_0^{T/2}\int_{S_t}\|u\|^2 dydt \le \frac{C}{R}e^{-KRT^2}\int_0^{T/2}\int_{S_t}\|Dv\|^2 dydt, \end{aligned} \tag{8}$$

$K = 21/100$, $R \to \infty$.

To derive the Carleman inquality set $v_0 = e^{R(T-t)^2/2}v$ then we want

$$\int_0^T \int_{S_t}\|v_0\|^2 dydt \le C\int_0^T\int_{S_t}\|\frac{\partial v_0}{\partial t} + (B_t + v_0)v_0 + R(T-t)v_0\|^2 dydt = \alpha. \tag{9}$$

Then

$$\alpha = \int\int \|\frac{\partial v_0}{\partial t} v_0 + C_t v_0\|^2 + \int\int \|B_t v_0 + R(T-t)v_0\|^2 + \beta, \tag{10}$$
$$\beta = 2\operatorname{Re}\int\int \langle \frac{\partial v_0}{\partial t} v_0 + C_t v_0, B_t v_0 + R(T-t)v_0\rangle.$$

Now

$$\begin{aligned}\beta &= -2\operatorname{Re}\int\int \langle v_0, \partial_t(B_t + R(T-t))v_0\rangle - 2\operatorname{Re}\int\int \langle v_0, C_t B_t v_0\rangle \\ &= \int\int \langle v_0, -\partial_t(B_t)v_0 + Rv_0\rangle + \int\int \langle v_0, [B_t, C_t]v_0\rangle = R\int\int \|v_0\|^2 + \gamma, \\ \gamma &= \int\int \langle v_0, -\partial_t(B_t)v_0 + [B_t, C_t]v_0\rangle\end{aligned} \tag{11}$$

and

$$\gamma \le \frac{1}{2}\left(\int\int \|v_0\|^2 + \int\int \|(B_t + R(T - t(v_0\|^2\right) \tag{12}$$

6.2. **Zuoqin Wang:- (notes as well as talk) Heat kernel.** Let (M, g) be n-dimensional Riemannian manifold, Δ be the scalar Laplacian acting on functions on M. In local coordinates it is given by

$$\Delta = \sum_{i,j}^{n} g^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b_i(x)\partial_{x_i}. \tag{13}$$

Definition 1. A **heat kernel** is a function $k \in C^\infty((0,\infty) \times M \times M)$ satisfying

$$\begin{aligned}(\partial_t - \Delta_x)k(t,x,y) &= 0, \\ \lim_{t\to 0+} k(t,x,y) &= \delta_y(x).\end{aligned} \tag{14}$$

We begin by considering $M = \mathbb{R}^n$, then by Fourier transform one can solve and get the heat kernel explicitly

$$k(t,x,y) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}. \tag{15}$$

Remark: The above formula is true not only for the standard inner product on $\mathbb{R}^n$, but also for any inner product given by positive definite matrix.

Unfortunately for general Riemannian manifold M, it is usually impossible to find such an exact formula for the heat kernel. However, for many problems, an approximate solution is sufficient.

The main theorem is

Theorem 4 (Minakshisundaram-Pleijel)**.** *Suppose M is compact without boundary, then there exists a unique heat kernel. More over, for each $x \in M$ there is a complete asymptotic expansion*

$$k(t,x,x) \sim (4\pi t)^{-n/2}(a_0(x) + a_1(x)t + a_2(x)t^2 + \cdots), \qquad t \to 0. \tag{16}$$

where the a_j's are smooth functions on M.

The standard proof of the Theorem use either Riemannian normal coordinates or the theory of pseudodifferential operators with parameter. In this lecture we will develop a heat calculus proof, appeared in Daniel Grieser's online notes [1], which was inspired by Melrose's treatment in [2].

6.2.1. *heat calculus.* We will use the notation

$$C^\infty([0,\infty)_{1/2}) = \{f \mid f(t) = g(\sqrt{t}), \forall t \geq 0 \text{ for some } g \in C^\infty(\mathbb{R})\}$$

and the symbol $D^\alpha_{\sqrt{t},x,y}$ means differential in variables $\sqrt{t}, x, y$.

Definition 2. Let M be a manifold and $s \leq 0$. The **heat space** $\Psi^s_H(M)$ is the set of functions A on $(0,\infty) \times M \times M$ satisfying
(a) A is smooth.
(b) For $x \neq y$, $D^\alpha_{t,x,y} A(t,x,y) = O(t^\infty)$ as $t \to 0$. ('Off diagonal decay')
(c) Locally $A(t,x,y) = t^{-\frac{n+2}{2}-s} \tilde{A}(t, \frac{x-y}{\sqrt{t}}, y)$ for some $\tilde{A} \in C^\infty([0,\infty)_{1/2} \times \mathbb{R}^n \times U)$, which is rapidly decaying in the second variable:

$$|D^\alpha_{\sqrt{t},X,y} \tilde{A}(t,X,y)| = O(|X|^{-\infty}), \quad |X| \to \infty \tag{17}$$

for all α, uniformly for bounded t and y.

For such a function A, we can define an operator, still defined by A, by

$$Af(t,x) = \int_M A(t,x,y) f(y)\, dy. \tag{18}$$

Definition 3. The **'top symbol'** of A is defined to be $\sigma^s_H(A)(X,y) = \tilde{A}(0,X,y)$.

Definition 4. The **convolution product** is

$$(A * B)(t,x,y) = \int_0^t \int_M A(t-s,x,z) B(s,z,y)\, dz ds. \tag{19}$$

The next proposition shows that the heat calculus is very similar to the standard pseudodifferential calculus: the Ψ^s_H also forms a filtered algebra, the 'top symbol' corresponds to the principal symbol for PsDO, there is a similar short exact sequence, and we also has the asymptotic summation.

Proposition 1. (a) $\Psi^{s-1/2}_H(M) \subset \Psi^s_H(M)$.
(b) $\sigma^s_H(A)$ is well-defined and defined invariantly as a function on TM, rapidly decaying in the fiber direction: $\sigma^s_H(A) \in C^\infty_{S(fibers)}(TM)$.
(c) We have the following short exact sequence

$$0 \to \Psi^{s-1/2}_H(M) \to \Psi^s_H(M) \to C^\infty_{S(fibers)}(TM) \to 0.$$

(d) If $A \in \Psi^s_H(M), B \in \Psi^t_H(M)$, with $s,t < 0$, then $A * B \in \Psi^{s+t}_H(M)$, and

$$\sigma(A*B)(X,y) = \int_0^1 \int_{\mathbb{R}^n} (1-\xi)^{-\frac{n+2}{2}-s} \xi^{-\frac{n+2}{2}-t} \sigma(A)\left(\frac{X-Z}{\sqrt{1-\xi}}, y\right) \sigma(B)\left(\frac{Z}{\sqrt{\xi}}, y\right) dZ d\xi. \tag{20}$$

For proof, see [1].

The following is the central calculation for the heat kernel construction.

Lemma 5. *Let $A \in \Psi^s_H(M)$ with $s \leq -1$, then $(\partial_t - \Delta_x)A \in \Psi^{s+1}_H(M)$, and*

$$\sigma((\partial_t - \Delta_x)A)(X,y) = \left[-\frac{n+2}{2} - s - \frac{1}{2}\sum_i X_i \partial_{X_i} - \sum_{ij} g^{ij}(y) \partial_{X_i} \partial_{X_j} \right] \sigma(A)(X,y). \tag{21}$$

Proof: Denote $l = (n+2)/2 + s$, then

$$\begin{aligned}\partial_t A = \partial_t (t^{-l} \tilde{A}(t, \frac{x-y}{\sqrt{t}}, y)) &= -l t^{-l-1} \tilde{A} - \sum_i t^{-l} \frac{x_i - y_i}{2t^{3/2}} \partial_{X_i} \tilde{A} + t^{-l} \partial_t \tilde{A} \\ &= t^{-l-1}(-l - \frac{1}{2} \sum_i X_i \partial_{X_i}) \tilde{A} + t^{-l} \partial_t \tilde{A}\end{aligned}$$

where $t^{-l}\partial_t \tilde{A} \in \Psi_H^{s+1/2}(M)$ since $\partial_t = \frac{1}{2\sqrt{t}} \partial_{\sqrt{t}}$.

Next, by

$$\partial_{x_i} A = \partial_{x_i}(t^{-l}\tilde{A}) = t^{-l-1/2} \partial_{X_i} \tilde{A}$$

and Taylor expansion

$$g^{ij}(x) = g^{ij}(y) + h^{ij}(x,y)(x-y)$$

we get

$$\begin{aligned}\Delta_x A &= t^{-l-1} \sum_{i,j} g^{ij}(x) \partial_{X_i} \partial_{X_j} \tilde{A} + t^{-l-1/2} \sum_i b_i(x) \partial_{X_i} \tilde{A} \\ &= t^{-l-1} \sum_{i,j} g^{ij}(y) \partial_{X_i} \partial_{X_j} \tilde{A} + t^{-l-1/2} (\sum_{i,j} h^{ij}(y + X\sqrt{t}, y) X \partial_{X_i} \partial_{X_j} + \sum_i b_i(y + X\sqrt{t}) \partial_{X_i}) \tilde{A}\end{aligned}$$

where the last term also contained in $\Psi_H^{s+1/2}(M)$. Combine the above, we get the result. **Q.E.D.**

Remark: The following observation is very important: when computing the top symbol of $(\partial_t - \Delta_x)A$ at y, one may forget the lower order part of Δ, the x-dependence of the leading term of Δ, and the t-dependence of $\tilde{A}$!

Lemma 6. *If $A \in \Psi_H^{-1}(M)$ and $f \in C^\infty(M)$, then $Af \in C^\infty([0,\infty)_{\sqrt{t}} \times M)$, and*

$$Af(0,x) = f(x) \int_{T_x M} \sigma(A)(X,x)\, dX. \tag{22}$$

In particular, if $A \in \Psi_H^s(M)$ with $s < -1$, then $Af(0,x) = 0$.

Proof:

$$\begin{aligned}Af(0,x) &= \lim_{t \to 0+} t^{-n/2} \int_{\mathbb{R}^n} \tilde{A}(t, \frac{x-y}{\sqrt{t}}, y) f(y)\, dy \\ &= \lim_{t\to 0+} \int_{\mathbb{R}^n} \tilde{A}(t, X, x - X\sqrt{t}) f(x - X\sqrt{t})\, dX \\ &= f(x) \int_{\mathbb{R}^n} \sigma(A)(X,x)\, dX. \qquad \text{Q.E.D.}\end{aligned}$$

6.2.2. *Constructing of Heat Kernel.* To construct the Heat kernel in general case, a natural way is 1) first construct a approximate heat kernel k_1, 2) then correct it step by step.

In view of the heat kernel in $\mathbb{R}^n$, it is very natural to take

$$k_1(t,x,y) = (4\pi t)^{-n/2} e^{-|x-y|^2_{g(y)}/4t}. \tag{23}$$

Obviously $k_1 \in \Psi_H^{-1}$ with leading symbol

$$\sigma(k_1)(X,y) = (4\pi)^{-n/2} e^{-|X|^2_{g(y)}/4}.$$

By Lemma 2, $\lim_{t\to 0+} k_1 = \delta_y(x)$. Let $r = (\partial_t - \Delta_x)k_1$, then Lemma 1 tell us that $r \in \Psi_H^0(M)$ with top symbol $\sigma_H^0(r) = 0$. So in fact $r \in \Psi_H^{-1/2}$. Now we only need to prove

Claim 1. *The Volterra series*

$$k_1 - k_1 * r + k_1 * r * r - \cdots \tag{24}$$

converges in $C^\infty((0,\infty)\times M^2)$ *to a heat kernel* $k \in \Psi_H^{-1}(M)$.

We will use the following

Lemma 7. *For any* $A \in \Psi_H^a(M)$ *with* $a < 0$, *we have*

$$\begin{aligned}(\partial_t - \Delta_x)(k_1 * A) &= A + r * A,\\ \lim_{t\to 0+} k_1 * A &= 0.\end{aligned} \tag{25}$$

<u>*'Proof' of Lemma:*</u> The second identity comes from Lemma 2. For the first one,

$$\begin{aligned}\partial_t(k_1 * A)(t,x,y) &= \partial_t \int_0^t \int_M k_1(t-s,x,z)A(s,z,y)\ dzds\\ &= \int_M k_1(0,x,z)A(t,z,y)dz + \int_0^t \int_M \partial_t k_1(t-s,x,z)A(s,z,y)\ dzds\\ &= A(t,x,y) + \int_0^t \int_M (\Delta_x k_1(t-s,x,z) + r(t-s,x,y))A(s,z,y)\ dzds\\ &= A(t,x,y) + \Delta_x(k_1 * A)(t,x,y) + (r * A)(t,x,y). \quad \textbf{Q.E.D.}\end{aligned}$$

<u>*Proof of Claim:*</u> For fixed $N \ge \frac{n}{2} + 1$, $s = r^{*N} \in \Psi_H^{-n/2-1}$ is bounded for bounded t. Let $C = \sup_{(0,t)\times M\times M} |s|$, then

$$\begin{aligned}|s^{*(m+1)}| &= \left| \int_{\Delta_m(t)} \int_{M^m} s(t-t_1,x,z_1)s(t_1-t_2,z_1,z_2)\cdots s(t_{m-1}-t_m,z_{m-1},z_m)s(t_m,z_m,y)\ d\hat{z}d\hat{t} \right|\\ &\le \frac{t^m(\mathrm{vol}M)^m C^{m+1}}{m!}\end{aligned}$$

where $\Delta_m(t)$ is the set $0 \le t_1 \le \cdots \le t_m \le t$, and we used $\mathrm{vol}(\Delta_m(t)) = t^m/m!$. Note that k_1 is uniformly bounded for bounded t,

$$|k_1 * r^{*(i+(m+1)N)}| \le (C')^m/m!, \qquad \forall i = 0,\cdots,N-1, \forall m \in \mathbb{N}$$

for some constant C'. Thus the Volterra series converges in C^0. Similar estimates holds with l derivatives, with N replaced by $N \ge n/2 + 1 + l$ instead. So the Volterra series converges in C^∞. Let k be the limit. Then by lemma 3, k is a heat kernel, since $(\partial_t - \Delta_x)(k_1 * r^{*m}) = r^{*m} + r^{*(m+1)}$. **Q.E.D.**

6.2.3. *Properties of Heat Kernel.*

Proposition 2 (Uniqueness). Suppose $\{\phi_i\}$ is an orthonormal basis of $L^2(M)$ with $\Delta\phi_i = -\lambda_i\phi_i$, then

$$k(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y). \tag{26}$$

In particular, the heat kernel k is unique.

Proof: Write $k(t,x,y)=\sum f_i(t,y)\phi_i(x)$, then $f_i(t,y)=\int_M k(t,x,y)\phi_i(x)\ dx$. Thus

$$\partial_t f_i(t,y)=\int_M \Delta_x k(t,x,y)\phi_i(x)\ dx=\int_M k(t,x,y)\Delta_x\phi_i(x)\ dx=-\lambda_i f_i(t,y).$$

so there exists function $k_i(y)$ such that $f_i(t,y)=k_i(y)e^{-\lambda_i t}$. On the other hand, for any function $f(x)=\sum a_i\phi_i(x)$, we have

$$\begin{aligned}f(y)&=\lim_{t\to 0+}\int_M k(t,x,y)f(x)\ dx=\lim_{t\to 0+}\int_M\sum_i e^{-\lambda_i t}k_i(y)\phi_i(x)\sum_j a_j\phi_j(x)\ dx\\&=\lim_{t\to 0+}\sum_i e^{-\lambda_i t}k_i(y)a_i=\sum_i k_i(y)a_i\end{aligned}$$

Thus $k_i(y)=\phi_i(y)$ and thus $k(t,x,y)=\sum e^{-\lambda_j t}\phi_j(x)\phi_j(y)$. **Q.E.D.**

Remark: There are two other ways to prove the uniqueness. One is by energy estimate, see [3], the other is by adjoint operators, see [2].

Proposition 3 (Asymptotics). We have the following asymptotic expansion

$$k(t,x,y)\sim(4\pi t)^{-n/2}(a_0(x,y)+a_1(x,y)t+a_2(x,y)t^2+\cdots).$$

In particular,

$$k(t,x,x)\sim(4\pi t)^{-n/2}(a_0(x)+a_1(x)t+a_2(x)t^2+\cdots)$$

Before proving this theorem, let's first introduce an definition:

Definition 5. Suppose $s\in-\mathbb{N}/2$. Call an element $K\in\Psi_H^\alpha(M)$ **even** if for the Taylor coefficients $k_j(X,y)$ of $\widetilde{K}(t,X,y)\sim\sum_{j=0}^\infty k_j(X,y)t^{j/2}$ is even function in X for $j/2+s\in\mathbb{Z}$ and odd in X otherwise.

Lemma 8. *a) K is even $\Rightarrow\partial_t K,\partial_{x_i}K,f(x)K$ are even.*
b) K_1,K_2 are even $\Rightarrow K_1*K_2$ is even.

Proof of Theorem 3: Since k_1 is even, by the above lemma, the heat kernel k constructed in the last section is even. So the terms in the Taylor expansion vanishes for odd j. This gives the required expansion. **Q.E.D.**

Let $a_i=\int_M a_i(x)\ dx$. By the construction, we can see that $a_0(x,y)$ comes from k_1, and $a_0(x)=1$, thus $a_0=Vol(M)$. The next term a_1 comes from k_1,k_1*r and k_1*r*r, and can by computation $a_1(x)=\frac{1}{6}s_M(x)$ is the scalar curvature, thus $a_1=\frac{1}{6}\int_M s_M(x)\ dx$. We omit the computation here. In general the coefficients a_i are certain algebraic expression in metrics and connection coefficients and their derivatives.

6.2.4. *Applications and Generalizations.* **Applications to Spectral Geometry**

By (14), we have

$$Z(t)=\int_M k(t,x,x)dx=\sum_{i=1}^\infty e^{-\lambda_i t}=\mathrm{tr}(e^{-t\Delta}).$$

On the other hand, by (4) we have

$$Z(t)\sim(4\pi t)^{-n/2}(a_0+a_1t+a_2t^2+\cdots)$$

with $a_i=\int_M a_i(x)\ dx$. Thus

$$(4\pi t)^{n/2}\sum_i e^{-\lambda_i t}\sim a_0+a_1t+a_2t^2+\cdots. \tag{27}$$

Corollary 1. The spectrum of Δ determines the dimension n, the volume a_1, the total scalar curvature a_2 and other geometric quantities a_i's.

We can also use (15) in the other direction: knowing the geometric quantities a_i's, we can discover information about the spectrum! The following Weyl's theorem is well known:

Corollary 2 (Weyl theorem). Let $N(\lambda)$ denote the number of eigenvalues of Δ less than λ, then

$$N(\lambda) \sim \frac{\mathrm{vol}(M)}{(4\pi)^{n/2}\Gamma(n/2+1)}\lambda^{n/2}. \tag{28}$$

'Proof': From (15) we get

$$t^{n/2}\sum_i e^{-\lambda_i t} \to (4\pi)^{-n/2}\mathrm{vol}(M), \qquad \text{as } t \to 0.$$

Now (16) follows from an abstract Tauberian theorem of Karamata. For details, see [3] or [4]. **Q.E.D.**

In other words, the k^{th} eigenvalue of Δ has an asymptotic estimate

$$\lambda_k \sim 4\pi\left(\frac{\mathrm{vol}(M)}{\Gamma(n/2+1)}\right)^{2/n} k^{2/n}. \tag{29}$$

As a corollary, the zeta function

$$\zeta(s) = \sum_k \lambda_k^{-s} \tag{30}$$

is well defined for $\mathrm{Re}s > \frac{n}{2}$. This will go to another story $\cdots$

Applications to Index theorem

Let $E(\lambda) = \{\phi \mid D^*D\phi = \lambda\phi\}$ and $F(\lambda) = \{\phi \mid DD^*\phi = \lambda\phi\}$, then for $\lambda \neq 0$, the map $D : E(\lambda) \to F(\lambda)$ is an isomorphism:

$$D^*D\phi = \lambda\phi \Longrightarrow (DD^*)D\phi = \lambda D\phi,$$
$$D\phi = 0 \Longrightarrow 0 = \|D\phi\|^2 = \langle D^*D\phi, \phi\rangle \Longrightarrow \phi \in \ker(D^*D).$$

Thus

$$\mathrm{ind}(D) = \dim\ker D^*D - \dim\ker DD^* = \sum_{\lambda_j=0} e^{-\lambda_j t} - \sum_{\mu_j=0} e^{-\mu_j t} = \mathrm{tr}(e^{-tD^*D}) - \mathrm{tr}(e^{-tDD^*})$$
$$= \sum_{j=0}^{\infty} a_j(D^*D)t^{-n/2+j} - \sum_{j=0}^{\infty} a_j(DD^*)t^{-n/2+j} = a_{n/2}(D^*D) - a_{n/2}(DD^*).$$

As a corollary, $\mathrm{ind}(D) = 0$ for odd n.

Generalizations

- Note that the lower order terms of Δ do not affect any things, so the heat kernel expansion holds for generalized Laplacian.
- Moreover, Δ may be replaced with any Petrovski-elliptic(all eigenvalues of the principal symbol have negative real part – a condition to ensure the 'model solutions' are rapidly decaying off the diagonal) self-adjoint differential operator P of order $d > 0$. Essentially the same procedure works, with $\sqrt{t}$ replaced by $t^{1/d}$.

- For manifold with boundary, one can also develop a b-heat calculus to show that there exists a unique heat kernel which admits a similar asymptotic expansion. For detail, see [1], [2].
- It also can be shown that a smooth family of generalized Laplacians will give a smooth family of heat kernels. For details, see §2.7 of [4].

References:
[1] Daniel Grieser, *Notes on Heat Kernel Asymptotics*, Online notes at http://www.math.uni-bonn.de/people/grieser/wwwlehre/heat.pdf
[2] Richard Melrose, *The Atiyah-Patodi-Singer index theorem.* §7.1-§7.5.
[3] John Roe, *Elliptic Operators, Topology and Asymptotic Methods.*
[4] N.Berline, E.Getzler and M.Vergne, *Heat Kernels and Dirac Operators.*

7. 9 March, 2006

7.1. **Zuoqin Wang:- (more of his notes) Heat kernel continued.**

7.2. **Maksim Lipyanskiy:- Heat kernel.** I failed to take notes. I will try a reconstruction.

8. 14 March, 2006

8.1. **William Lopes:- Periodicity.** Periodicity for complex K-theory. Show that

$$K(X \times \mathbb{S}^2) \equiv K(X) \otimes K(\mathbb{S}^2). \tag{1}$$

Definition of $K(X)$. Consider vector bundles over X, taken compact. The direct sum $E \oplus F$ and tensor product $E \otimes F$ are well-defined vector bundles. Consider the isomorphism classes of such vector bundles. It has an additiive identity 0 and multiplicative, $\mathbb{C}$, the trivial line bundle. Consider pairs (E, F), representing $E \ominus F$ and declare $(E, F) \sim (G, H)$ if there exists a bundle P such that

$$E \oplus H \oplus P \equiv F \oplus G \oplus P. \tag{2}$$

Theorem 5. $K(X \times \mathbb{S}^2 \equiv K(X)[t]/\left\{(t-1)^2 = 0\right\}.$

In fact if $L \longrightarrow X$ is a line bundle then

$$K(P(L \oplus \mathbb{C})) = K(X)[t]/\left\{(Lt-1)(t-1) = 0\right\} \tag{3}$$

Subsets of $\mathbb{S}^2 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$:

$$D^0 = \{|z| \le 1\},\ D^\infty = \{|z| \ge 1\| \cup \{\infty\},\ \mathbb{S} = \{|z| = 1\}. \tag{4}$$

Given bundles $E^0 \longrightarrow X \times D^0$ and $E^\infty \longrightarrow X \times D^\infty$ and an isomorpism $f : E^0 \longrightarrow E^\infty$ over $X \times \mathbb{S}$, this gives a vector bundle over $X \times \mathbb{S}^2$, f is called the clutdching function. Homotopic isomorphisms f, f' give isomorphic bundles.

Consider the projection $z : X \times \mathbb{S} \longrightarrow \mathbb{S}$ and its powers z^k, $z \in \mathbb{Z}$. If $a_k : E^0 \longrightarrow E^\infty$ is a bundle map over $X \times \mathbb{S}$ then we can form the bundle map

$$\sum_{k=-n}^{n} a_k z^k \text{ over } X \times \mathbb{S}. \tag{5}$$

Since our bundles E^0 and E^∞ are bundles over $X \times D$ they are both isomorphic to the pull-back of bundles over X. For any f take the Fourier coefficients and get bundle maps (over $X \times \mathbb{S}$)

$$S_n = \sum_{|k| \le n} a_k(x),\ a_k(x) = \frac{1}{2\pi i} \int_{\mathbb{S}} f(x,z) z^{-k-1} dz \tag{6}$$

The Cesaro mean $f_n = \frac{1}{n} \sum_{k=0}^{n} S_k$ converges uniformly to f. So for large enough n, f_n is an isomorphism homotopic to f and hence gives an isomorphic bundle.

The map $z : X \times \mathbb{S} \longrightarrow \mathbb{S} \in \mathrm{GL}(1, \mathbb{C})$ is a clutching function for the trivial bundle and

$$(\mathbb{C}, z, \mathbb{C}) = H^* \tag{7}$$

is the canonical line bundle over $X \times \mathbb{S}^2$.

Given a polynomial clutching function p for E^0, E^∞ then we can find bundles $V^0 = (n+1)E^0$, and $V^\infty = (n+1)E^\infty$ and a linear clutching function p' between them such that

$$(E^0, p, E^\infty) + n(E^0, 1, E^0) \simeq (V^0, p', V^\infty) \tag{8}$$

8.2. **Yakov Shapiro:- Chern-Weil theory.** Let $E \longrightarrow M$ be a vector bundle, $\Omega^k(M)$ the k forms on M,

$$\Omega^k(E) = \Omega^k(M) \otimes \mathcal{C}^\infty(E)$$

A connection on E is a linear map

$$\nabla : \mathcal{C}^\infty(E) \longrightarrow \Omega^1(E) \tag{9}$$

such that

$$\nabla(fs) = df \otimes s + f \nabla s. \tag{10}$$

In local coordinates $\nabla = d + \omega$ where ω is a matrix of 1-forms. A connection can be extendded to a superconnection

$$\Omega^k(E) \longrightarrow \Omega^{k+1}(E),\ \nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \otimes \nabla s. \tag{11}$$

The curvature of ∇ is

$$F = \nabla^2 : \Omega^k(E) \longrightarrow \Omega^{k+2} \tag{12}$$

and in terms of a local trivialization

$$F = d\omega + \omega \cdot \omega \tag{13}$$

is a matrix of 2-forms.

Let P is a polynomial in one indeterminate then $P(F)$ is a matrix and $\mathrm{Tr}(P(F))$ is the Chern-Weil character of ∇ corresponding to P, in the sense that it defines a cohomology class independent of the choice of ∇.

Theorem 6. *$P(F)$ is closed and its cohomology class does not depend on the ∇.*

Proof.

Lemma 9. *If $B \in \Omega^k(\mathrm{End}(E))$ is a homomorphism valued in k-forms then*

$$d\,\mathrm{Tr}(B) = \mathrm{Tr}\left(\nabla B + (-1)^{k+1} B \nabla\right) \tag{14}$$

Proof. Computing in terms of a local trivialization
(15)
$$\nabla Bs+(-1)^{k+1}B\nabla s = (d+\omega)B+(-1)^{k+1}B(d+\omega)s = (dB)s+\omega BS+(-1)^{k+1}B\omega s$$

Taking the trace gives (14). Hence $d\,\mathrm{Tr}(P(F)) = \mathrm{Tr}(\nabla F) = 0$ by the Bianchi identity. □

Lemma 10. *If ∇_t is a family of connections for $t \in [0,1]$ then*

$$\frac{\partial}{\partial t}\mathrm{Tr}(P(F_t)) = d\,\mathrm{Tr}(\frac{\partial \nabla_t}{\partial t}P'(F_t)). \tag{16}$$

□

9. 16 March, 2006

9.1. **Yakov Shaprio:- Chern-Weil continued.** Chern-Weil form for a connection on a vector bundle $\mathrm{Tr}(P(F))$ for $P(z)$ a formal power series in one variable. The class $[P(F)] \in H^{\mathrm{even}}$.

9.2. **Maksim Lipyanskiy:- Impromptu discussion of McKean-Singer.** Index:- Even dimensional Dirac operator. The index is

$$\mathrm{ind}(D^+) = \dim\mathrm{null}(D^+) - \dim\mathrm{null}(D^-) = \mathrm{Tr}(e^{-tD^-D^+} - e^{-tD^-D^+}). \tag{1}$$

Discussion of eigenvalues and cancellation. Local index theorem.

9.3. **Zuoqin Wang:- Notes (not a talk) on McKean-Singer.** Let (M, g) be n-dimensional compact Riemannian manifold without boundary, Δ be a generalized Laplacian. Then the heat kernel is given by the Volterra series

$$k_1 - k_1 * r + k_1 * r * r - \cdots \tag{2}$$

where $k_1(t,x,y) = (4\pi t)^{-n/2}e^{-|x-y|^2_{g(y)}/4t}$ and $r = (\partial_t - \Delta_x)k_1$. Thus a smooth family of generalized Laplacians will give a smooth family of heat kernels.

The heat kernel generate a semigroup of operators

$$e^{-t\Delta}s(x) = \int_M k(t,x,y)s(y)\ dy. \tag{3}$$

Moreover, suppose $\{\phi_i\}$ is an orthonormal basis of $L^2(M)$ with $\Delta\phi_i = -\lambda_i\phi_i$, then the heat kernel has the following expansion

$$k(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y). \tag{4}$$

Thus

$$\mathrm{tr}(e^{-t\Delta}) = \int_M k(t,x,x)dx = \sum_{i=1}^{\infty} e^{-\lambda_i t}. \tag{5}$$

Now suppose D is a Dirac operator. Denote the eigenvalues of D^*D by λ_i, eigenvalues of DD^* by μ_i. Let $E(\lambda) = \{\phi \mid D^*D\phi = \lambda\phi\}$, $F(\mu) = \{\phi \mid DD^*\phi = \lambda\phi\}$.

Lemma 11. *For $\lambda \neq 0$, the map $D : E(\lambda) \to F(\lambda)$ is bijective, with inverse $\lambda^{-1}D^*$.*

<u>Proof:</u> Suppose $\phi \in E(\lambda)$, then $(DD^*)D\phi = \lambda D\phi$, i.e. $D\phi \in F(\lambda)$. By symmetry, D^* maps $F(\lambda)$ to $E(\lambda)$. Moreover, $\lambda^{-1}D^*D = 1$ on $E(\lambda)$, and $\lambda^{-1}DD^* = 1$ on $F(\lambda)$. This proves the lemma. **Q.E.D.**

Theorem 7 (McKean-Singer)**.** *ind*$(D) = tr(e^{-tD^*D}) - tr(e^{-tDD^*})$.

<u>*Proof:*</u> Obviously $\ker(D) \subset \ker(D^*D)$. On the other hand, $\ker(D^*D) \subset \ker(D)$ since $\|D\phi\|^2 = \langle D^*D\phi, \phi\rangle$. Thus $\ker(D) = \ker(D^*D)$. By symmetry, $\ker(D^*) = \ker(DD^*)$. So

$$\begin{aligned}\operatorname{ind}(D) &= \dim \ker D - \dim \ker D^* \\ &= \dim \ker D^*D - \dim \ker DD^* \\ &= \sum_{\lambda_j = 0} e^{-\lambda_j t} - \sum_{\mu_j = 0} e^{-\mu_j t} \\ &= \sum_{\lambda_j} e^{-\lambda_j t} - \sum_{\mu_j} e^{-\mu_j t} \\ &= \operatorname{tr}(e^{-tD^*D}) - \operatorname{tr}(e^{-tDD^*}). \qquad \textbf{Q.E.D.}\end{aligned}$$

Remark: Moreover, **let** f **be any rapidly decreasing smooth function on** $\mathbb{R}^+$ **with** $f(0) = 1$**, then ind**$(D) = \mathbf{tr}(f(D^*D)) - \mathbf{tr}(f(DD^*))$. In fact, let $g(x) = f(x) - e^{-tx}$, it suffices to show that $\operatorname{tr}(g(D^*D)) - \operatorname{tr}(g(DD^*))$ vanishes. But now g is rapidly decreasing with $g(0) = 0$. So $g(x) = xh(x)$ for some rapidly decreasing function h, which implies $\operatorname{tr}(g(D^*D)) - \operatorname{tr}(g(DD^*)) = \sum_{\lambda_j} \lambda_j h(\lambda_j) - \sum_{\mu_j} \mu_j h(\mu_j) = 0$.

Corollary 3. Let D_t be a continuous family of Dirac operators, then $\operatorname{ind}(D_0) = \operatorname{ind}(D_1)$.

<u>*Proof:*</u> Since the heat kernels of D_t varies continuous w.r.t. t, while the index is a integer. **Q.E.D.**

A very useful fact about heat kernel is that on the diagonal it admits the following asymptotic expansion

$$k(t, x, x) \sim (4\pi t)^{-n/2}(1 + a_1(x)t + a_2(x)t^2 + \cdots) \tag{6}$$

Write $a_n = \int_M a_n(x)\, dx$.

Corollary 4. The index of D is zero when n is odd, is $(4\pi)^{-n/2}(a_{n/2}(D^*D) - a_{n/2}(DD^*))$ when n is even.

<u>*Proof:*</u>

$$\begin{aligned}\operatorname{ind}(D) &= \operatorname{tr}(e^{-tD^*D}) - \operatorname{tr}(e^{-tDD^*}) \\ &= (4\pi t)^{-n/2} \left(\sum_{j=0}^{\infty} a_j(D^*D) t^{-n/2+j} - \sum_{j=0}^{\infty} a_j(DD^*) t^{-n/2+j} \right) \\ &= (4\pi)^{-n/2}(a_{n/2}(D^*D) - a_{n/2}(DD^*)). \qquad \textbf{Q.E.D.}\end{aligned}$$

Corollary 5. If $\tilde{M}$ is a k-fold covering of M, then $\operatorname{ind}(\tilde{D}) = k\operatorname{ind}(D)$.

<u>*Proof:*</u> Since $a_{n/2}$ is a local expression which is the same on M as on $\tilde{M}$. **Q.E.D.**

References:

[1] John Roe, *Elliptic Operators, Topology and Asymptotic Methods.*
[2] N.Berline, E.Getzler and M.Vergne, *Heat Kernels and Dirac Operators.*

9.4. **William Lopes:- End of periodicity.** From last time:- $p = \sum_{k=0}^{n} a_k z^k$: $E^0_{X\times\mathbb{S}} \longrightarrow E^\infty_{X\times\mathbb{S}}$ is clutching data for a bundle over $X \times \mathbb{S}^2$. Found a linear map $\mathcal{L}(p) : V^0 \longrightarrow V^\infty$ where $V^0 = (n + 1)E^0$ and $V^\infty = (n + 1)V^\infty$.

Now we can see directly that in K-theory

$$(V^0_{n+1}, \mathcal{L}^{n+1}(zp), V^\infty_{n+1}) = (V^0_n, \mathcal{L}^n(p), V^\infty_n) \oplus (E^0, z, E^\infty) \tag{7}$$

and

$$(V^0_{n+1}, \mathcal{L}^{n+1}(p), V^\infty_{n+1}) = (V^0_n, \mathcal{L}^n(p), V^\infty_n) \oplus (E^0, 1, E^0) \tag{8}$$

where H is the dual of the canonical line bundle. This shows that $(H - 1)^2 = 0$ in $K(X \times \mathbb{S}^2)$ then

$$K(X)[t]/(t - 1)^2 \longrightarrow K(X \times \mathbb{S}^2),\ t \longmapsto H \tag{9}$$

is well defined.

Lemma 12. *For V a vector bundle, $T \in \operatorname{End}(V)$ S a circle in $\mathbb{C}$ missing all the eigenvalues then*

$$Q = \int_S (z - T)^{-1} dz \tag{10}$$

is a projection and commutes with T. If $V^+ = QV$, $V^- = (\operatorname{Id} - Q_V$ then $V = V^+ \oplus V^-$ splits $T = T^+ \oplus T^-$.

This extends to bundles with linear clutching functions, provided the circle is fixed, so $p = P_+ + p_-$ and $p_\pm : E^0_\pm \longrightarrow E^\infty_\pm$ then

$$(E^0, p, E^\infty) = (E^0_+, z, E^\infty_+) \oplus (E^0_+, 1, E^\infty_+). \tag{11}$$

Applied to the general clutching construction this gives

$$(E^0, p, E^\infty) = [V^0_n(E^0, p, E^\infty)][H^2 - 1] + [E^0][1] \tag{12}$$

Finally then if f is any rational clutching function $f = \sum_{-n}^{n} a_k z^k$ $p_n = z^n f(z)$ put

$$v_n(f) = V_{2n}(E^0, p, E^\infty)(t^{n-1} - t^n) - e^0 t^n \in K(X)[t]. \tag{13}$$

For large n this is independent of n and denpends, modulo $(t - 1)^2$ only on the homotopy class of f. Can define $v(E)$ as the image of $v_n(f)$ in $K(x)[t]/(t - 1)^2$ so

$$\mu\nu(E) = \dots \tag{14}$$

so proves the isomorphism.

10. 21 March, 2006

10.1. **Ricardo Andrade:- Mehler's formula and scaling.** Euclidean vector space V, orthonormal basis e_i, A a finite-dimensional commutative $\mathbb{C}$ algebra (to be the even part of the exterior algebra). Let R be an $n \times n$ antisymmetric matrix with entries in A and consider

$$\omega = \frac{1}{4} \sum_{i,j} R_{ij} x_j dx_j \in \Lambda^1(V, A) \tag{1}$$

and the connection

$$\nabla_i = \partial_i + \omega(\partial_i) = \partial_i + \sum_j R_{ij} x_j$$

acting on $\mathcal{C}^\infty(V, A)$. THen for F another $N \times N$ matrix with values in A,

$$H = -\sum_i \nabla_i + F = -\sum_i (\partial_i + \sum_j R_{ij} x_j)^2 + F \tag{2}$$

want existence and uniqueness of solutions of the heat equation associated to this. Set

$$j_V(R) = \det\left(\frac{e^{R/2} - e^{-R/2}}{R/2}\right),\ j_V(0) = 1,\ j_V(tR) = 1 \sum_{k\geq 1} t^k b_k(R) \tag{3}$$

is holomorphic in $|t| < 1$ as is

$$\frac{tR}{2}\coth(\frac{R}{2}).$$

Proposition 2. *The function*

$$f_t(x, R, F) = (4\pi t)^{-\frac{n}{2}} j_V^{-\frac{1}{2}}(tR) \cdot \exp\left(-\frac{1}{4t}\langle x|\frac{tR}{2}\coth(\frac{tR}{2})|x\rangle \exp(-tF)\right) \tag{4}$$

is a solution of

$$(\partial_t + H_x) f_t(x) = 0. \tag{5}$$

Proof. Consider

$$q_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4t}}. \tag{6}$$

If $\Phi_t(x)$ is a formal power series in t then

$$(\partial_t + H_x) q_t \Phi_t) = q_t(\partial + t^{-1} r + H)\Phi_t,\ r = \sum_i x_i \partial_i \tag{7}$$

so is a formal solution if and only if

$$(\partial_t + t^{-1} r + H)\Phi_t = 0. \tag{8}$$

□

10.2. **Fangyun Yang:- Eta invariant.** For B a self-adjoint elliptic differential operator on a compact manifold X let λ_j be the eigenvalues then

$$\eta(s) = \sum_{\lambda_j \neq 0} \operatorname{sgn}(\lambda_j)|\lambda_j|^{-s}. \tag{9}$$

On an even-dimensional manifold for a Dirac operator $\eta(s) = 0$. In the odd-dimensional case $\eta(s)$ is meromorphic and regular at $s = 0$. If A is a positive self-adjoint pseudodifferential operator the zeta function is

$$\zeta_A(s) = \sum_{\mu > 0} \mu^{-s} = \operatorname{Tr}(A^{-s}). \tag{10}$$

By Seeley the sum converges for $\operatorname{Re}(s) > n/m$ and extends to be meromorphic in the complex plane with poles only at

$$) = \int \alpha_0 \zeta_A(s) = \sum_{k=-n,\neq 0}^{N} \frac{a_k}{s + k/m} + \phi_N(s) \tag{11}$$

with $\phi_N(s)$ holomorphic in $\operatorname{Re}(s) \geq -N/m$ with

$$a_k = \int \alpha_k,\ \zeta_A(0. \tag{12}$$

For a smooth family of such operators, A_u, the expansion is smooth in u. Even if A_u has zero eigenvalues at some point the coefficients in the expansion of $\zeta(A_u)$ remain smooth since they are determined by the symbol.

Now B is a self-adjoint Dirac operator on can deduce the meromorphy of the eta function. The convergence follows from the convergence of the zeta function for B^2 in $\mathrm{Re}(s) > n$. In fact 0 is a possible pole since if we write

$$B_1 = \frac{3}{2}|B| + \frac{1}{2}B,\ B_2 = \frac{3}{2}|B| - \frac{1}{2}B. \tag{13}$$

These are both non-negative and if u, $-v$ are positive and negative eigenvalues of B then $2u$, v are eigenvalues of B_1 and u, $2v$ are the eigenvalues of B_2. Thus

$$\zeta_{B_1}(s) - \zeta_{B_2}(s) = (2^{-s} - 1)\eta_B(s). \tag{14}$$

So, a priori, η_B has a simple pole at $s = 0$ with residue

$$R(B) = \frac{-1}{\log 2}(\zeta_{B_1}(0) - \zeta_{B_2}(0)) = \int w. \tag{15}$$

In fact this vanishes. It is homotopy invariant $\frac{d}{du}B_u = 0$. It follows that $R(B)$ descends to a function $R : K^1(T^*X) \longrightarrow \mathbb{R}$.

11. 23 March, 2006

11.1. **Ricardo Andrade:- Local index formula.** M Riemannian manifold, Clifford module, E, Clifford connection, ∇. Heat kernel

$$e^{-tD^2} = K_t \in \Gamma(E \boxtimes E^*) \text{ over } \mathbb{R}^+ \times M \times M). \tag{1}$$

Restricted to the diagonal, $\mathrm{Diag} : M \longrightarrow M \times M$ $E \boxtimes E^* = E \otimes E^* = \hom(E) = \mathrm{End}(E)$. Then we can take the pointwise trace

$$I(t, x) = \mathrm{str}(K_t(x, x)) \text{ is smooth down to } t = 0 \text{ and } \mathrm{ind}(D^+) = \int_M I(0, x)dx. \tag{2}$$

Lichnerowicz formula

$$D^2 = \nabla^*\nabla + R = \nabla^*\nabla + \frac{r_M}{4} + \mathrm{cl}(F), \tag{3}$$

where R is valued in $\Lambda^2(\mathrm{End}(E))$ and locally in terms of an orthonormal bundle

$$R = \frac{1}{2}\sum_{j,k} \mathrm{cl}(e_j)\,\mathrm{cl}(e_k)R(e_j, e_k) \tag{4}$$

and

$$\mathrm{cl}(F) = \sum_{i<j} F(e_i, e_j)\,\mathrm{cl}(e_i)\,\mathrm{cl}(e_j). \tag{5}$$

Work near $x_0 \in M$ in Riemannian normal coordinates in U with $E = E_0$. Then E is trivial using radial translation and the Clifford action of the Riemannian orthonormal basis is constant, i.e. trivial. Then

$$\mathrm{End}(E) = \mathrm{Cl}(T) \otimes \mathrm{End}'(E) \tag{6}$$

where the second part commutes with the Clifford action. There is a corresponding splitting of the bundle $E = S \otimes W$ locally, where S is the spinor bundle. Rescaling

$$\alpha \in \mathcal{C}^\infty(\mathbb{R}^+ \times U; \Lambda^*T \otimes \mathrm{End}'(E),\ \delta_u\alpha(t, x) = \sum_{i=0}^{n} u^{-\frac{i}{2}}\alpha(ut, u^{\frac{1}{2}}x)_{(i)},\ 0 < u \leq 1. \tag{7}$$

Thus we get a rescaled heat kernel

$$r(u,t,x) = u^{\frac{n}{2}}(\delta_u K)(t,x). \tag{8}$$

For the action of the

$$\begin{aligned} \delta_u \partial_t \delta_u^{-1} &= u^{-1}\partial_t, \\ \delta_u \partial_i \delta_u^{-1} &= u^{-\frac{1}{2}}\partial_t, \\ \delta_u \phi(x)\delta_u^{-1} &= \phi(u^{\frac{1}{2}}x),\ \phi \in \mathcal{C}^\infty(U), \\ \delta_u(\alpha\wedge\cdot\delta_u^{-1} &= u^{-\frac{1}{2}}(\alpha\wedge\cdot, \\ \delta_u(\alpha\langle\cdot\delta_u^{-1} &= u^{\frac{1}{2}}(\alpha\langle\cdot, \alpha \in T^*. \end{aligned} \tag{9}$$

Now, if $L(u) = u\delta_u D^2\delta_u^{-1}$,

$$(\partial_t + D^2)K = 0 \Longrightarrow (\partial_t + L(u))r = 0. \tag{10}$$

Then $L(u)$ is smooth down to $u = 0$ and

$$(\partial_t + L(0))r_0 = 0 \tag{11}$$

where r_0 is the leading term in the expansion in u of r.

Lemma 13. *The connection becomes*

$$\nabla_{e_i} = e_i + \frac{1}{4}\sum_{j,k<l} R_{klij}x_j + \sum_{k<l} f_{ikl}(x)\,\mathrm{cl}(e_k)\,\mathrm{cl}(e_l) + g_i(x), \tag{12}$$

where the $f_\bullet$ vanish quadratically at $x = 0$ and g_i vanishes linearly but is valued in $\mathrm{End}'(E)$.

Rescaling the connection it follows that

$$L(0) = -\sum_i (e_i - \frac{1}{4}\sum_j \theta_{ij}xPj)^2 + F \tag{13}$$

where

$$\theta_{ij} = \langle R_0 e_i, e_j\rangle \in \Lambda^2 T.$$

Now the expansion of the heat kernel shows that

$$r = q)t(x)\sum_{i=-n}^{2N} u^{i/2}\gamma_i(t,x) + O(u^N) \tag{14}$$

uniformly in all variables, where the γ_i are polynomials and the γ_i for $i > 0$ vanish at 0. The leading term B_l must satisfy

$$(\partial_t + L(0))B_l(t,x) = 0. \tag{15}$$

and from the uniquess of solutions to this all the singular terms are zero and so the leading term is in fact the u^0 term. From Mehler's formula this leading term is

(16)

$$r_0 = q_t(x)\gamma_0(t,x) = (4\pi t)^{-\frac{n}{2}}\det^{\frac{1}{2}}\left(\frac{t\theta/2}{\sinh(t\theta/2)}\right)\exp\left(-\frac{1}{4t}\langle x, \frac{t\theta}{2}\coth(t\theta/2)x\rangle\right)\exp(-tF).$$

Computing the supertrace at $x = 0$ we just get

$$\inf(D) = \int_M \widehat{A}\,\mathrm{Ch}'(E). \tag{17}$$

11.2. **Index theorem outlined again.** On a $2d$-dimensional compact manifold.

- Clifford algebra, filtered by degree (minimum number of factors),

$$\mathbb{C}\mathrm{l} = \bigcup_{j=0}^{2d} \mathbb{C}\mathrm{l}_{(j)}, \tag{18}$$

- $Z \in \mathbb{C}\mathrm{l}_{(2d)}$ chiral element, $\mathrm{str}(a) = \mathrm{tr}(Za)$, Patodi' Lemma

$$\begin{gathered} 0 \longrightarrow \mathbb{C}\mathrm{l}_{(2d-1)} \longrightarrow \mathbb{C}\mathrm{l} \xrightarrow{\mathrm{str}} \mathbb{C} \longrightarrow 0 \text{ exact}, \\ \bigoplus_j (\mathbb{C}\mathrm{l}_{(j)}/\mathbb{C}\mathrm{l}_{(j-1)} = \Lambda^* \text{ as algebras.} \end{gathered} \tag{19}$$

- Hermitian Clifford module E, unitary Clifford connection, filtering of

$$\hom(E) = \sum_j \hom_{(j)} = \sum_j \mathbb{C}\mathrm{l}_{(j)} \otimes \hom'(E),$$

twisting Chern character $\mathrm{Ch}'(E)$.

- Dirac operators, $\mathbb{Z}$-grading, ellipticity

$$\mathrm{ind}(\eth^+) = \dim \mathrm{null}(\eth^+) - \dim \mathrm{null}(\eth^-). \tag{20}$$

- Heat kernel, McKean-Singer formula

$$\mathrm{ind}(\eth^+) = \mathrm{Str}(e^{-t\eth^2}) \ \forall \ t > 0. \tag{21}$$

- Local index theorem, Getzler's rescaling

$$\begin{gathered} k(t,x) = e^{-t\eth^2}(t,x,x) \in \mathcal{C}^\infty([0,\infty); \hom(E)), \\ k = \sum_{j=0}^{2d} t^{-d+j} k_j(t,x), \ \ k_j \in t^{-d}\mathcal{C}^\infty([0,\infty); \hom_{2j}(E)), \\ \sum_j [k_j] = \frac{1}{(4\pi)^{2d}} \det{}^{\frac12}\left(\frac{R/2}{\sinh(R/2)}\right) \mathrm{Ch}'(E) \in \Lambda^*. \end{gathered} \tag{22}$$

- Atiyah-Singer

$$\mathrm{ind}(\eth^+) = \int_M \widehat{A}\,\mathrm{Ch}'(E).$$

12. 4 April, 2006

12.1. **Fangyun Yang:- Eta invariant cont.** P an elliptic self-adjoint invertible operator.

$$\eta(s,P) = \sum_{\lambda\neq 0} \mathrm{sgn}(\lambda)|\lambda|^{-s} = \mathrm{Tr}(P(P^2)^{-\frac{s+1}{2}}). \tag{1}$$

For $\mathrm{Re}(s) >> 0$ this is holomorphic

(2)
$$\int_0^\infty t^{\frac{s-1}{2}} \lambda e^{-\lambda^2 t} dt = \Gamma(\frac{s+1}{2})\,\mathrm{sgn}(\lambda)|\lambda|^{-s}, \eta(s,P) = \frac{1}{\Gamma(\frac{s+1}{2})}^{-1} \int_0^\infty t^{\frac{s-1}{2}}\,\mathrm{Tr}(Pe^{-tP^2})dt$$

Then

$$\mathrm{Tr}(Pe^{-tP^2}) \sim \sum_{n=0}^\infty \tag{3}$$

Splitting the integral into $t > 1$ and $t \leq 1$ give an entire and a meromorphic splitting and the expansion of the heat kernel

$$\eta(s, P) = \frac{1}{\Gamma(\frac{s+1}{2})}^{-1} \sum_{n \leq n_0} \frac{2d}{ds + n - m} a_n + \text{regular}. \tag{4}$$

In principle $\eta(s, P)$ may have a simple pole at $s = 0$.

For a smooth family P_a of such invetible self-adjoint operators

$$\frac{d}{da}\eta(s, P_a) = -s \operatorname{Tr}(\dot{P}(P^2)^{-\frac{s+1}{2}} \tag{5}$$

by differentiating the definition. It follows that the residue at $s = 0$ is homotopy invariant. Define $R(P) = d \times$residue. This satisfies

$$\begin{aligned} R(P(P^2)^\sigma, s) &= \eta(P, (2\sigma + 1)s), \\ R(P \oplus Q) = R(P) + R(Q),\ \operatorname{ord}(P) &= \operatorname{ord}(Q), \\ R(P_t) &= R(P_0), \\ R(P) = 0,\ P &> 0. \end{aligned} \tag{6}$$

The principal symbol $P_d : S^*M \longrightarrow \operatorname{End}(V)$. Splits into positive an negative parts $V = V_+ \oplus V_-$ and $V_+ xs$ defines an element of $K(S^*M)$. This defines a homomorphism

$$r : K(S^*M) \longrightarrow \mathbb{R},\ r(V_+) = R(P). \tag{7}$$

This satisfies $r(\pi^* W) = 0$ and hence on trivial bundles. Complementing V stabilizes with a bundle of the same rank (after stabilizing) gives a symbol which generates V as its negative part.

By looking at generators we can see that in fact $r = 0$. If M is odd dimensional consider the signature operator $(d + \delta)_+$ on $N = M \times (0, 1)$. The twist by bundles on M, generates $K(S^*M)$ over $\mathbb{Q}$.

13. 6 April, 2006

13.1. **Maksim Lipyanskiy – Index theorem again.** Consider a principal bundle with compact Lie group G as structure group

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & M. \end{array} \tag{1}$$

Take a connection ω on G Given and Ad-invariant metric on G and a metric on M gives a metric on P. If X is a vector field on M then $\tilde{X}$ is its horizontal lift to P and the curvature is

$$\Omega(X, Y) = [\tilde{X}, \tilde{Y}] - \widetilde{[X, Y]}. \tag{2}$$

Then

$$\begin{aligned} \langle \nabla_{\tilde{X}} a, \tilde{Y} \rangle = \langle \nabla_a \tilde{X}, \tilde{Y} \rangle &= \frac{1}{2} \langle \Omega(X, Y), a \rangle \\ \nabla_a b &= \frac{[}{a}, b] \\ \langle \nabla_{\tilde{Y}} \tilde{X}, a \rangle &= -\frac{1}{2} \langle a, \Omega(X, Y) \rangle. \end{aligned} \tag{3}$$

Proof:- use the definition of the Levi-Civita connection on M. For a in the Lie algebra $\Omega \cdot a$ is an endomorphism of TM.

The idea. Take an action of G on V. The connection Laplacian on V is

$$\varDelta^V = \sum_i \nabla^2_{e_i} \tag{4}$$

and the Laplacian on P for the metric is

$$\varDelta^P = -\sum_i \nabla^2_{\tilde{e}_i} - \sum_i \nabla^2_{a_i}. \tag{5}$$

Proposition 3. *$\varDelta^P - \varDelta^V = \mathrm{Cas}$ on $f : P \longrightarrow V$ is the Casimir of the representation.*

Proof. Computing the action of the connection of f. □

Note that the Casimir commutes with $\varDelta^P$.

Proposition 4. *The lift k_t of the heat kernel of the V-Laplacian satisfies*

$$k_t(p_1g_1, p_2g_2) = \rho(g_1^{-1})k_t(p_1, p_2)\rho(g_2) \tag{6}$$

and

$$k_t(p_1, p_2) = e^{-t\,\mathrm{Cas}} \int_G h_t(p_1, p_2g)\rho(g^{-1})dg. \tag{7}$$

Now pass to the spin case, $G = \mathrm{Spin}$, $V = S^+ \oplus S^-$. Then

$$\mathrm{Str}(\rho(e^a)) = (-2i)^{\frac{n}{2}}\,\mathrm{Pf}(a) j_V^{\frac{1}{2}}(a),\ j_V(a) = \det(\frac{\sinh(\frac{a}{2})}{\frac{a}{2}}) \tag{8}$$

and $\mathrm{Pf}(a)$, the Pfaffian of a is the top order term in the expansion of $\frac{\exp(a)}{n!}$.

13.2. **Zuoqin Wang:- Zeta function.** ★ Let (M, g) be n-dimensional compact Riemannian manifold without boundary. It is well known that the spectrum of the (positive) Laplacian Δ is real and discrete which accumulate only at $+\infty$. Thus we can enumerate them as

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots \to +\infty.$$

The well-known Weyl asymptotic law asserts that the k^{th} eigenvalue of Δ has an asymptotic estimate

$$\lambda_k \sim 4\pi \left(\frac{\mathrm{vol}(M)}{\Gamma(n/2+1)}\right)^{2/n} k^{2/n}. \tag{9}$$

Thus the zeta function

$$\zeta_\Delta(s) = \mathrm{Tr}(\Delta^{-s}) = \sum_k \lambda_k^{-s} \tag{10}$$

is well defined for $\mathrm{Re} s > \frac{n}{2}$.

The first important result for these spectral zeta functions is that, like the original Riemannian zeta function, they can be extend to a meromorphic function on the whole complex plane.

Theorem 8. *$\zeta_\Delta(s)$ is holomorphic for Re $s > \frac{n}{2}$ and can be extend to a meromorphic function with at worst isolated simple poles at $n/2 - \mathbb{N}_0$, and is holomorphic at $s = 0, -1, -2, \cdots$.*

Proof: By definition of the Riemannian gamma function,

$$\lambda_n^{-s}\Gamma(s) = \int_0^\infty e^{-t\lambda_n} t^{s-1}\, dt,$$

thus

$$\Gamma(s)\zeta_\Delta(s) = \int_0^\infty t^{s-1}\mathrm{tr}(e^{-t\Delta})\, dt = \left(\int_0^1 + \int_1^\infty\right) t^{s-1}\mathrm{tr}(e^{-t\Delta})\, dt \tag{11}$$

In view of (1), the $\int_1^\infty$ part converges to a holomorphic function. On the other hand, by heat kernel expansion

$$\mathrm{tr}(e^{-t\Delta}) = \int_M k(t,x,x)dx \sim (4\pi t)^{-n/2}(a_0 + a_1 t + a_2 t^2 + \cdots), \tag{12}$$

we get

$$\begin{aligned} \int_0^1 t^{s-1}\mathrm{tr}(e^{-t\Delta})\, ds &= (4\pi)^{-n/2}\int_0^1 \sum_{k=0}^\infty a_k t^{s-1+k-n/2}\, ds \\ &= (4\pi)^{-n/2}\sum_{k=0}^\infty \frac{a_k}{s+k-n/2}. \end{aligned} \tag{13}$$

Now the result comes from the fact that $\Gamma(x)$ has isolated simple poles at the points $s = 0, -1, -2, \cdots$. **Q.E.D.**

Corollary 6. (1) $\zeta_\Delta(0) = (4\pi)^{-n/2}a_{n/2}$. In particular, $\zeta_\Delta(0) = 0$ if n is odd.
(2) Suppose D is dirac operator, then $\mathrm{index}(D) = \zeta_{D^*D}(0) - \zeta_{DD^*}(0)$.

In general, suppose P is a self-adjoint positive elliptic operator of degree d on M, then all the above properties preserved and the zeta function

$$\zeta_P(s) = \mathrm{Tr}(P^{-s}) = \sum_k \lambda_k^{-s} \tag{14}$$

is well defined for $\mathrm{Re}s > \frac{n}{d}$. As in the case above, $\zeta_P(s)$ can be extended to be a meromorphic function on $\mathbb{C}$.

Even more general, suppose P is a self-adjoint positive elliptic operator of order $d > 0$, and Q is a differential operator of order a, then we have the "heat trace expansion" (cf.[1])

$$\mathrm{Tr}(Qe^{-tP}) \sim \sum_{k=0}^\infty a_k(P,Q)t^{(k-n-a)/d}. \tag{15}$$

Thus we have

Theorem 9. *The "generalized zeta function"*

$$\zeta_P(s,Q) := Tr(QP^{-s}) = \sum_{\lambda>0} Tr(\pi_P(\lambda)Q)\lambda^{-s} \tag{16}$$

($\pi_P(\lambda)$ is the projection onto the λ-eigenspace of P.)
has a meromorphic extension to $\mathbb{C}$ with isolated simple poles at $s = (n+a-k)/d$ for $k = 0, 1, 2, \cdots$.

Some Remarks:
1. Given elliptic differential operator P of order $d > 0$, there are three operator families: **Resolvent** $(P-\lambda)^{-1}$, **Heat operator** e^{-tP} and **Power operator** P^{-s} (defined as 0 on $\ker(P)$). They can be obtained from one another, Say

$$P^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tP}\, dt = \frac{1}{2\pi i} \int_{\mathcal{C}} \lambda^{-s} (\lambda - P)^{-1}\, d\lambda, \tag{17}$$

So each one of the spectrum, the trace of heat kernel, and the zeta function determines the other two. Since zeta function is meromorphic, the value of the zeta function on any open set in $\mathbb{C}$ will determines the whole spectrum of P. In other words, there is no possible to compute the zeta function on any open set explicitly in general!
2. We can release the positivity condition to the semi-bounded condition $\langle Pf, f\rangle \geq C\langle f, f\rangle$. In this case the zeta function is defined as the summation over nonnegative eigenvalues. Then the negative part is finite sum, thus will not affect the meromorphic continuation. One of the difference is that $\zeta_P(0)$ will decrease by $-\dim \operatorname{Ker} P$ since $\Gamma(s)\zeta_P(s) = \int_0^\infty t^{s-1} \operatorname{tr}(e^{-tP} - \Pi_0)\, dt$ in this case. One of the amazing corollary is that the nonzero spectrum of P determinants the multiplicity of the zero spectrum in odd dimension!
3. We can even release P, Q to be pseudo-differential operators instead of differential operators. In this case we have the heat trace expansion (See Seeley, Grubb and Seeley)

$$\operatorname{Tr}(Qe^{-tP}) \sim \sum_{k\geq 0} a_k t^{(k-n-a)/d} + \sum_{k\geq 0} (b_k \log t + c_k) t^k$$

and thus the (meromorphic extension of) zeta function has pole structure

$$\Gamma(s)\zeta_P(s, Q) \sim \sum_{k\geq 0} \frac{a_k}{s + (k-n-a)/d} + \sum_{k\geq 0} \left(\frac{b_k}{(s+k)^2} + \frac{c_k}{s+k} \right).$$

So in this general case, 0 is not a regular point.
4. The eta function of a self-adjoint elliptic pseudo-differential operator is just the generalized zeta function $\eta_P(s) = \zeta_P(s, P|P|^{-1})$.

★ Given the meromorphic extension, one of the natural problems is to compute the residues at these poles. Of course the residue can be computed via the heat coefficients a_k's, but this is almost of no use since we don't have an efficient way to compute a_k. Now we will relate the zeta function residues to the *noncommutative residues* developed by Wodzicki and Guillemin.

Lemma 14. *Suppose P is positive, T is a pseudo-differential operator, then for $z \in \mathbb{C}$,*

$$[P^{-z}, T] = \binom{-z}{1} T^{(1)} P^{-z-1} + \binom{-z}{2} T^{(2)} P^{-z-2} + \cdots + \binom{-z}{k} T^{(k)} P^{-z-k} + \cdots \tag{18}$$

where $T^0 = T$ and $T^k = [P, T^{k-1}]$.

Proof: We shall use Cauchy's formula

$$\binom{-z}{p} P^{-z-p} = \frac{1}{2\pi i} \int \lambda^z (\lambda - P)^{-p-1}\, d\lambda. \tag{19}$$

Use the resolvent expression of the power operator, we compute

$$
\begin{aligned}
&[P^{-z}, T] = \frac{1}{2\pi i}\int \lambda^{-z}[(\lambda - P)^{-1}, T]\ d\lambda \\
&= -\frac{1}{2\pi i}\int \lambda^{-z}(\lambda - P)^{-1}[\lambda - P, T](\lambda - P)^{-1}\ d\lambda \\
&= \frac{1}{2\pi i}\int \lambda^{-z}(\lambda - P)^{-1}T^{(1)}(\lambda - P)^{-1}\ d\lambda \\
&= T^{(1)}\frac{1}{2\pi i}\int \lambda^{-z}(\lambda - P)^{-2}\ d\lambda + \frac{1}{2\pi i}\int \lambda^{-z}[(\lambda - P)^{-1}, T^{(1)}](\lambda - P)^{-1}\ d\lambda \\
&= \binom{-z}{1}T^{(1)}P^{-z-1} + \frac{1}{2\pi i}\int \lambda^{-z}(\lambda - P)^{-1}T^{(2)}(\lambda - P)^{-2}\ d\lambda \\
&= \cdots \\
&= \binom{-z}{1}T^{(1)}P^{-z-1} + \binom{-z}{2}T^{(2)}P^{-z-2} + \cdots + \binom{-z}{k}T^{(k)}P^{-z-k} + \cdots \quad \blacksquare
\end{aligned}
$$

Recall that a *trace functional* on a associated algebra is a linear functional that vanishes on commutators.

Theorem 10. *The residue functional*

$$\tau(T) = Res_{z=0}\, Tr(TP^{-z}) \tag{20}$$

is a trace on the algebra of (classical) pseudo-differential operators.

Proof: We only need to prove

$$\mathrm{Res}_{z=0}\mathrm{Tr}(TSP^{-z}) = \mathrm{Res}_{z=0}\mathrm{Tr}(STP^{-z}).$$

This amounts to show that

$$\mathrm{Res}_{z=0}\mathrm{Tr}(SP^{-z}T) = \mathrm{Res}_{z=0}\mathrm{Tr}(STP^{-z}).$$

Use the previous lemma, we get

$$SP^{-z}T - STP^{-z} = \sum_{j=1}^{\infty}\binom{-z}{j}ST^{(j)}P^{-z-j},$$

so

$$\mathrm{Res}_{z=0}\mathrm{Tr}(SP^{-z}T - STP^{-z}) = \sum_{j=1}^{\infty}\mathrm{Res}_{z=0}\left(\binom{-z}{j}\mathrm{Tr}\left(ST^{(j)}P^{-z-j}\right)\right).$$

Note that order of $T^{(j)}P^{-z-j}$ decreased by 1 as j increased by 1, the sum is finite sum. Moreover, since each trace function has at worst simple pole, which is canceled out by the factor z in $\binom{-z}{j}$, all the residues in the sum is 0. This proves the theorem.$\blacksquare$

Now we can apply the well known fact that up to a multiplication constant there is only one trace functional on the algebra of (classical) pseudo-differential operators, given by

$$\mathrm{res}(A) = \frac{1}{(2\pi)^n}\int_{S^*M} a_{-n} d\sigma,$$

where a_{-n} is the degree $-n$ term in the expansion of symbol of P:

$$\sigma(A)(x,\xi) \sim a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

where $a_{m-j}(x, t\xi) = t^{m-j}a(x,\xi)$ for $|\xi| \geq 1, t \geq 1$.

It was proved by Wodzocki that the multiplication constant is the order d of P:

$$\mathrm{res}|_{s=\frac{n-k}{d}}\zeta_P(s) = \frac{1}{d}\mathrm{res}(P^{-\frac{n-k}{d}}), \quad 1 \le k \le n$$

As a corollary, we get an expression of heat coefficient via noncommutative residue (or verse vise):

$$a_k(P) = \frac{1}{d}\,\Gamma\left(\frac{n-k}{d}\right)\mathrm{res}\left(P^{-\frac{n-k}{d}}\right), \quad 1 \le k \le n.$$

★ Via these zeta functions, Ray and Singer define a *zeta function determinant* as

$$\det{}_\zeta(P) := \begin{cases} e^{-\zeta_P'(0)} & , \quad P \text{ is invertible} \\ 0 & , \quad \text{else} \end{cases}. \tag{21}$$

Why we call this "determinant"? Well, formally,

$$\zeta_P(s) = \sum_k \lambda_k^{-s} \Longrightarrow \zeta_P'(s) = -\sum_k \lambda_k^{-s}\log\lambda_k \Longrightarrow e^{-\zeta_P'(0)} = e^{\sum_k \log\lambda_k} = \prod_k \lambda_k.$$

Note that the zeta function determinant only depend on the eigenvalues of P, and not depend on the underline Riemannian metric. The following property is immediate from the definition:

Proposition 4. 1) If P has real coefficients, $\det_\zeta(P)$ is nonnegative real number.
2) $\det_\zeta(\rho P) = \rho^{\zeta_P(0)}\det_\zeta(P)$ for $\rho > 0$.
3) $\det_\zeta(P^s) = \det_\zeta(P)^s$ for $s > 0$.
4) $\det_\zeta(A \oplus B) = \det_\zeta(A) + \det_\zeta(B)$.
5) Suppose A is invertible, then $\det_\zeta(A^{-1}PA) = \det_\zeta(P)$.

Proof: 1) This comes from the fact $\overline{\zeta_P(\bar{s})} = \zeta_P(s) \Longrightarrow \zeta_P'(0) \in \mathbb{R}$.
2) For $\rho > 0$,

$$\zeta_{\rho P}'(0) = (\rho^s\zeta_P(s))'|_{s=0} = \zeta_P(0)\log(\rho) + \zeta_P'(0),$$

thus

$$\det{}_\zeta(\rho P) = \rho^{\zeta_P(0)}\det{}_\zeta(P). \tag{22}$$

(This shows that the dimension of the infinitely dimension space has been "renormalized" to $\zeta_P(0)$.)
3) Since $\zeta_{P^s}(t) = \zeta_P(st) \Longrightarrow \zeta_{P^s}'(0) = s\zeta_P'(0)$.
4) This is obvious.
5) The eigenvalues of P is the same as eigenvalues of $A^{-1}PA$. ■

But, the zeta function determinant is not a determinant, i.e. in general

$$\det{}_\zeta(AB) \ne \det{}_\zeta(A)\det{}_\zeta(B).$$

This comes from the fact that $\zeta_P(s) = \mathrm{Tr}(P^{-s})$ is not a trace:

$$\zeta_{A+B}(s) \ne \zeta_A(s) + \zeta_B(s).$$

One of the applications of zeta function determinant is that it gives the Quillen's metric for determinant line bundle:

Theorem 11. *The function $\det_\zeta(D^*D)$ is smooth as a function of D.*

Proof: Suppose D' is not invertible. The eigenvalues $\lambda_n(D^*D)$ are smooth as a function of D, and each D^*D has discrete spectrum, thus we can find $\mu > 0$ and neighborhood U of D' such that μ is not an eigenvalue for any $D \in U$. Now $\zeta_{D^*D}(s) = \zeta_{<\mu}(s) + \zeta_{>\mu}(s)$, thus

$$e^{-\zeta_{D^*D}(0)} = e^{-\zeta'_{<\mu}(0)} e^{-\zeta'_{>\mu}(0)} = \det(D^*D|_{F_\mu}) e^{-\zeta'_{>\mu}(0)}$$

where F_μ is the span of eigenfunctions with eigenvalue less than μ. The first term tends to 0 as D tends to D', and the second term is bounded for U sufficient small. Thus proves the theorem. ■

References:

[1] P.Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, second edition.
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14. 11 April, 2006

14.1. **Yakov Shapiro:- Determinant bundle.** Families index setting

(1)
$$\begin{array}{ccc} & & E \\ & & \downarrow \\ Z & \text{——} & M \\ & & \downarrow \\ & & B \end{array}$$

a fibration with a vector bundle over the today space. Given a family of Dirac operators on E, can one find a family of finite rank smoothing operators, smooth on B so that $\eth_x + A_x$ is invertible. For each fixed x, $\text{null}(\eth_+)$ and $\text{null}(\eth_-)$ are finite dimensional; the vanishing of the numerical index is equivalent to the existence of such a perturbation for each point.

Suppose $\mathbb{H} = (H^+, H^-)$ is a superbundle over B, with $\dim(H^+) = \dim(H^-)$. Suppose $D_+ : H^+ \longrightarrow H^-$ is a homomorphism then set

(2) $$\det(\mathbb{H}) = \Lambda^n H^- \otimes (\Lambda^N H^+) * .$$

Then $\det(D_+) : \Lambda^n H^= \longrightarrow \Lambda^n H^-$ and hence $\det(D_+) \in \det(\mathbb{H})$ is well defined.

For the Dirac case the null spaces may jump in dimension, so one cannot proceed so simply. Set

(3) $$U_\lambda = \{x \in B; \lambda \text{ no an eigenvalue of } \eth^-\eth^+\},\ \lambda > 0.$$

Let $H^\pm_{[0,\lambda)}$ be the sum of the eigenspaces of $\eth^\mp\eth^\pm$ for eigenvalues in $[0, \lambda)$. This is a bundle over U_λ, with possibly different dimensions over the components.

Suppose $\mu > \lambda > 0$. Then

(4) $$H^\pm_{0,\mu)} = H^\pm_{[0,\lambda)} \oplus H^\pm_{(\lambda,\mu)} \text{ over } U_\lambda \cap U_\mu.$$

In the finite-dimensional case if $\mathbb{H}_1$ and $\mathbb{H}_2$ are two superbundles then

(5) $$\det(\mathbb{H}_1 \oplus \mathbb{H}_2) = \det(\mathbb{H}_1) \otimes \det(\mathbb{H}_2).$$

So

$$\det(\mathbb{H}_{[0,\mu)} = \det(\mathbb{H}_{[0,\mu)} \otimes \det(\mathbb{H}_{(\lambda,\mu)} \tag{6}$$

so we need to construct a trivialization of $\det(\mathbb{H}_{(\lambda,\mu)}$; but in fact $\det(\eth_+)$ gives a section. The cocycle condition follows from the multiplicativity of the usual determinant. This defines the determinant bundle of the Dirac operator, $\mathrm{Det}(\eth)$.

There is also a metric on $\mathrm{Det}(\eth)$. The $H^{\pm}_{[0,\lambda)}$ have metrics induced by the L^2 norm, and hence on $\det(\mathbb{H}^{\pm})$ over each U_λ. We need to introduce another factor into the metric to remove the dependence on λ. If $\lambda < \mu$ and λ_i, $i = 1, \dots, m$ are the eigenvectors of $\eth^-\eth^+$ between μ and λ then the identification of $\det(\mathbb{H}_{[0,\lambda)}$ and $\det(\mathbb{H}_{([0,\mu)}$ multiplies the metrics by $\sqrt{\lambda_1}\dots\sqrt{\lambda_m}$. We need to use the fact that

$$\zeta'(0, \eth^-\eth^+, \lambda) = \zeta'(0, \eth^-\eth^+, \mu) + \sum_i \log(\lambda_i). \tag{7}$$

So the Quillen norm

$$|v|_{H^{\pm}_{[0,\lambda)}} = \exp(-\frac{1}{2}\zeta'(0, \eth^-\eth^+, \lambda)|v|_{H^{\pm}_{[0,\lambda)}}. \tag{8}$$

14.2. **Zuoqin Wang:- Zeta function continued.**

15. 20 April, 2006

15.1. **Ricardo Andrade: Atiyah, Patodi and Singer.** Give a generalization of the signature theorem to the case $\partial X \neq \emptyset$. So take $\dim X = 4$. If $\partial X = \emptyset$ then

$$\operatorname{sign}(X) = \frac{1}{3}\int_X p_1,\ p_1 = -\operatorname{Ch}_2(TX \otimes \mathbb{C}). \tag{1}$$

In the case of a manifold with boundary with a product-type metric near the boundary then

$$\operatorname{sign}(X) = \frac{1}{3}\int_X p_1 + f(\partial X). \tag{2}$$

Here $f(\partial X)$ is a spectral invariant, essentially the eta invariant, which changes sign under reversal of orientation. In fact

$$f(\partial X) = -\frac{1}{2}\eta_{d+\delta}(0).$$

Consider an 'elliptic boundary value problem' D on X, where near the boundary

$$D = \sigma(\frac{\partial}{\partial u} + A) \tag{3}$$

where u is an inward normal coordinate and A is an elliptic selfadjoint differential operator. The boundary condition on $Y = \partial X$ is

$$\phi\big|_{\partial X} \in \{\text{Span of negative eigenspaces}\}. \tag{4}$$

So the general setting is an elliptic first-order differential operator on a compact manifold with boundary

$$D : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F) \tag{5}$$

for bundles E and F and takes the form (3) near the boundary. So the domain is

$$\left\{u \in \mathcal{C}^\infty(X; E); P_-(f\big|_Y) = 0\right\} \tag{6}$$

and as an operator from this domain to $\mathcal{C}^\infty(X;F)$ it is Fredholm and the index is

(7)

where α_0 is the contant term in the asymptotic expansion of the heat kernel on the double, restricted to X, $h=\dim\operatorname{null}(A)$ and η is the eta invariant of A.

In particular in our standard setting of Dirac operators, with the metric, Clifford action and connection product near the boundary

$$\operatorname{ind}(\eth,P_-)=\int_X \widehat{A}\operatorname{Ch}'(E)-\frac{h+\eta}{2}. \tag{8}$$

Specializing to the signature operator, take $d+\delta$ acting on Λ^* with grading induced by $\tau=i^{p(p-1)+2p}\star$ on λ^p. Then

$$\operatorname{sgn}(X)=\operatorname{ind}(d+\delta)=\int_X L(p)-\frac{1}{2}\eta \tag{9}$$

and eta invariant can be simplified. The Hirzebruch polynomials are determined by homogeneity, multiplicativity and normalization.

Remarks on proof.

16. 25 April, 2006

16.1. **Fangyun Yang:- Families index.** Setup. $\pi:M\longrightarrow B$ is Riemannian fibration, a fibration with typical fibre X (even dimensional) and $\mathcal{E}\longrightarrow M$ is a Hermitian bundle over M with a fibrewise (smooth) Clifford module structure for the fibrewise Riemann metrics, cl :* $(M/B)\longrightarrow\hom(\mathcal{E})$ compatible with Lev-Civita. Choose a connection on the fibration

$$TM=T(M/B)\oplus T_MB,\ P:TM\longrightarrow T(M/B). \tag{1}$$

Choose a metric on B and combine with the connection to get a metric on M

$$g_M=g_{M/B}+\pi^*g_B. \tag{2}$$

The connection on the vertical tangent bundle

$$\nabla^{M/B}=P\nabla^gP \tag{3}$$

is independent of the choice of metric on B and only depends on the fibre metrics and the connection on the fibration.

So, the objective is to compute the Chern character of the index bundle of the family of Dirac operators on the fibres. Let $\pi_*\mathcal{E}$ be the infinite dimensional bundle over B with fibre over z $\mathcal{C}^\infty(M_z;\mathcal{E})$. By definition a smooth section is just a smooth section over M. Set

$$\mathcal{A}(B;\pi_*\mathcal{E})=\Gamma(M;\pi^*(\Lambda^*T^*B)\otimes\mathcal{E})=\Gamma(M;\mathbb{E}). \tag{4}$$

A superconnection on this infinite dimensional $\mathbb{A}$ is a differential operator acting on $\mathcal{A}(B,\pi_*\mathcal{E})$ which is odd with respect to the $\mathbb{Z}_2$ graing and satisfies

$$\mathbb{A}(\nu\cdot s)=d_B\nu\wedge s+(-1)^{|\nu|}\nu\mathbb{A}s \tag{5}$$

If $\mathbb{A}_0=\eth$ then $\mathbb{A}$ is sait to be associated to $\eth$.

Bismut gives a particular superconnection. Set

$$\nabla^{\oplus}=\nabla^{M/B}+\pi^*\nabla^B \text{ on } TM. \tag{6}$$

Compare this to the Levi-Civita connection on M then

$$g(\nabla^g_XY,Z)=g(\nabla^{\oplus}_XY,Z)+w(X)(Y,Z) \tag{7}$$

where $w \in \mathcal{A}^1(M; \Lambda^2 T^* M)$ is given by
(8)

$$w(X)(Y,Z) = S(X,Z)(Y) - S(X,Y)(Z) + \frac{1}{2}(\Omega(X,Y),Z) - \frac{1}{2}(\Omega(X,Z),Y) + \frac{1}{2}(\Omega(Z,Y),X)$$

where $\Omega(X,Y) = -P([X,Y])$ and

$$2(S(X,Y),Z) = Z(X,Y) - (P[Z,X],Y) - (P[(Z,Y],X). \tag{9}$$

Rescale the metric and Clifford structure corresponding to

$$g_u = g_{M/B} + u^{-1} g_B \text{ on } TM. \tag{10}$$

Then for the rescaled objects

$$\nabla^{g_u} = \nabla^{\oplus} + \frac{1}{2} T_u(w). \tag{11}$$

On $\mathbb{E}$ over M consider

$$\nabla^{\mathbb{E},\oplus} = \pi^* \nabla^B \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\mathcal{E}},\ \nabla^{\mathbb{E},u} = \nabla^{\mathbb{E},\oplus} + \frac{1}{2} m_u(\omega) \tag{12}$$

where

$$m_u : C_u(M) \longrightarrow \mathrm{End}(\mathbb{E}),\ m_u(\alpha) = \alpha \wedge -u\iota_\alpha \tag{13}$$

on the first factor and the usual action on the second factor. This has a limit as $u \to 0$ with m_0 given by wedge product on the first factor.

16.2. **Zuoqin Wang:- Analytic torsion(his notes).** Suppose (M, g) is a n-dimensional compact oriented Riemannian manifold without boundary, (E, ∇) is a flat vector bundle over M. Let $\Omega^*(M; E) = \Omega^*(M) \otimes E$ denote the space of E-valued C^∞ differential forms on M. We have the usual exterior derivative $d : \Omega^*(M; E) \to \Omega^{*+1}(M; E)$ which gives us the twisted deRham complex

$$0 \to \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0. \tag{14}$$

From the Riemannian structure, we can define a $*$-operator $* : \Omega^p \to \Omega^{n-p}$, which provides Ω^p an inner product

$$\langle \omega_1, \omega_2 \rangle := \int_M \omega_1 \wedge *\omega_2. \tag{15}$$

Relative to this inner product, d has the formal adjoint $\delta = (-1)^{np+n+1} * d *$. Now the Laplacian on p-forms is defined to be

$$\Delta_p = d\delta + \delta d : \Omega^p \to \Omega^p. \tag{16}$$

It is well known that Δ_p is self-adjoint nonnegative operator, which has a pure point spectrum. We assume that Δ_p are positive for all p. Note that by Hodge theory, $\mathrm{Ker}(\Delta_p) \cong H^p(M; E)$. So this amounts to say the the twisted deRham complex (1) is acyclic. As in the scalar Laplacian case, we can define the zeta function

$$\zeta_p(s) = \mathrm{Tr}(\Delta_p^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}(e^{-t\Delta_p})\, dt \tag{17}$$

which can be extended to a meromorphic function on the whole plane $\mathbb{C}$ and which is holomorphic at the origin 0. The zeta regularized determinant is

$$\det(\Delta_p) := e^{\zeta_p'(0)} \tag{18}$$

Now we can give the marvelous definition of Analytic Torsion due to Ray and Singer:

Definition 6. Suppose the twisted deRham complex (1) is acyclic. The analytic torsion T is defined to be

$$\log T = -\frac{1}{2}\sum_p (-1)^p p \log\det(\Delta_p). \tag{19}$$

Remark: This definition is an analytic analogue of the Reidemeister-Franz torsion (R-torsion). The latter one is a secondary topological invariant, i.e. only defined when the cohomology groups vanish (thus defined at the cochain level). The R-torsion is originally used to distinguish lens spaces which has the same cohomology groups and homotopy groups but non-homeomorphic. It was proved by Ray and Singer that the Analytic torsion possessed many of the key properties of R-torsion (See theorem 1 - theorem 3 below). Later, Cheeger, Muller, Bismut and Zhang, Braverman etc each proved independently (by using different methods) that the analytic torsion coincides with the R-torsion. We will discuss this in the future.

Theorem 12. *Suppose the dimension n is even, then the analytic torsion $T = 1$.*

Proof: We will show that if n is even, then

$$\sum_p (-1)^p p\, \zeta_p(s) = 0, \qquad \forall s, \tag{20}$$

which of course imply the result. In fact by definition it is easy to check that $*\Delta_p = \Delta_{n-p}*$, but $*$-operator is an isomorphism, thus Δ_p and Δ_{n-p} are isospectral. This implies that $\zeta_p(s) = \zeta_{n-p}(s)$, so

$$\sum_p (-1)^p p\, \zeta_p(s) = \frac{n}{2}\sum_p (-1)^p \zeta_p(s).$$

On the other hand, $\Delta_{p+1} d = d\Delta_p$ implies that d maps the λ-eigenspace E^p_λ of Δ_p to the λ-eigenspace E^{p+1}_λ of Δ_{p+1}, thus we get a complex

$$0 \to E^0_\lambda \xrightarrow{d} E^1_\lambda \xrightarrow{d} \cdots \xrightarrow{d} E^n_\lambda \to 0. \tag{21}$$

Moreover, this sequence is exact, since if $\omega \in E^p_\lambda$ is d-closed, then $\omega = \frac{1}{\lambda}\Delta_p \omega = d(\frac{1}{\lambda}\delta\omega)$, and $\frac{1}{\lambda}\delta\omega \in E^{p-1}_\lambda$. As a result, we get

$$\sum_p (-1)^p \dim E^p_\lambda = 0.$$

So the coefficient of λ^s in $\sum_p (-1)^p \zeta_p(s)$ is 0 for any eigenvalue λ, which implies $\sum_p (-1)^p \zeta_p(s) = 0$. **Q.E.D.**

Theorem 13. *The analytic torsion is independent of the metric g on M.*

Proof: In view of the last theorem, we can assume that $n = \dim M$ is odd.

Suppose g_0 and g_1 are two arbitrary Riemannian metrics on M, then $g_r = (1-r)g_0 + rg_1 (0 \le r \le 1)$ are all Riemannian metrics on M. Let $\Delta^{(r)}_p$ and T_r denote the Laplacian (on $p-$forms) and the Analytic Torsion corresponding to g_r. To prove T is independent of g, we only need to show $\frac{d}{dr}T_r = 0$. The function

$$f(r,s) = \frac{1}{2}\sum_{p=0}^{n} (-1)^p p \int_0^\infty t^{s-1} \mathrm{Tr}(e^{-t\Delta^{(r)}_p})\, dt, \tag{22}$$

is well defined for $\mathrm{Re}(s)$ big enough, and can be extended to a meromorphic function in the $s-$plane. Note that near $s=0$, we have

$$\Gamma(s)\zeta_p^{(r)}(s) = \Gamma(s)\left(\zeta_p^{(r)}(0) + s\zeta_p^{(r)\prime}(0) + O(s^2)\right) \xrightarrow{s\to 0} \zeta_p^{(r)\prime}(0),$$

where we used the fact $\zeta_p^{(r)}(0)=0$ and $\lim_{s\to 0} s\Gamma(s)=1$. So by rewriting $f(r,s)$ as

$$f(r,s) = \frac{1}{2}\sum_p (-1)^p p\Gamma(s)\zeta_p^{(r)}(s),$$

we can see that $\log T_r = -f(r,0)$. So we have to show that $\frac{d}{dr}f(r,0)=0$.

To compute the derivative, formally we have

$$\frac{d}{dr}\mathrm{Tr}(e^{-t\Delta_p^r}) = -t\mathrm{Tr}(\dot{\Delta}_p^{(r)}e^{-t\Delta_p^{(r)}}). \tag{23}$$

This can be proved via some computation on heat kernel, see [1] Prop. 6.1 or [2] Prop. 5.14. So we have

$$\frac{d}{dr}f(r,s) = -\frac{1}{2}\sum_{p=0}^{n}(-1)^p p\int_0^\infty t^s\mathrm{Tr}(\dot{\Delta}_p^{(r)}e^{-t\Delta_p^{(r)}})\,dt \tag{24}$$

for $\mathrm{Re}(s)$ big enough. We omit this proof of (10) and proceed to compute $\dot{\Delta}_p^{(r)}$.

In the definition of $\Delta_p^{(r)}$, only the $*$-operator depends on the metric. We write $\alpha = *^{-1}\dot{*} = -\dot{*}*^{-1}$, then

$$\begin{aligned}
\dot{\Delta}_p^{(r)} &= \frac{d}{dr}(d\delta+\delta d) = \frac{d}{dr}((-1)^{np+n+1}d*d*+(-1)^{n(p+1)+n+1}*d*d)\\
&= (-1)^{np+n+1}\frac{d}{dr}(d*d*-*d*d)\\
&= (-1)^{np+n+1}(d\dot{*}d*+d*d\dot{*}-\dot{*}d*d-*d\dot{*}d)\\
&= -d\alpha\delta + d\delta\alpha - \alpha\delta d + \delta\alpha d.
\end{aligned} \tag{25}$$

To compute $\mathrm{Tr}(\dot{\Delta}_p^{(r)}e^{-t\Delta_p^{(r)}})$, note that $e^{-t\Delta_p^{(r)}}$ is of trace class, and $\alpha\delta de^{-t\Delta_p^{(r)}}$ is bounded on L^2, thus

$$\mathrm{Tr}(\delta\alpha de^{-t\Delta_p^{(r)}}) = \mathrm{Tr}(e^{-\frac{t}{2}\Delta_p^{(r)}}\delta\alpha de^{-\frac{t}{2}\Delta_p^{(r)}}) = \mathrm{Tr}(\alpha de^{-t\Delta_p^{(r)}}\delta) = \mathrm{Tr}(\alpha d\delta e^{-t\Delta_{p+1}^{(r)}}),$$

and similarly

$$\mathrm{Tr}(d\alpha\delta e^{-t\Delta_p^{(r)}}) = \mathrm{Tr}(\alpha d\delta e^{-t\Delta_{p-1}^{(r)}}), \qquad \mathrm{Tr}(d\delta\alpha e^{-t\Delta_p^{(r)}}) = \mathrm{Tr}(\alpha d\delta e^{-t\Delta_p^{(r)}}).$$

As a result,

$$\mathrm{Tr}(\dot{\Delta}_p^{(r)}e^{-t\Delta_p^{(r)}}) = -\mathrm{Tr}(\alpha\delta de^{-t\Delta_p^{(r)}})+\mathrm{Tr}(\alpha d\delta e^{-t\Delta_{p+1}^{(r)}})-\mathrm{Tr}(\alpha\delta de^{-t\Delta_{p-1}^{(r)}})+\mathrm{Tr}(\alpha d\delta e^{-t\Delta_p^{(r)}}),$$

Note that $\mathrm{Tr}(\alpha\delta de^{-t\Delta_n^{(r)}})=0$, we get

$$\begin{aligned}
\sum_{p=0}^{n}(-1)^p p\mathrm{Tr}(\dot{\Delta}_p^{(r)}e^{-t\Delta_p^{(r)}}) &= \sum_{p=0}^{n}(-1)^p(\mathrm{Tr}(\alpha\delta de^{-t\Delta_p^{(r)}}) + \mathrm{Tr}(\alpha d\delta e^{-t\Delta_p^{(r)}}))\\
&= \sum_{p=0}^{n}(-1)^p\mathrm{Tr}(\alpha\Delta_p^{(r)}e^{-t\Delta_p^{(r)}})\\
&= -\frac{d}{dt}\sum_{p=0}^{n}(-1)^p\mathrm{Tr}(\alpha e^{-t\Delta_p^{(r)}}).
\end{aligned}$$

Plugging this into (11) and integrating by parts, we get

$$(26)\qquad \begin{aligned} \frac{d}{dr} f(r,s) &= \frac{1}{2}\sum_{p=0}^{n}(-1)^p \int_0^\infty t^s \frac{d}{dt}\mathrm{Tr}(\alpha e^{-t\Delta_p^{(r)}})\, dt \\ &= -\frac{1}{2}s\sum_{p=0}^{n}(-1)^p \int_0^\infty t^{s-1}\mathrm{Tr}(\alpha e^{-t\Delta_p^{(r)}})\, dt, \end{aligned}$$

where the boundary terms vanishes for $\mathrm{Re}(s)$ large because $\mathrm{Tr}(\alpha e^{-t\Delta_p^{(r)}})$ decreases exponentially for large t and is $O(t^{-N/2})$ for small t. Use the heat expansion

$$(27)\qquad \mathrm{Tr}(\alpha e^{-t\Delta_p^{(r)}}) \sim \left(\sum_{k=0}^{\infty}\int_M \mathrm{tr}(\alpha u_k^p(r))\right) t^{k-\frac{n}{2}}$$

we know that the function

$$s\int_0^\infty t^{s-1}\mathrm{Tr}(\alpha e^{-t\Delta_p^{(r)}})\, dt$$

has a meromorphic continuation to $\mathbb{C}$ whose value at 0 is $\int_M \mathrm{tr}(\alpha u_{n/2}^p(r))$, and thus vanishes for n odd. This proves the theorem. **Q.E.D.**

Theorem 14. *Suppose M' is oriented compact simply connected manifold without boundary. Then*

$$(28)\qquad \log T_{M\times M'} = \chi(M')\log T_M,$$

where $\chi(M')$ is the Euler characteristic o M'.

Sketch of Proof:
Step 1: By the last theorem, we can choose the product metric on $M\times M'$. Now the Laplacian on $M\times M'$ comes from the Laplacians on M and M', and $\Delta_r^{M\times M'}$ have eigenvalues $\lambda+\mu$ with multiplicities $N_p(\lambda,M)N_q(\mu,M')$, where λ, μ are eigenvalues of Δ_p^M and $\Delta_q^{M'}$ with multiplicities $N_p(\lambda,M)$ and $N_q(\mu,M')$ respectively, and $p+q=r$. As a result,

$$\zeta_r^{M\times M'}(s) = \sum_{\lambda,\mu}\sum_{p+q=r}(\lambda+\mu)^{-s}N_p(\lambda,M)N_q(\mu,M').$$

Step 2: Similar to the proof of Theorem 1, we can show

$$\sum_{p=0}^{n_1}(-1)^p N_p(\lambda,M) = 0$$

for all λ ($M\times M'$ is acyclic and $\pi_1(M')=0$ implies M is acyclic), and

$$\sum_{q=0}^{n_2}(-1)^q N_q(\mu,M') = 0$$

for nonzero eigenvalues μ of $\Delta_q^{M'}$.
Step 3: Now we have

$$\begin{aligned}\sum_{r=0}^{n_1 n_2}(-1)^r r\zeta_r^{M\times M'}(s) &= \sum_{\lambda,\mu}(\lambda+\mu)^{-s}\left(\sum_{p=0}^{n_1}(-1)^p pN_p(\lambda,M)\right)\left(\sum_{q=0}^{n_2}(-1)^q N_q(\mu,M')\right)\\ &\quad+\sum_{\lambda,\mu}(\lambda+\mu)^{-s}\left(\sum_{p=0}^{n_1}(-1)^p N_p(\lambda,M)\right)\left(\sum_{q=0}^{n_2}(-1)^q qN_q(\mu,M')\right)\\ &= \sum_{\lambda}\lambda^{-s}\left(\sum_{p=0}^{n_1}(-1)^p pN_p(\lambda,M)\right)\left(\sum_{q=0}^{n_2}(-1)^q N_q(0,M')\right)\\ &= \chi(M')\sum_{p=0}^{n_1}(-1)^p\, p\,\zeta_p^M(s).\end{aligned}$$

References:

[1] D. Ray and I. Singer, *R-Torsion and the Laplacian on Riemannian Manifolds*, Adv. in Math. 7, 145-210.
[2] S. Rosenberg, *The Laplacian on a Riemannian manifold*.

17. 27 April, 2006

17.1. **Zuoqin Wang:- Analytic torsion (continued).**

17.2. **Fangyun Yang:- Families index.** The rescaled superconnection

$$\nabla^{\mathbb{E},u} = \nabla^{\mathbb{E},0} + \frac{1}{2}m_u(w) \tag{1}$$

is a Clifford connection for the Clifford action m_u

$$[\nabla_X^{\mathbb{E},u}, m_u(\alpha)] = m_u(\nabla_X^{T^*M,u}\alpha$$

by considering separately horizontal and vertical vector fields. So we can define a 'Dirac operator' associated to it, which is the Bismut superconnection

$$\mathbb{A} : \mathcal{A}(B,\pi_*\mathcal{E}) \longrightarrow \mathcal{A}(B,\pi_*\mathcal{E}),\ \mathbb{A} = \sum_a m_0^a \nabla_a^{\mathbb{E},0}. \tag{2}$$

Then $\mathbb{A}$ is odd with respect to the grading and

$$\mathbb{A}(\nu S) = (d_B\nu)S + (-1)^{|\nu|}\nu\mathbb{A}S,\ \nu\in\mathcal{A}(B),\ S\in\Gamma(M,\mathbb{E}). \tag{3}$$

Splitting into horizontal and vertical parts

$$\mathbb{A} = \sum_i c^i\nabla_i^{\mathbb{E},0} + \sum_\alpha \epsilon^\alpha\nabla_\alpha^{\mathbb{E},0} + \frac{1}{4}\sum_{abc} w(e_a)(e_b,e_c)m_0^a m_0^b m_0^c = \eth + \mathbb{A}_{[1]} + \mathbb{A}_{[2]}. \tag{4}$$

Let $\varDelta^{M/B}\in\Gamma(B,\operatorname{End}(\pi_*\mathcal{E}))$ be the family of fibrewise connection Laplacians.

Theorem 15. *The curvature is*

$$\mathbb{A}^2 = \varDelta^{M/B} + \frac{1}{4}r_{M/B} + \sum_{a<b} m_0^a m_0^b F^{\mathcal{E}/S}(e_a,e_b),\ \operatorname{Ch}(\operatorname{ind}_\eth) = \operatorname{Str}(e^{-\mathbb{A}^2}). \tag{5}$$

Now $e^{-t\mathbb{A}^2}$ acts on $\Gamma(M,\mathbb{E})$ and has a kernel

$$K_t(x,y)\in\Gamma(M\times_\pi M;\pi^*\mathcal{A}\otimes\mathcal{E}\boxtimes_\pi\mathcal{E}^*). \tag{6}$$

Theorem 16. *On the diagonal*

$$K_t(x,x) \sim (4\pi t)^{-n/2}\sum_{i=0}^{\infty} t^i K_i(x),\ K_i \in \sum_{j\le 2i} \mathcal{A}^{2j}(M;\operatorname{End}(\mathcal{E})). \tag{7}$$

Rescaling gives a superconnection

$$\mathbb{A}_t = t^{\frac12}\delta_t^B\mathbb{A}(\delta_t^B)^{-1} = t^{\frac12}\mathbb{A}_{[0]} + \mathbb{A}_{[1]} + t^{-\frac12}\mathbb{A}_{[2]}. \tag{8}$$

Then the limit as $t \to 0$ *of* $\operatorname{Str}(e^{-\mathbb{A}_t^2})$ *exists and gives the explicit families index*

(9)

$$\operatorname{Ch}(\mathbb{A}_t) = \int_X \delta_t^B \operatorname{Str}(K_t(x,x))dx,\ \delta_t^B \operatorname{Str}(K_t(x,x)) \longrightarrow (2\pi i)^{-\frac{n}{2}}\int \widehat{A}(M/B)\operatorname{Ch}(\mathcal{E}/S) \text{ as } t \to 0.$$

18. 2 May, 2006

18.1. **Yakov Shapiro:- Connection on the determinant bundle.** Usual families setting of a fibration and the infinite dimensional bundle $\pi_*\mathcal{E}$ and Dirac operators on the fibre. Determinant bundle defined earlier from $\mathcal{H}^{\pm}_{[0,\lambda)}$ over

$$U_\lambda = \{z \in B; \lambda \text{ is not an eigenvalue of } \eth_z\}.$$

Quillen metric, obtained by modifying the L^2 metric with a factor

$$|\cdot|_Q = e^{-\zeta'(0,\eth^-\eth^+,\lambda)}|\cdot|_{L^2}.$$

To get a metric connection start from the local super connections induced by the superconnection on $\pi_*\mathcal{E}$. Define

$$\mathbb{A}_\lambda = P_{(\lambda,\infty)}\eth + \nabla^{\pi_*\mathcal{E}},\ \mathbb{A}_{\lambda,s} = s^{\frac12}P_{(\lambda,\infty)}\eth + \nabla^{\pi_*\mathcal{E}}.$$

Then set

$$\alpha^{\pm} = \operatorname{Tr}_{\pi_*\mathcal{E}}\left(\frac{\partial\mathbb{A}_{\lambda,s}}{\partial s}e^{-\mathbb{A}^2_{\lambda,s}}\right) \tag{1}$$

First claim is that the 1-form components are equal

$$\alpha^+_{[1]} = \alpha^-_{[1]}. \tag{2}$$

There are exapnsions as $s \to 0$

$$\alpha^{\pm} \sim \sum_{k=-n}^{\infty}\alpha_k^{\pm}s^{\frac{s}{2}}.$$

Consider

$$\int_t^\infty \alpha^{\pm}(\lambda,s) = \sum_{k-=n}^{\infty} w_k^{\pm}t^{\frac{k}{2}} + \tilde{\omega}^{\pm}\log t. \tag{3}$$

Set

$$\beta^{\pm} = \omega_0^{\pm} - \gamma\tilde{\omega}^{\pm},\ \beta^+ + \beta^- \in \mathbb{R},\ \beta^+ - \beta^- \in i\mathbb{R}. \tag{4}$$

Next claim is that

$$d\zeta'(0,\eth^-\eth^+,\lambda) = -(\beta^+_\lambda + \beta^-_\lambda). \tag{5}$$

The local connection over U_λ on the determinant bundle of $\mathcal{H}_{[0,\lambda)}$

$$\nabla + \beta^+_\lambda.$$

This patches to give a global connection which is compatible with the Quillen metric. Moreover its curvature can be computed and is the 2-form part, in cohomology, of the Chern character of the index bundle.

18.2. **Jonathan Campbell:- Suspension and η (his notes).** The following is an exposition of the first half of [1].

We want to be able to define an eta invariant for a larger class of operators. We have two definitions of the eta invariant already that work for self-adjoint differential operators and dirac operators and can be extended to other classes on certain manifolds. Namely, these definitions are

$$\eta(\eth) = \sum_i \operatorname{sgn}(\lambda_i)|\lambda_i|^{-s}\Big|_{s=0}$$

and

$$C\int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}(\eth e^{-t\eth^2})dt.$$

There is another way of defining the eta invariant that with some finessing will work for pseudo-differential operators of any order, provided these pseudodifferential operators come from a sufficiently nice space. Before I actually define what space they come from, I'll give the general form of what we want our eta invariant to look like, and explain why this makes sense in some way. The way we want our eta invariant to look is something like

$$\eta(A) = \frac{1}{i\pi}\int_{\mathbb{R}} \operatorname{Tr}\left(A^{-1}(\tau)\frac{dA}{d\tau}\right)$$

where A is some operator depending on a parameter. Unfortunately, in general this thing isn't well defined, there are some problems we'll see later.

To see why this definition makes sense, we'll look at it operating on some very very nice set of operators. Let A be finite rank, self-adjoint and invertible. Then the eigenvalues all exist. Consider the family of operators $\widetilde{A} = A + i\tau$ associated with A. Then by the definition of our eta invariant

$$\eta(\widetilde{A}) = \frac{1}{\pi i}\int_{\mathbb{R}} \operatorname{Tr}(A^{-1}(\tau)i)d\tau = \frac{1}{\pi}\int_{\mathbb{R}} \operatorname{Tr}(A^{-1}(\tau))d\tau.$$

Suppose the eigenvalues for A are $\lambda_1, \dots, \lambda_n$, then the eigenvalues for $\widetilde{A} = A + i\tau$ are $\lambda_1 + i\tau, \dots, \lambda_n + i\tau$, and we have

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{R}} \operatorname{Tr}(A^{-1}(\tau)) &= \frac{1}{\pi}\int_{\mathbb{R}}\sum_j(\lambda + i\tau)^{-1} = \frac{1}{\pi}\sum_j\int_{\mathbb{R}}(\lambda_j + i\tau)^{-1} \\ &= \lim_{T\to\infty}\frac{1}{\pi}\sum_j \log(\lambda_j + i\tau)\Big|_{-T}^{T} = \frac{1}{\pi}\lim_{T\to\infty}\sum_j \log\left(\frac{\lambda_j + iT}{\lambda_j - iT}\right)\end{aligned}$$

Now we'll just look at each term in the summation

$$\lim_{T\to\infty}\left(\frac{\lambda_j + iT}{\lambda_j - iT}\right) = \lim_{T}\log\left(\frac{\frac{\lambda_j}{T} + i}{\frac{\lambda_j}{T} - i}\right) = \lim_{T\to\infty}\log\left(\frac{\frac{\lambda_j^2}{T^2} + \frac{2\lambda_j i}{T} - 1}{|\lambda_j/T^2 - i|^2}\right).$$

Note the stuff inside the parenthesis is going to -1, *however*, when $\lambda_j > 0$, it approaches -1 from above (i.e. in quadrant II), and when $\lambda_j < 0$ it approaches +1 form below (quadrant III). Taking the logarithm as the principal branch we have that

$$\lim_{T\to\infty}\left(\frac{\lambda_j + iT}{\lambda_j - iT}\right) = \begin{cases}\pi & \lambda_j > 0\\ -\pi & \lambda_j < 0\end{cases}.$$

Thus

$$\eta(A) = \frac{1}{\pi}\sum_j \pi\,\mathrm{sgn}(\lambda_j) = \mathrm{signature}(A).$$

So at the very least, for a very very nice case, this version of eta invariant gives us something nice. Now we'll extend this invariant to some larger class of operators.

By virtue of our definition, we'll want the operators to depend on a paramter, but in a nice way. Let Y be a compact manifold, we'll start off looking at $\Psi^*(Y\times\mathbb{R})$, the space of classical pseudodifferential operators (i.e. they'll just have a formal development).

We want to turn this into an algebra (which it isn't now), and we want it to behave nicely. Note that for the kernels are maps $A : C_c^\infty(Y\times\mathbb{R}) \to C^\infty(Y\times\mathbb{R})$. In order for it to behave nicely, we just declare that the kernels will behave nicely, off of a compact set:

$$A \in C_c^{-\infty}(Y^2\times\mathbb{R};\Omega Y) + \mathcal{S}(Y^2\times\mathbb{R};\Omega Y).$$

In order for the kernels to be defined on $Y^2\times\mathbb{R}$ we'll require that the operators be translation invarint, i.e. if $T_\tau : Y\times\mathbb{R}\to Y\times\mathbb{R}$ is translation in t, $T(y,t) = T(y,t-\tau)$ then

$$T_\tau^* A f = A T_\tau^* f \qquad \forall\tau\in\mathbb{R}, f\in C_c^\infty(Y\times\mathbb{R}).$$

Very informally, this makes it so that the kernel of $A\in\Psi^*(Y\times\mathbb{R})$, $A(y,y',t,s)$ depends only on $t-s$ (since it doesn't matter if we translate then integrate, or integrate and then translate), so the operator should have the form

$$Af(y,t) = \int_Y\int_{\mathbb{R}} A(y,y',t-s)f(y',s)ds.$$

This makes A into a convolution operator, i.e. its of the form $\int A(t-s)u(s)ds$. If A, B are two operators whose kernels decrease quickly (are Schwarz) and are convolution operators, then $A*B$ is of the same form. So we've made $\Psi^*(Y\times\mathbb{R})$ into an algebra by giving it some restrictions.

Definition 1. We define $\Psi^m_{\mathrm{sus}}(Y)\subset\Psi^m(Y\times\mathbb{R})$ to be the set of pseudodifferential operators that satisfy the conditions above. Also define the set of pseudodifferential operators action on sections of a bundle E over Y by

$$\Psi^m_{\mathrm{sus}}(Y;E) = \Psi^m_{\mathrm{sus}}(Y)\otimes_{C^\infty(Y)} C^\infty(Y^2;\mathrm{Hom}(E))$$

The key property of $\Psi^*_{\mathrm{sus}}(Y;E)$ is that the symbol of each operator is a symbol in both y and t variables.

From $A\in\Psi^m_{\mathrm{sus}}(Y;E)$ we define the indicial family, as the family of operators

$$\hat{A}(\tau)g = e^{-it\tau}A(e^{it\tau}g)$$

or in terms of kernels

$$\hat{A}(\tau) = \int e^{-it\tau}A(y,y',t)dt.$$

This is a smooth 1-paramter family of pseudodifferential operators.

Proposition 5. *$A \in \Psi^m_{sus}(Y;E)$ elliptic, then $\hat{A}(\tau)$ is elliptic in $\Psi^m(Y;E)$ for all $\tau \in \mathbb{R}$ and is invertible with inverse $\Psi^{-m}(Y;E)$.*

Proposition 6. *$A \in \Psi^m_{sus}(Y;E)$ elliptic and $\hat{A}(\tau)$ is invertible for all $\tau \in \mathbb{R}$ then A is invertible with inverse in $\Psi^{-m}_{sus}(Y;E)$.*

We'll now define a notion of the trace of A, as it will obviously be necessary in order to extend our formula for $\eta(A)$. The **regularized trace** will be give in terms of the indicial family

$$\overline{\mathrm{Tr}}(A) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\,\hat{A}(\tau)d\tau.$$

This is only really defined in $m < -\dim Y - 1$, but we'll want to extend it somehow. The way we'll do this is to consider derivatives

$$h_p(\tau) = \mathrm{Tr}\left(\frac{d^p\hat{A}(\tau)}{d\tau^p}\right)$$

it is not too hard to show, and its done in [1], that

$$A \in \Psi^m_{\mathrm{sus}}(Y;E) \Longrightarrow \frac{d^p\hat{A}(\tau)}{d\tau^p} \in \Psi^{m-p}(Y;E).$$

We'll basically differentiate repeatedly until the the operator is sufficiently smoothing, and then we'll be able to integrate. What is really important is the following

Lemma 15. *$h_p(\tau)$ has a complete asymptotic expansion as $\tau \to \pm\infty$,*

$$h_p(\tau) \sim \sum_l h^{\pm}_{p,l}|\tau|^{m-p+\dim Y-1}$$

Proof. Trust me. Or see [1]. □

Then we'll define the trace of A as

$$\frac{1}{2\pi}\left[\int_{-\tau}^{\tau}\int_0^{\tau_p}\cdots\int_0^{\tau_1} h_p(\tau)drd\tau_1\ldots d\tau_p\right] \qquad p > m + \dim Y$$

as $\tau \to \infty$. This has an asymptotic expansion

$$g_p(\tau) \sim \sum_{j\geq 0} g_j\tau^{m+1+\dim Y-j} + g'(\tau) + g''(\tau)\log\tau. \tag{6}$$

Lemma 16. *The above does not depend on p. Also for $m < -\dim Y - 1$ the above is equivalent to $2\pi\overline{Tr}(A)$.*

Proof. We'll show that the expression is invariant when we increase p by 1. Then

$$g_{p+1}(\tau) = \int_{-\tau}^{\tau}\int_0^{\tau_p}\cdots\int_0^{\tau_1}(h_p(r) + g_0(r))drd\tau_1\ldots d\tau_p.$$

We get this from integrating $h_{p+1}(\tau) = \frac{d}{d\tau}h_p(\tau)$. $g_0(r)$ is actually a constant polynomial, and the $p+1$ other integrals turn this into a polynomial, but without constant term, because of the $\int_{-\tau}^{\tau}$ term. So the coefficient of τ^0 is the same in $g_{p+1}(\tau)$ and $g_p(\tau)$. □

A desirable property of a trace functional is that it must vanish on commutators. Next we'll show that this is indeed the case

Lemma 17. *$\overline{\mathrm{Tr}}$ vanishes on commutators.*

Proof. We want that $\overline{\mathrm{Tr}}([A,B]) = 0$, or

$$\mathrm{coeff}(\tau_0)^{\tau\to\infty}\left[\int_{-\infty}^{\infty}\int_0^{\tau_p}\cdots\int_0^{\tau_1}\frac{d^p}{dr^p}\widehat{[A,B]}(r)drd\tau_1\dots d\tau_p\right] = 0.$$

Since A, B are convolution opertors, $\widehat{[A,B]} = [\hat{A},\hat{B}]$, and

$$\frac{d}{d\tau}[\hat{A}(\tau),\hat{B}(\tau)] = \left[\frac{d\hat{A}(\tau)}{d\tau},\hat{B}(\tau)\right] + \left[\hat{A}(\tau),\frac{d\hat{B}(\tau)}{d\tau}\right].$$

Notice that differentiating like this reduces the order of the operator by 1. Thus if we differentiate enough times, it should happen that $\frac{d^p}{d\tau^p}[\hat{A}(\tau),\hat{B}(\tau)] = 0$. Then the trace will be 0, by the above lemma. □

Finally, we can define the eta invariant.

Definition 2. For $A \in \Psi^m_{\mathrm{sus}}(Y;E)$ elliptic and invertible

$$\eta(A) = 2\pi\overline{Tr}([A,t]A^{-1})$$

or, equivalently since the indicial family of $[A,\tau]$ is $\frac{1}{i}\frac{\partial\hat{A}(\tau)}{\partial\tau}$,

$$\mathrm{coeff}(\tau^0)_{\tau\to\infty}\left[\int_{-\tau}^{\tau}\int_0^{\tau_p}\cdots\int_0^{\tau_1}\mathrm{Tr}\left(\frac{d}{d\tau}\right)^p\left(\frac{\partial\hat{A}(s)}{\partial\tau}\hat{A}(s)^{-1}\right)dsd\tau_1\dots d\tau_p\right]$$

Lemma 18. *$\eta : \mathrm{Inv}(\Psi^*_{sus}(Y;E)) \to \mathbb{C}$ is an additive homomorphism.*

Proof. Fun with commutators! $A, B \in \mathrm{Inv}(\Psi^*_{\mathrm{sus}}(Y;E))$ then

$$[AB,t](AB)^{-1} = [A,t]A^{-1} + A[B,t]B^{-1}A^{-1}.$$

Since $\overline{\mathrm{Tr}}$ vanishes on commutators, it is invariant under conjugation. Take the trace of both sides and we see that $\eta(AB) = \eta(A) + \eta(B)$. □

We will now show that if we define a suspended dirac operator by passing to the "most trivial" family of operators that one could associate with it, we get the eta invariant of APS back.

Theorem 17. *Let*

$$\eth^{\pm}_{sus} = \eta \pm \frac{\partial}{\partial t} \in \Psi^1_{sus}(Y;E).$$

Then the $\eth^{\pm}_{sus}$ are invertible if $\eth$ is invertible and

$$\eta_\nu(\eth^{\pm}_{sus}) = \pm\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-\frac{1}{2}}\,\mathrm{Tr}(\eth e^{-t\eth^2})dt = \pm\eta(\eth)$$

Proof (Quick, Sloppy, and Wrong).

$$
\begin{aligned}
\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-\frac{1}{2}}\operatorname{Tr}(\eth e^{-t\eth^2})dt &= \int_0^\infty\int_{-\infty}^\infty \operatorname{Tr}(\eth e^{-t(\eth^2+\tau^2)})d\tau dt \\
&= \int_{-\infty}^\infty\int_0^\tau \operatorname{Tr}((\eth+i\tau)e^{-t(\eth^2+\tau^2)}dtd\tau \\
&= \int_{-\infty}^\infty \operatorname{Tr}\left(\frac{\eth+i\tau}{\eth^2+\tau^2}\right)d\tau \\
&= \int_{-\infty}^\infty \operatorname{Tr}((\eth-i\tau)^{-1}\frac{d}{d\tau}(\eth-i\tau))d\tau \\
&= \int_{-\infty}^\infty \operatorname{Tr}(\hat{\eth}_{\text{sus}}^{-1}\frac{d}{d\tau}(\hat{\eth}_{\text{sus}}))d\tau
\end{aligned}
$$

□

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19. 9 May, 2006

19.1. **Maksim Lipyanskiy – Bott peridicity and index.** For a locally compact Hausdorff space

$$K(X)=\ker(p),\ p:K(X^+)\longrightarrow K(+),\ +\hookrightarrow X^+. \tag{1}$$

For a product

$$0\longrightarrow K(X\times Y)\longrightarrow K(X^+\times Y^+)\longrightarrow K(X^+)\oplus K(Y^+),\ (X\times Y)^+=X^+\times Y^+/(X^+\times +)\cup(+\times X^+). \tag{2}$$

Set

$$K^{-n}(X)=K(\mathbb{R}^n\times X) \tag{3}$$

and Bott periodicity becomes

$$
\begin{gathered}
K^{-n-2}(X)\simeq K^{-n}(X),\\
K(X)\times K(\mathbb{C}\times X)=K(\mathbb{R}^2\times X).
\end{gathered} \tag{4}
$$

Explicity maps are given by the Bott map

$$\beta:K(X)\longrightarrow K^{-2}(X),\ v\longmapsto b\boxtimes v \tag{5}$$

where the Bott class $b\in K(\mathbb{R}^2)$ generates the K-theory. Want to construct and inverse $\alpha:K^{-2}(X)\longrightarrow K(X)$.

Suppose one can construct α so that

(1) $\alpha(b)=1$ in $K(\{\text{pt}\})$.
(2) The diagramme

$$
\begin{array}{ccc}
K^{-2}(X\times Y) & \longleftarrow & K^{-2}(X)\otimes K(Y)\\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha}\\
K(X\times Y) & \longleftarrow & K(X)\otimes K(Y).
\end{array} \tag{6}
$$

The diagramme is just

$$K(\mathbb{R}^2\times X)\otimes K(Y) \longrightarrow K(\mathbb{R}^2\times X\times Y) \quad (7)$$
$$\downarrow\alpha \qquad\qquad \downarrow\alpha$$
$$K(X)\otimes K(Y)\longrightarrow K(X\times Y)$$

If $X=\{\mathrm{pt}\}$ then this implies

$$\alpha(b\cdot u)=\alpha(b)\cdot u=u \quad (8)$$

so α is surjective.

Next, take $Y=\mathbb{R}^2$ giving

$$K(\mathbb{R}^2\times X)\otimes K(\mathbb{R}^2) \longrightarrow K(\mathbb{R}^2\times X\times \mathbb{R}^2) \quad (9)$$
$$\downarrow\alpha \qquad\qquad \downarrow\alpha$$
$$K(X)\otimes K(\mathbb{R}^2)\longrightarrow K(X\times \mathbb{R}^2)$$

Now, $\sigma:(v,x,w)\longmapsto(w,x,v)$ is properly homotopic to the identity. Then

$$\alpha(ub)=\sigma(b\tilde{u})=\tilde{u} \quad (10)$$

where, if $u\in K(\mathbb{R}^2\times X)$ then $\tilde{u}\in K(X\times\mathbb{R}^2)$ is the corresponding element. Thus α is injective.

Thus it is only necessary to construct α, and only for compact X.

For a compact manifold M and compact Hausdorff space X consider $K(M\times X)$. This can be represented by bundles which are smooth over M. If $d:E\longrightarrow F$ is an elliptic differential operator then the index is an element of $K(X)$. Can twist by any vector bundle over $M\times X$. This constructs a map $K(M\times X)\longrightarrow K(X)$ given by the index of d_Q. This is a homomorphism over the module action of $K(X)$.

So, to construct d consider $\overline{\partial}:\Lambda^{0,0}(\mathbb{S}^2)\longrightarrow\Lambda^{0,1}$ has index 1. Then consider the anti-canonical bundle $\mathcal{O}(-1)$ then $\overline{\partial}_b$ has index 0. Now, $1-b\in K(\mathbb{R}^2)$ corresponding to $K(\mathbb{R}^2)\longrightarrow K(\mathbb{S}^2)\longrightarrow\mathbb{Z}$. So we get α by composing

$$K(\mathbb{R}^2\times X)\longrightarrow K(\mathbb{S}^2\times X)\longrightarrow K(X) \quad (11)$$

has all the desired properties.

20. 11 May, 2006

21. Zuoqin Wang:- Cheeger-Müller Theorem (his notes)

I. Statement of the theorem

• (M,g) compact odd dim Riemannian manifold, (F,g^F) flat vector bundle over M.

• If E is a n-dimensional vector space, set $\det E=\Lambda^n(E)$.

• If $(V^\bullet,\partial)\ :\ 0\to V^0\overset{\partial}{\to}V^1\overset{\partial}{\to}\cdots\overset{\partial}{\to}V^n\to 0$ be a chain complex of finite dimensional vector spaces, $H^\bullet(V)=\oplus_{i=0}^n H^i(V)$ be its cohomology, set

$$\det V^\bullet=\otimes_{i=0}^n(\det V^i)^{(-1)^i},\quad \det H^\bullet(V)=\otimes_{i=0}^n(\det H^i(V))^{(-1)^i}.$$

• Claim: There is a canonical isomorphism $\det V^\bullet\simeq\det H^\bullet(V)$.

Proof: For $0\le i\le n$ we have short exact sequences

$0\to\partial(V^i)\to\mathrm{Ker}(\partial_{i+1})\to H^{i+1}\to 0,\ 0\to\mathrm{Ker}(\partial_{i+1})\to V^{i+1}\to\partial(V^{i+1})\to 0.$

These exact sequences give canonical isomorphisms
$\det(V^{i+1}) \simeq \det(\mathrm{Ker}(\partial_{i+1}))\otimes\det(\partial(V^{i+1}))$, $\det(\mathrm{Ker}(\partial_{i+1})) \simeq \det(\partial(V^i))\otimes\det(V^{i+1})$.
This gives the required isomorphism.

• Given metrics on V^i, which induces metrics $\|\cdot\|_{\det V^i}$ on $\det V^i$, we can canonically construct a metric on $\det(V^\bullet)$ by $\|\cdot\|_{\det V^\bullet} = \otimes_{i=0}^n \|\cdot\|_{(\det V^i)^{(-1)^i}}$. Let $\|\cdot\|_{\det H^\bullet(V)}$ be the corresponding metric on $\det H^\bullet(V)$ by the above isomorphism.

• Let ∂^* be the adjoint of ∂. By Hodge theory, $H^i(V^\bullet,\partial) \simeq \{v \in V^i \mid \partial v = \partial^* v = 0\}$. Denote $|\cdot|_{\det H^\bullet(V)}$ be the metric inherited from the metric on $V^\bullet$.

• Set $\Delta = \partial\partial^* + \partial^*\partial$, and let $\Delta^i = \Delta|_{V^i}$. Define $\det'(\Delta^i)$ to be the product of the nonzero eigenvalues of Δ^i.

• The **torsion** $T^{\mathcal{M}}$ of the complex $(V^\bullet,\partial)$ is $\log T^{\mathcal{M}} = \frac{1}{2}\sum_{i=0}^n (-1)^i i \log\det'\Delta^i$.

• Claim: $\|\cdot\|_{\det H^\bullet(V)} = |\cdot|_{\det H^\bullet(V)} \cdot T^{\mathcal{M}}$.

• Let $f : M \to \mathbb{R}$ be a Morse function satisfying the Thom-Smale transversality conditions, i.e. for any two critical points x, y of f, the stable manifold $W^s(x)$ and the unstable manifold $W^u(y)$ (w.r.t ∇f) intersects transversely. Let B be the set of critical points. For $x \in B$, let $[W^u(x)]$ be the real line generated by $W^s(x)$.

• Set $C^i(W^u, F) = \oplus_{x\in B,\mathrm{ind}(x)=i}[W^u(x)]^* \otimes_{\mathbb{R}} F_x$. By result of Thom-Smale, there are well-defined operators $\partial : C^i(W^u,F) \to C^{i+1}(W^u,F)$ such that $(C^\bullet(W^u,F),\partial)$ is a chain complex, and there is a canonical identification of $\mathbb{Z}-$graded vector spaces $H^\bullet(C^\bullet(W^u,F),\partial) \simeq H^\bullet(M,F)$. Thus $\det H^\bullet(M,F) \simeq \det C^\bullet(W^u,F)$.

• The metric g^F induces a metric on $C^\bullet(W^u,F)$, thus a metric on $\det C^\bullet(W^u,F)$. By above isomorphism, we get the **Milnor metric** $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ on the line $\det H^\bullet(M,F)$. Milnor proved that this metric coincides with the Reidemeister metric, thus doesn't depend on f, g, g^F.

• Let $(\Omega^\bullet(M,F), d^F)$ be the deRham complex of smooth sections of $\Lambda(T^*M)\otimes F$. The cohomology of this complex is $H^\bullet(M,F)$, which is isomorphic to the space of harmonic forms in $\Omega^\bullet(M,F)$. Thus the L^2-metric on $\Omega^\bullet(M,F)$ induces a metric $|\cdot|^{RS}_{\det H^\bullet(M,F)}$ on $\det H^\bullet(M,F)$.

• The **Ray-Singer metric** is defined to be $\|\cdot\|^{RS}_{\det H^\bullet(M,F)} = |\cdot|^{RS}_{\det H^\bullet(M,F)} \cdot T^{RS}$, where T^{RS} is the corresponding Ray-Singer torsion.

Now we can state the theorem:

Theorem 18 (Cheeger-Muller). $\|\cdot\|^{RS}_{\det H^\bullet(M,F)} = \|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$.

• The pair (g^{TM}, f) is called a generalized triangulation of M if f is a Morse function satisfying the Thom-Smale transversality condition, and in a neighborhood of each critical point x, one can introduce local coordinate $(y_1,\cdots,y_n)$ with g^{TM} Euclidean in it, such that $f(y) = f(x) - \frac{1}{2}(y_1^2 + \cdots + y_k^2) + \frac{1}{2}(y_{k+1}^2 + \cdots + y_n^2)$.

We only need to prove the theorem for (g, f) being a generalized triangulation.

II. Technical Lemmas

Set $d_t^F = e^{-tf} d^F e^{tf}$, then $d_t^{F^*} = e^{tf}\delta^F e^{-tf}$. The Witten Laplacian is defined to be $\Delta_{f,t} = d_t^F d_t^{F^*} + d_t^{F^*} d_t^F$. Denote $\Delta^i_{f,t} = \Delta_{f,t}|_{\Omega^i(M,F)}$. Let $T^{RS}(f,t)$ be the corresponding Ray-Singer torsion.

On the other hand, consider the metric $g_t^F = e^{-2tf} g^F$ on F. Let $|\cdot|^{RS}_{\det H^\bullet(M,F),f,t}$ be the corresponding L^2-metric on $\det H^\bullet(M,F)$. It is easy to see $\delta_t^F = e^{2tf}\delta^F e^{-2tf}$, thus $d^F + \delta_t^F = e^{tf}(d_t^F + d_t^{F^*})e^{-tf}$, so the corresponding new Laplacian satisfies $\tilde{\Delta}_{f,t} = (d^F + \delta_t^F)^2 = e^{tf}(d_t^F + d_t^{F^*})^2 e^{-tf} = e^{tf}\Delta_{f,t}e^{-tf}$. Since the map $\alpha \mapsto e^{-tf}\alpha$

is isomorphism on the space of forms, the new Laplacian and the Witten Laplacian is isospectral. Thus $T^{RS}(f,t)$ equals the Ray-Singer torsion with respect to g_t^F and g^{TM}. It follows that $\|\cdot\|^{RS}_{\det H^\bullet(M,F)} = |\cdot|^{RS}_{\det H^\bullet(M,F),f,t}\cdot T^{RS}(f,t)$.

It is well-known that $\mathrm{spec}(\Delta_{f,t})\subset[0,e^{-|t|C'})\cup(C''|t|,\infty)$ for t big. Let $T^{RS}_{sm}(f,t)$ be the Torsion of the complex with V^i the space with small eigenvalues, $T^{RS}_{la}(f,t)$ be the "Ray-Singer" Torsion corresponding to $\zeta_{la}(s)$, the zeta function defined by big eigenvalues. Clearly, we have $T^{RS}(f,t)=T^{RS}_{sm}(f,t)\cdot T^{RS}_{la}(f,t)$ for $|t|>t_0$.

Set $R(M,F,f)=\log\|\cdot\|^{RS}_{\det H^\bullet(M,F)}-\log\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$. Note that $R(M,F,f)$ is in fact independent of f, and we only need to prove $R(M,F,f)=0$.

Lemma 19 (Bismut-Zhang)**.** *Suppose (g,f) is a generalized triangulation, then*
$$\log T^{RS}_{la}(f,t)=R(M,F,f)+t\,\mathrm{rank}(F)\mathrm{Tr}^B_s[f]-\tfrac{1}{2}\tilde\chi'(F)\log\left(\tfrac{t}{\pi}\right)+o(1),$$
as $t\to+\infty$, where $Tr^B_s[f]=\sum_{x\in B}(-1)^{\mathrm{ind(x)}}f(x)$, $\tilde\chi'(F)=\mathrm{rank}(F)\sum_{x\in B}(-1)^{\mathrm{ind(x)}}\mathrm{ind}(x)$.

Lemma 20 (Braverman)**.** *Let $M,\tilde M$ be Riemannian manifolds with same odd dimension, $F,\tilde F$ be flat vector bundles with same rank, $f:M\to\mathbb{R}$ and $\tilde f:\tilde M\to\mathbb{R}$ be Morse functions with same critical point structure, i.e. there exists $U\subset M,\tilde U\subset\tilde M$ neighborhoods of the sets of critical points such that $f=\tilde f\circ\phi$ for some isometry $\phi:U\to\tilde U$. Then*
$$\log T^{RS}_{la}(f,t)-\log T^{RS}_{la}(\tilde f,t)=\textstyle\sum_{j=0}^\infty\left(a_j(t/|t|)t^j+b_j(t/|t|)t^j\log|t|\right)+o(1),$$
as $t\to\infty$, and the free term satisfies $a_0(1)+a_0(-1)=0$.

Lemma 21 (Milnor)**.** *There exists generalized triangulation $(g^{M\times S^2},f_1)$ of $M\times S^2$ and $(g^{M\times S^1\times S^1},f_2)$ of $M\times S^1\times S^1$ such that f_1 and f_2 have same critical structure.*

III. Proof of the theorem.

By definition we have $\Delta_{f,-t}=\Delta_{-f,t}$, thus $T^{RS}_{la}(f,-t)=T^{RS}_{la}(-f,t)$. So by Lemma 1, $R(M,F,f)-R(\tilde M,\tilde F,\tilde f)$ equals the free term of the asymptotic expansion of $\log T^{RS}_{la}(f,t)-\log T^{RS}_{la}(\tilde f,t)$. Hence from Lemma 2 we conclude

$$(1)\qquad [R(M,F,f)-R(\tilde M,\tilde F,\tilde f)]+[R(M,F,-f)-R(\tilde M,\tilde F,-\tilde f)]=0.$$

But $R(M,F,f)$ is in fact independent of f, thus we have $R(M,F,f)=R(\tilde M,\tilde F,\tilde f)$.

On the other hand, suppose N is a compact manifold of even dimension, (g^{TN},f^N) is a generalized triangulation on N, then $\bar f(x,y)=f(x)+f^N(y)$ is a Morse function on $M\times N$, and by exactly the same proof of the theorem on Analytic torsion of product manifolds, we have $\log T^{RS}_{la}(\bar f,t)=\chi(N)\log T^{RS}_{la}(f,t)$. Thus by Lemma 1 we obtain $R(M\times N,\bar F,\bar f)=\chi(N)R(M,F,f)$. As a corollary, we have

$$(2)\qquad R(M\times S^2,\bar F,\bar f)=2R(M,F,f),\quad R(M\times S^1\times S^1,\bar F',\bar f')=0.$$

Now the theorem follows from Lemma 3.

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